

The Multiple Linear Regression Model

General model

$$Y_t = \beta_0 + \beta_1 X_{1t} + \dots + \beta_k X_{kt} + u_t$$

$$\beta_i = \frac{\partial Y_t}{\partial X_{it}}$$

for each of the observations, the general model can be written as

$$Y_1 = \beta_0 + \beta_1 X_{11} + \dots + \beta_k X_{k1} + u_1$$

$$Y_2 = \beta_0 + \beta_1 X_{12} + \dots + \beta_k X_{k2} + u_2$$

$$Y_3 = \beta_0 + \beta_1 X_{13} + \dots + \beta_k X_{k3} + u_3$$

$\vdots$

$$Y_T = \beta_0 + \beta_1 X_{1T} + \dots + \beta_k X_{kT} + u_T$$

using a matrix notation

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_T \end{bmatrix} = \begin{bmatrix} 1 & X_{11} & X_{12} & \dots & X_{1k} \\ 1 & X_{21} & & & X_{2k} \\ \vdots & & & & \\ 1 & & & & X_{Tk} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_k \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_T \end{bmatrix}$$

$\begin{matrix} \times & & \times & & \beta & & u \end{matrix}$

$\begin{matrix} T \times (k+1) & (k+1) \times 1 & (T \times 1) \end{matrix}$

$$Y = X\beta + u$$

Assumptions

We maintain all the basic assumptions hold

$$E(u_t) = 0$$

$$\text{Var}(u_t) = \sigma^2 \quad (\text{homoscedasticity, no heteroscedasticity})$$

$$\text{Cov}(u_t, u_s) = 0 \quad (\text{no autocorrelation, no serial correlation})$$

$$X_i \rightarrow Y \quad \text{but not } Y \rightarrow X_i$$

No perfect linear relationship between X_i and X_i

$$\text{Var}(X_{it}) \neq 0 \rightarrow \text{Some observations for } X_i \text{ are different from others}$$

Estimation:

$$Y_t = \beta_0 + \beta_1 X_{1t} + \dots + \beta_k X_{kt} + u_t$$

$$Y_t = \beta_0 + \sum_{i=1}^k \beta_i X_{ti} + u_t$$

estimate:

$$\hat{Y}_t = \hat{\beta}_0 + \hat{\beta}_1 X_{t1} + \dots + \hat{\beta}_k X_{tk}$$

$$\hat{Y}_t = \hat{\beta}_0 + \sum_{i=1}^k \hat{\beta}_i X_{ti}$$

$$\hat{u}_t = Y_t - \hat{Y}_t$$

$$\text{Minimise } SSR = \sum \hat{u}_t^2 = \sum (Y_t - \hat{Y}_t)^2$$

w.r.t. $\hat{\beta}_0, \hat{\beta}_1, \dots, \hat{\beta}_k$

$$SSR = \sum (Y_t - \hat{\beta}_0 - \hat{\beta}_1 X_{t1} - \dots - \hat{\beta}_k X_{tk})^2$$

min SSR

First order conditions

$$\frac{\partial SSR}{\partial \hat{\beta}_0} = -2 \sum (Y_t - \hat{\beta}_0 - \hat{\beta}_1 X_{t1} - \dots - \hat{\beta}_k X_{tk}) = 0$$

$\sum \hat{u}_t = 0$

$$\frac{\partial SSR}{\partial \hat{\beta}_1} = -2 \sum X_{t1} (Y_t - \hat{\beta}_0 - \hat{\beta}_1 X_{t1} - \dots - \hat{\beta}_k X_{tk}) = 0$$

⋮

$$\frac{\partial SSR}{\partial \hat{\beta}_k} = -2 \sum X_{tk} (Y_t - \hat{\beta}_0 - \hat{\beta}_1 X_{t1} - \dots - \hat{\beta}_k X_{tk}) = 0$$

$\sum X_{tk} \hat{u}_t = 0$

normal equations

$$N_1 \quad T \hat{\beta}_0 + \hat{\beta}_1 \sum X_{t1} + \hat{\beta}_2 \sum X_{t2} + \dots + \hat{\beta}_k \sum X_{tk} = \sum Y_t$$

$$N_2 \quad \hat{\beta}_0 \sum X_{t1} + \hat{\beta}_1 \sum X_{t1}^2 + \hat{\beta}_2 \sum X_{t1} X_{t2} + \dots + \hat{\beta}_k \sum X_{t1} X_{tk} = \sum X_{t1} Y_t$$

⋮

$$N_{k+1} \quad \hat{\beta}_0 \sum X_{tk} + \hat{\beta}_1 \sum X_{t1} X_{tk} + \dots + \hat{\beta}_k \sum X_{tk}^2 = \sum X_{tk} Y_t$$

re-arrange in matrix form

$$\begin{bmatrix} T & \sum X_{t1} & \sum X_{t2} & \dots & \sum X_{tk} \\ \sum X_{t1} & \sum X_{t1}^2 & \sum X_{t1} X_{t2} & \dots & \sum X_{t1} X_{tk} \\ \sum X_{t2} & \sum X_{t1} X_{t2} & \sum X_{t2}^2 & \dots & \sum X_{t2} X_{tk} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sum X_{tk} & \sum X_{t1} X_{tk} & \sum X_{t2} X_{tk} & \dots & \sum X_{tk}^2 \end{bmatrix} \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \vdots \\ \hat{\beta}_k \end{bmatrix} = \begin{bmatrix} \sum Y_t \\ \sum X_{t1} Y_t \\ \vdots \\ \sum X_{tk} Y_t \end{bmatrix}$$

$$X'X \hat{\beta} = X'Y$$

pre-multiplying both sides by $(X'X)^{-1}$

$$\underbrace{(X'X)^{-1} X'X}_{K} \beta = (X'X)^{-1} X'Y$$

$$\hat{\beta} = (X'X)^{-1} X'Y$$

* If x_i are not linearly independent $\Rightarrow$

$X'X$ will be singular $\Rightarrow$

$\hat{\beta}$ cannot be estimated

sampling properties: bias

$$\begin{aligned} \hat{\beta} &= (X'X)^{-1} X'Y \\ &= (X'X)^{-1} X'(X\beta + u) \\ &= (X'X)^{-1} X'X\beta + (X'X)^{-1} X'u \\ \hat{\beta} &= \beta + (X'X)^{-1} X'u \end{aligned}$$

$$E(\hat{\beta}) = \beta \quad \text{since } E(u) = 0$$

$\hat{\beta}$ is unbiased

Variance - Covariance

$$\begin{aligned} \hat{\beta} &= (X'X)^{-1} X'Y \\ &= (X'X)^{-1} X'(X\beta + u) \\ &= (X'X)^{-1} X'X\beta + (X'X)^{-1} X'u \\ \hat{\beta} &= \beta + (X'X)^{-1} X'u \end{aligned}$$

$$E(\hat{\beta}) = \beta \quad \text{since } E(u) = 0 \quad \hat{\beta} \text{ is unbiased}$$

variance - covariance

$$\begin{aligned}
 \text{Var-cov}(\hat{\beta}) &= E[\hat{\beta} - E(\hat{\beta})][\hat{\beta} - E(\hat{\beta})]' \\
 &= E(\hat{\beta} - \beta)(\hat{\beta} - \beta)' \\
 &= (X'X)^{-1} X' \underbrace{U U' X}_{\delta^2 I} (X'X)^{-1}
 \end{aligned}$$

$$\rightarrow = \delta^2 \overbrace{(X'X)^{-1} X' X (X'X)^{-1}}^I$$

using the fact that $(X'X)$ is a symmetric and independent matrix

$$\rightarrow (X'X)^{-1} = [(X'X)^{-1}]'$$

$$\text{Var-cov}(\hat{\beta}) = \delta^2 (X'X)^{-1}$$

estimated variance - covariance

$$\text{Var-cov}(\hat{\beta}) = \delta^2 (X'X)^{-1}$$

$$\text{Est. Var-cov}(\hat{\beta}) = \hat{\delta}^2 (X'X)^{-1}$$

consider

$$\begin{aligned}
 &\text{estimated var-cov}(\hat{\beta}) = \begin{bmatrix} a_1 & a_2 & \dots & a_k \\ b_1 & b_2 & \dots & b_k \\ \vdots & \vdots & \ddots & \vdots \\ k_k & \vdots & \vdots & \vdots \end{bmatrix} \\
 &\quad \nearrow \text{est cov}(\hat{\beta}_0, \hat{\beta}_1) \quad \quad \quad \nearrow \text{Est cov}(\hat{\beta}_0, k_k) \\
 &\quad \quad \quad \searrow \text{Var}(\hat{\beta}_k)
 \end{aligned}$$

General Linear Model II

$$k=2$$

$$Y_t = \beta_0 + \beta_1 X_{t1} + \beta_2 X_{t2} + u_t$$

$$\hat{Y}_t = \hat{\beta}_0 + \hat{\beta}_1 X_{t1} + \hat{\beta}_2 X_{t2}$$

$$\hat{u}_t = Y_t - \hat{Y}_t$$

$$\min \sum \hat{u}_t^2 = \sum (Y_t - \hat{Y}_t)^2 \text{ wrt. } \hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2$$

normal equations (form FOC's) $\Rightarrow$

$$(N_1) \quad \frac{\partial SSR}{\partial \hat{\beta}_0} \rightarrow T \hat{\beta}_0 + \hat{\beta}_1 \sum X_{t1} + \hat{\beta}_2 \sum X_{t2} = \sum Y_t$$

$$(N_2) \quad \frac{\partial SSR}{\partial \hat{\beta}_1} = \hat{\beta}_0 \sum X_{t1} + \hat{\beta}_1 \sum X_{t1}^2 + \hat{\beta}_2 \sum X_{t1} X_{t2} = \sum X_{t1} Y_t$$

$$(N_3) \quad \frac{\partial SSR}{\partial \hat{\beta}_2} = \hat{\beta}_0 \sum X_{t2} + \hat{\beta}_1 \sum X_{t1} X_{t2} + \hat{\beta}_2 \sum X_{t2}^2 = \sum X_{t2} Y_t$$

from (N1)

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}_1 - \hat{\beta}_2 \bar{X}_2$$

↓

insertions to N_2 and N_3 gives

$$(N_2^*) \quad \hat{\beta}_1 \sum X_{t1}^2 + \hat{\beta}_2 \sum X_{t1} X_{t2} = \sum X_{t1} Y_t$$

$$(N_3^*) \quad \hat{\beta}_1 \sum X_{t1} X_{t2} + \hat{\beta}_2 \sum X_{t2}^2 = \sum X_{t2} Y_t$$

from N_2^* & N_3^* after some reorganization

$$\hat{\beta}_1 = \frac{\sum \hat{u}_{t2} Y_t}{\sum \hat{u}_{t1}^2}$$

where, $\hat{u}_{ti}$: the residuals from the regression of its explanatory variable

on the other

$$Y_t = \beta_0 + \beta_1 X_{t1} + \beta_2 X_{t2} + U_t$$

$$\Rightarrow \hat{\beta}_1 = \frac{\sum \hat{U}_{t1} Y_t}{\sum \hat{U}_{t1}^2}$$

where $\hat{U}_{t1}$: the residuals from

$$X_{t1} = c_{10} + c_{12} X_{t2} + U_{t1}$$

the part of X_{t1} not explained by X_{t2}

$$Y_t = \beta_0 + \beta_1 X_{t1} + \beta_2 X_{t2} + U_t$$

$$\hat{\beta}_2 = \frac{\sum \hat{U}_{t2} Y_t}{\sum \hat{U}_{t2}^2}$$

where $\hat{U}_{t2}$ the residual from

$$X_{t2} = c_{20} + c_{21} X_{t1} + U_{t2}$$

the part of X_{t2} not explained by X_{t1}

Assignment

consider

$$(1) Y_t = \beta_0 + \beta_1 X_{t1} + \beta_2 X_{t2} + U_{t1}$$

$$(2) Y_t = \gamma_0 + \gamma_1 X_{t1} + U_{t2}$$

$$(3) Y_t = \alpha_0 + \alpha_1 X_{t2} + U_{t3}$$

$$1- \text{ is } \hat{\beta}_1 = \hat{\gamma}_1$$

$$\text{is } \hat{\beta}_2 = \hat{\alpha}_1$$

Y_t output
 X_{t1} = labor
 X_{t2} = capital

answer

consider the normal equations derived earlier

$$(N_2^*) \quad \hat{\beta}_1 \sum x_{t1}^2 + \hat{\beta}_2 \underbrace{\frac{\sum x_{t1} x_{t2}}{\sum x_{t1}^2}}_{\hat{C}_{21}} = \underbrace{\frac{\sum x_{t1} y_t}{\sum x_{t1}^2}}_{\hat{d}_1}$$

$$(N_3^*) \quad \hat{\beta}_1 \underbrace{\frac{\sum x_{t1} x_{t2}}{\sum x_{t2}^2}}_{\hat{C}_{12}} + \hat{\beta}_2 \frac{\sum x_{t2}^2}{\sum x_{t2}^2} = \frac{\sum x_{t2} y_t}{\sum x_{t2}^2} \hat{d}_2$$

$$\hat{\beta}_1 + \hat{\beta}_2 \hat{C}_{21} = \hat{d}_1$$

$$\hat{\beta}_1 \hat{C}_{12} + \hat{\beta}_2 = \hat{d}_2$$

where

$$x_{t2} = C_{20} + C_{21} x_{t1} + v_{t2}$$

$$x_{t1} = C_{10} + C_{12} x_{t2} + v_{t1}$$

$$\hat{\beta}_1 = \hat{d}_1 \quad \text{iff} \quad \hat{C}_{21} = 0$$

$$\hat{\beta}_2 = \hat{d}_2, \quad \hat{C}_{12} = 0$$