

Econ 205

Continuous Random Variables and Probability
Distributions – 1

Course Outline

- **Probability**

- Probability and Inference
- The Discrete Case
- Calculating the Probability of an Event
- Conditional Probability and the Independence of Events
- Laws of Probability
- The Law of Total Probability and Bayes' Rule
- Introduction to Random Variables and their distributions

- **Discrete Random Variables and Their Probability Distributions**

- Discrete Random Variable
- The Probability Distribution for Discrete Random Variable
- The Expected Value of Random Variable
- Moments and Moment-Generating Functions
- The Bernoulli Probability Distribution
- The Binomial Probability Distribution
- The Geometric Probability Distribution
- The Negative Binomial Probability Distribution
- The Hypergeometric Probability Distribution
- The Poisson Probability Distribution
- Probability-Generating Functions

- **Continuous Random Variables and Their Probability Densities**

- Continuous Random Variable
- The Probability Distribution for Continuous Random Variable
- The Expected Value for Continuous Random Variable
- Moments and Moment-Generating Functions
- The Uniform Probability Distribution
- The Normal Probability Distribution
- Exponential Probability Distribution
- Chi-Square Probability Distribution
- Tchebysheff's Theorem

Course Outline (cont'd)

- **Multivariate Probability Distributions**

- Bivariate and Multivariate Probability Distributions
- Independent Random Variables
- Marginal and Conditional Probability Distributions
- The Expected Value of a Function of Random Variables
- Special Theorems
- The Covariance of Two Random Variables
- The Expected Value and Variance of Linear Functions of Random Variables
- The Multinomial Probability Distribution
- The Bivariate Normal Distribution
- Conditional Expectations

- **Functions of Random Variables**

- Finding the Probability Distribution of a Function of Random Variables
- The Method of Distribution Functions
- The Method of Transformations
- The Method of Moment-Generating Functions
- Multivariable Transformations Using Jacobians

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Content of this presentation

- **Continuous Random Variables and Probability Distributions**
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 - Continuous Random Variables
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 - Uniform Distribution
 - Cdfs
 - Using $F(x)$ to compute probabilities
 - Obtaining $f(x)$ from $F(x)$
 - Percentiles of a Continuous Distribution
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 - Percentiles of the Standard Normal Distribution
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 - The Normal Distribution and Discrete Populations
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- **Examples**

Continuous Random Variables

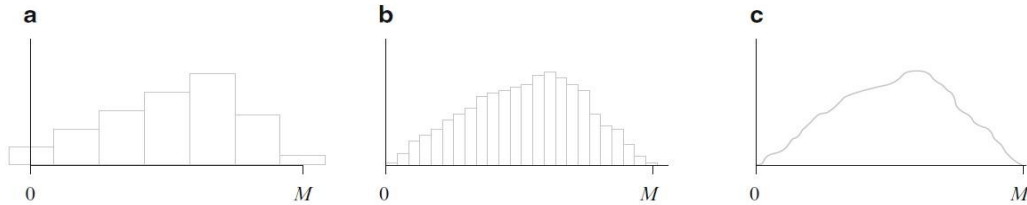


Figure 4.1 (a) Probability histogram of depth measured to the nearest meter; (b) probability histogram of depth measured to the nearest centimeter; (c) a limit of a sequence of discrete histograms

Recall:

A random variable is **continuous** if *both* of the following apply:

1. Its set of possible values consists either of all numbers in a single interval on the number line (possibly infinite in extent, e.g., from $-\infty$ to ∞) or all numbers in a disjoint union of such intervals (e.g., $[0, 10] \cup [20, 30]$).
2. No possible value of the variable has positive probability, that is, $P(X = c) = 0$ for any possible value c .

Definition from [WMS]:

A random variable Y is **continuous** if its (cumulative) distribution function $F(y)$ is continuous.

Remark: There are random variables which are neither discrete nor continuous but instead a mixture of the two types.

Probability Density Functions (Pdfs)

Definition:

Let X be a continuous rv. Then a **probability distribution** or **probability density function (pdf)** of X is a function $f(x)$ such that for any two numbers a and b with $a \leq b$,

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

That is, the probability that X takes on a value in the interval $[a, b]$ is the area above this interval and under the graph of the density function, as illustrated in Figure 4.2. The graph of $f(x)$ is often referred to as the *density curve*.

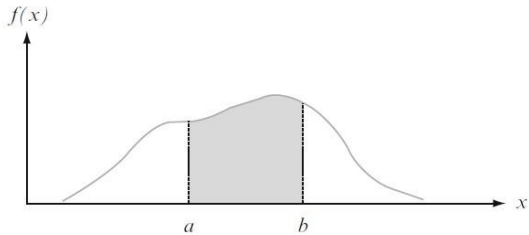


Figure 4.2 $P(a \leq X \leq b)$ = the area under the density curve between a and b

For $f(x)$ to be a legitimate pdf, it must satisfy the following two conditions:

1. $f(x) \geq 0$ for all x
2. $\int_{-\infty}^{\infty} f(x)dx = [\text{area under the entire graph of } f(x)] = 1$

Compare with the definitions for the discrete case:

The **probability distribution** or **probability mass function (pmf)** of a discrete rv is defined for every number x by $p(x) = P(X = x) = P(\text{all } s \in \mathcal{S}: X(s) = x)$.¹

Definition from [WMS] for the discrete case:

The **probability distribution** of a (discrete) random variable is the collection of probabilities for each value of a random variable. It can be represented by a formula, table or a graph that provides $p(y) = P(Y=y)$ for all y .

Compare with the properties for the discrete case:

- 1) $p(x) \geq 0$ for all x
- 2) $\sum_x p(x) = 1$

Pdfs and Uniform Distribution

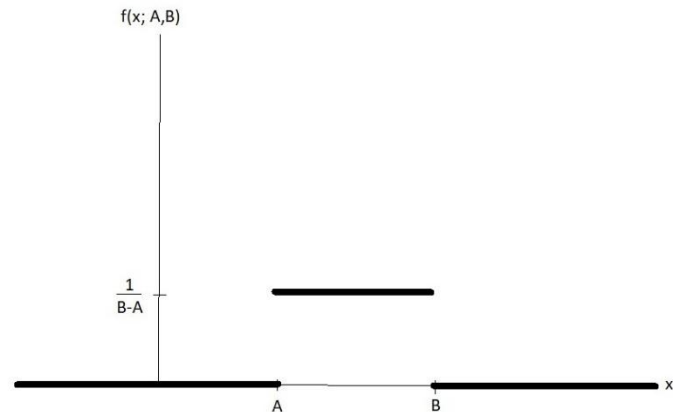
The fact that $P(X = c) = 0$ when X is continuous has an important practical consequence: The probability that X lies in some interval between a and b does not depend on whether the lower limit a or the upper limit b is included in the probability calculation:

$$P(a \leq X \leq b) = P(a < X < b) = P(a < X \leq b) = P(a \leq X < b)$$

Definition:

A continuous rv X is said to have a **uniform distribution** on the interval $[A, B]$ if the pdf of X is

$$f(x; A, B) = \begin{cases} \frac{1}{B-A} & A \leq X \leq B \\ 0 & \text{otherwise} \end{cases}$$



The Cumulative Distribution Function (Cdf) for Continuous Random Variables

Definition:

The **cumulative distribution function** $F(x)$ for a continuous rv X is defined for every number x by

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(y) dy$$

For each x , $F(x)$ is the area under the density curve to the left of x . This is illustrated in Figure 4.5, where $F(x)$ increases smoothly as x increases.

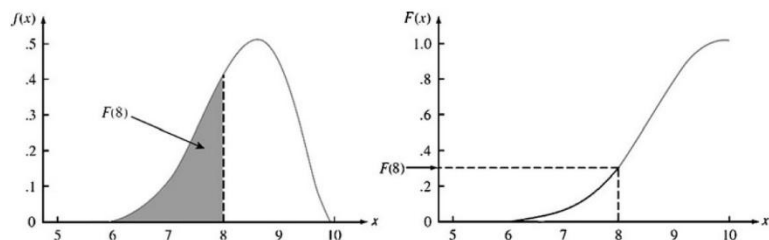


Figure 4.5 A pdf and associated cdf

Compare with the definition for the discrete case:

The **cumulative distribution function** (cdf) $F(x)$ of a discrete rv X with pmf $p(x)$ is defined for every number x by

$$F(x) = P(X \leq x) = \sum_{y: y \leq x} p(y)$$

For any number x , $F(x)$ is the probability that the observed value of X will be at most x .

Property:

For X a discrete rv, the graph of $F(x)$ will have a jump at every possible value of X and will be flat between possible values. Such a graph is called a **step function**.

Using $F(x)$ to Compute Probabilities:

Proposition:

Let X be a continuous rv with pdf $f(x)$ and cdf $F(x)$. Then for any number a ,

$$P(X > a) = 1 - F(a)$$

and for any two numbers a and b with $a < b$,

$$P(a \leq X \leq b) = F(b) - F(a)$$

Figure 4.8 illustrates the second part of this proposition; the desired probability is the shaded area under the density curve between a and b , and it equals the difference between the two shaded cumulative areas.

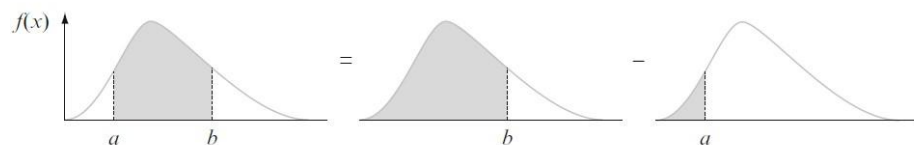


Figure 4.8 Computing $P(a \leq X \leq b)$ from cumulative probabilities

Compare with the Proposition for the DISCRETE case:

Let X be a discrete random variable.

For any two numbers a and b with $a \leq b$,

$$P(a \leq X \leq b) = F(b) - F(a-)$$

where $F(a-)$ represents the maximum of $F(x)$ values to the left of a . Equivalently, if a is the limit of values of x approaching from the left, then $F(a-)$ is the limiting value of $F(x)$. In particular, if the only possible values are integers and if a and b are integers, then

$$\begin{aligned} P(a \leq X \leq b) &= P(X = a \text{ or } a + 1 \text{ or } \dots \text{ or } b) \\ &= F(b) - F(a - 1) \end{aligned}$$

Taking $a = b$ yields $P(X = a) = F(a) - F(a - 1)$ in this case.

Obtaining $f(x)$ from $F(x)$:

For X discrete, the pmf is obtained from the cdf by taking the difference between two $F(x)$ values. The continuous analog of a difference is a derivative. The following result is a consequence of the Fundamental Theorem of Calculus.

Proposition:

If X is a continuous rv with pdf $f(x)$ and cdf $F(x)$, then at every x at which the derivative $F'(x)$ exists, $F'(x) = f(x)$.

Compare with the DISCRETE case:

Let X be a discrete random variable.

Obtaining the pmf from the cdf whenever cdf is available:

For example, if the possible values of X are 0, 1, 2, 3 and 4,

$$\begin{aligned} p(2) &= P(X=2) = [p(0) + p(1) + p(2)] - [p(0) + p(1)] \\ &= P(X \leq 2) - P(X \leq 1) \\ &= F(2) - F(1) \end{aligned}$$

Percentiles of a Continuous Distribution:

When we say that an individual's test score was at the 85th percentile of the population, we mean that 85% of all population scores were below that score and 15% were above.

Definition:

Let p be a number between 0 and 1. The **(100 p)th percentile** of the distribution of a continuous rv X , denoted by $\eta(p)$, is defined by

$$p = F[\eta(p)] = \int_{-\infty}^{\eta(p)} f(y)dy$$

$\eta(p)$ is that value on the measurement axis such that 100 p % of the area under the graph of $f(x)$ lies to the left of $\eta(p)$ and 100(1 - p)% lies to the right.

Definition:

The **median** of a continuous distribution, denoted by $\tilde{\mu}$, is the 50th percentile, so $\tilde{\mu}$ satisfies $.5 = F(\tilde{\mu})$. That is, half the area under the density curve is to the left of $\tilde{\mu}$ and half is to the right of $\tilde{\mu}$.

Expected Value of a Continuous Random Variable:

Definition:

The **expected** or **mean value** of a continuous rv X with pdf $f(x)$ is

$$\mu_X = E(X) = \int_{-\infty}^{\infty} x \cdot f(x) dx$$

This expected value will exist provided that $\int_{-\infty}^{\infty} |x|f(x) dx < \infty$

Definition:

If X is a continuous rv with pdf $f(x)$ and $h(X)$ is any function of X , then

$$E[h(X)] = \mu_{h(X)} = \int_{-\infty}^{\infty} h(x) \cdot f(x) dx$$

This expected value will exist provided that $\int_{-\infty}^{\infty} |h(x)|f(x) dx < \infty$

Compare with the Expected Value of a discrete random variable X:

Let X be a discrete rv with set of possible values D and pmf $p(x)$. The **expected value** or **mean value** of X , denoted by $E(X)$ or μ_X , is

$$E(X) = \mu_X = \sum_{x \in D} x \cdot p(x)$$

This expected value will exist provided that $\sum_{x \in D} |x| \cdot p(x) < \infty$

Compare with Expected Value of a function of a discrete random variable X:

If the rv X has a set of possible values D and pmf $p(x)$, then the expected value of any function $h(X)$, denoted by $E[h(X)]$ or $\mu_{h(X)}$, is computed by

$$E[h(X)] = \sum_D h(x) \cdot p(x)$$

assuming that $\sum_D |h(x)| \cdot p(x)$ is finite.

Variance of a Continuous Random Variable:

Definition:

The **variance** of a continuous random variable X with pdf $f(x)$ and mean value μ is

$$\sigma_X^2 = V(X) = \int_{-\infty}^{\infty} (x - \mu)^2 \cdot f(x) dx = E[(X - \mu)^2]$$

The **standard deviation** (SD) of X is $\sigma_X = \sqrt{V(X)}$.

Proposition: Shortcut Formula

$$V(X) = E(X^2) - [E(X)]^2$$

COMPARE WITH THE DISCRETE CASE:

Definition:

Let X have pmf $p(x)$ and expected value μ . Then the **variance** of X , denoted by $V(X)$ or σ_X^2 , or just σ^2 , is

$$V(X) = \sum_D (x - \mu)^2 \cdot p(x) = E(X - \mu)^2$$

The **standard deviation** (SD) of X is

$$\sigma_X = \sqrt{\sigma_X^2}$$

Proposition: Shortcut Formula

$$V(X) = \sigma^2 = \left[\sum_D x^2 \cdot p(x) \right] - \mu^2 = E(X^2) - [E(X)]^2$$

Proof of the Shortcut Formula Expand $(x - \mu)^2$ in the definition of σ^2 to obtain $x^2 - 2\mu x + \mu^2$, and then carry $\sum$ through to each of the three terms:

$$\begin{aligned} \sigma^2 &= \sum_D x^2 \cdot p(x) - 2\mu \cdot \sum_D x \cdot p(x) + \mu^2 \sum_D p(x) \\ &= E(X^2) - 2\mu \cdot \mu + \mu^2 = E(X^2) - \mu^2 \end{aligned}$$

■

Expected Value, Variance and Approximating the Mean Value and Variance of a function of a Random Variable

Proposition:

$$E(aX + b) = aE(X) + b \quad V(aX + b) = a^2 \sigma_X^2 \quad \sigma_{aX+b} = |a| \sigma_X$$

Proposition:

Suppose $h(x)$ is differentiable and that its derivative evaluated at μ satisfies $h'(\mu) \neq 0$. Then if the variance of X is small, so that the distribution of X is largely concentrated on an interval of values close to μ , the mean value and variance of $Y = h(X)$ can be approximated as follows:

$$E[h(X)] \approx h(\mu), \quad V[h(X)] \approx [h'(\mu)]^2 \sigma^2$$

Proof:

The justification for these approximations is a first-order Taylor series expansion of $h(X)$ about μ ; that is, we approximate the function for values near μ by the tangent line to the function at the point $(\mu, h(\mu))$:

$$Y = h(X) \approx h(\mu) + h'(\mu)(X - \mu)$$

Taking the expected value of this gives $E[h(X)] \approx h(\mu)$, which validates the first part of the proposition. The variance of the linear approximation is $V[h(X)] \approx [h'(\mu)]^2 \sigma_X^2$ as stated in the second part of the proposition.

COMPARE WITH THE DISCRETE CASE:

Proposition:

$$E(aX + b) = aE(X) + b \quad V(aX + b) = a^2 \sigma_X^2 \quad \sigma_{aX+b} = |a| \sigma_X$$

Recall the Variance of a function of a discrete random variable X:

$$V[h(X)] = \sigma_{h(X)}^2 = \sum_D \{h(x) - E[h(x)]\}^2 \cdot p(x)$$

Moment Generating Functions - 1

Definition:

The **moment generating function** (mgf) of a continuous random variable X is

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x) dx.$$

As in the discrete case, we will say that the moment generating function exists if $M_X(t)$ is defined for an interval of numbers that includes zero in its interior, which means that it includes both positive and negative values of t .

Property:

Just as before, when $t = 0$ the value of the mgf is always 1:

$$M_X(0) = E(e^{0X}) = \int_{-\infty}^{\infty} e^{0x} f(x) dx = \int_{-\infty}^{\infty} f(x) dx = 1.$$

COMPARE WITH THE DISCRETE CASE:

Definition:

The moment generating function (mgf) of a discrete random variable X is defined to be

$$M_X(t) = E(e^{tX}) = \sum_{x \in D} e^{tx} p(x)$$

where D is the set of possible X values. We will say that the moment generating function exists if $M_X(t)$ is defined for an interval of numbers that includes zero as well as positive and negative values of t (an interval including 0 in its interior).

Property:

When $t = 0$, for any random variable X

$$M_X(0) = E(e^{0X}) = \sum_{x \in D} e^{0x} p(x) = \sum_{x \in D} 1p(x) = 1$$

Moment Generating Functions - 2

All the propositions and theorems on this slide were stated for discrete random variables. **They also hold for continuous random variables.**

Proposition:

If the mgf exists and is the same for two distributions, then the two distributions are the same. That is, the moment generating function uniquely specifies the probability distribution; there is a one-to-one correspondence between distributions and mgf's.

Theorem:

If the mgf exists,

$$E(X^r) = M_X^{(r)}(0)$$

Sometimes, when doing the differentiation, defining $R_X(t) = \ln[M_X(t)]$ makes the computation easier.

We can show that if the mgf exists, then

$$\mu = E(X) = R'_X(0)$$

$$\sigma^2 = V(X) = R''_X(0)$$

Proposition:

Let X have the mgf $M_X(t)$ and let $Y = aX + b$. Then $M_Y(t) = e^{bt}M_X(at)$.

The Normal Distribution and The Standard Normal Distribution

Definition:

A continuous rv X is said to have a **normal distribution** with parameters μ and σ (or μ and σ^2), where $-\infty < \mu < \infty$ and $0 < \sigma$, if the pdf of X is

$$f(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)} \quad -\infty < x < \infty$$

The statement that X is normally distributed with parameters μ and σ^2 is often abbreviated $X \sim N(\mu, \sigma^2)$.

$E(X) = \mu$ and $V(X) = \sigma^2$, so the parameters are the mean and the standard deviation of X . Figure 4.13 presents graphs of $f(x; \mu, \sigma)$ for several different (μ, σ^2) pairs. Each resulting density curve is symmetric about μ and bell-shaped, so the center of the bell (point of symmetry) is both the mean of the distribution and the median. The value of σ is the distance from μ to the inflection points of the curve.

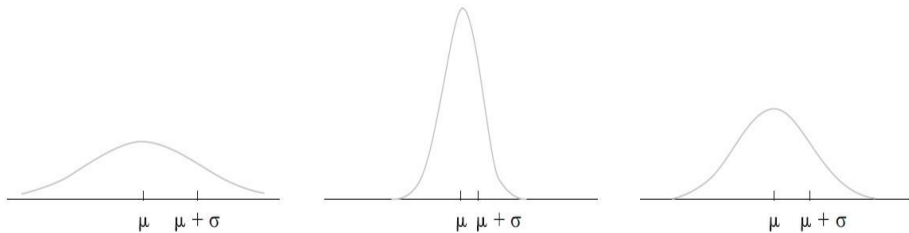


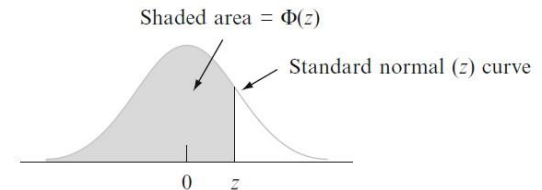
Figure 4.13 Normal density curves

Definition:

The normal distribution with parameter values $\mu = 0$ and $\sigma = 1$ is called the **standard normal distribution**. A random variable that has a standard normal distribution is called a **standard normal random variable** and will be denoted by Z . The pdf of Z is

$$f(z; 0, 1) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \quad -\infty < z < \infty$$

The cdf of Z is $P(Z \leq z) = \int_{-\infty}^z f(y; 0, 1) dy$, which we will denote by $\Phi(z)$.



To compute $P(a \leq X \leq b)$ when X is a normal rv with parameters μ and σ , we must evaluate

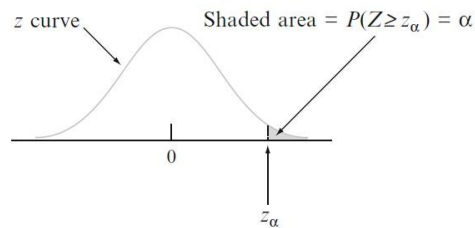
$$\int_a^b \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)} dx \quad (4.4)$$

None of the standard integration techniques can be used to evaluate Expression (4.4). Instead, for $\mu = 0$ and $\sigma = 1$, Expression (4.4) has been numerically evaluated and tabulated for certain values of a and b . This table can also be used to compute probabilities for any other values of μ and σ under consideration.

Percentiles of the Standard Normal Distribution

In general, the $(100p)$ th percentile is identified by the row and column of Appendix Table A.3 in which the entry p is found (e.g., the 67th percentile is obtained by finding .6700 in the body of the table, which gives $z = .44$). If p does not appear, the number closest to it is often used, although linear interpolation gives a more accurate answer. For example, to find the 95th percentile, we look for .9500 inside the table. Although .9500 does not appear, both .9495 and .9505 do, corresponding to $z = 1.64$ and 1.65 , respectively. Since .9500 is halfway between the two probabilities that do appear, we will use 1.645 as the 95th percentile and -1.645 as the 5th percentile.

z_α notation: z_α will denote the value on the measurement axis for which α of the area under the z curve lies to the right of z_α .



Since α of the area under the standard normal curve lies to the right of z_α , $1 - \alpha$ of the area lies to the left of z_α . Thus z_α is the $100(1 - \alpha)$ th percentile of the standard normal distribution. By symmetry the area under the standard normal curve to the left of $-z_\alpha$ is also α . The z_α 's are usually referred to as **z critical values**.

Table A.3 Standard Normal Curve Areas (*cont.*)

$$\Phi(z) = P(Z \leq z)$$

[illegible]

(Nonstandard) Normal Distributions

Proposition:

If X has a normal distribution with mean μ and standard deviation σ , then

$$Z = \frac{X - \mu}{\sigma}$$

has a standard normal distribution. Thus

$$\begin{aligned} P(a \leq X \leq b) &= P\left(\frac{a - \mu}{\sigma} \leq Z \leq \frac{b - \mu}{\sigma}\right) \\ &= \Phi\left(\frac{b - \mu}{\sigma}\right) - \Phi\left(\frac{a - \mu}{\sigma}\right) \end{aligned}$$

$$P(X \leq a) = \Phi\left(\frac{a - \mu}{\sigma}\right) \quad P(X \geq b) = 1 - \Phi\left(\frac{b - \mu}{\sigma}\right)$$

Standardizing amounts to nothing more than calculating a distance from the mean value and then re-expressing the distance as some number of standard deviations. For example, if $\mu = 100$ and $\sigma = 15$, then $x = 130$ corresponds to $z = (130 - 100)/15 = 30/15 = 2.00$. Thus 130 is 2 standard deviations above (to the right of) the mean value.

Empirical Rule:

If the population distribution of a variable is (approximately) normal, then

1. Roughly 68% of the values are within 1 SD of the mean.
2. Roughly 95% of the values are within 2 SDs of the mean.
3. Roughly 99.7% of the values are within 3 SDs of the mean.

Proposition: Percentiles of a Normal Distribution

$$\begin{array}{l} (100p)\text{th percentile} \\ \text{for normal}(\mu, \sigma) \end{array} = \mu + \left[\begin{array}{l} (100p)\text{th percentile} \\ \text{for standard normal} \end{array} \right] \cdot \sigma$$

Approximating the Binomial Distribution

Proposition:

Let X be a binomial rv based on n trials with success probability p . Then if the binomial probability histogram is not too skewed, X has approximately a normal distribution with $\mu = np$ and $\sigma = \sqrt{npq}$. In particular, for $x =$ a possible value of X ,

$$\begin{aligned} P(X \leq x) &= B(x; n, p) \approx (\text{area under the normal curve to the left of } x + .5) \\ &= \Phi\left(\frac{x + .5 - np}{\sqrt{npq}}\right) \end{aligned}$$

In practice, the approximation is adequate provided that both $np \geq 10$ and $nq \geq 10$.

If either $np < 10$ or $nq < 10$, the binomial distribution may be too skewed for the (symmetric) normal curve to give accurate approximations.

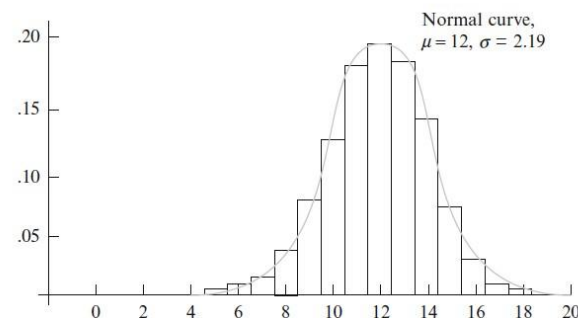


Figure 4.25 Binomial probability histogram for $n = 20$, $p = .6$ with normal approximation curve superimposed

The Normal Mgf

Proposition:

The moment generating function of a normally distributed random variable X is

$$M_X(t) = e^{\mu t + \sigma^2 t^2 / 2}$$

Proof Consider first the special case of a standard normal rv Z . Then

$$M_Z(t) = E(e^{tZ}) = \int_{-\infty}^{\infty} e^{tz} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(z^2 - 2tz)/2} dz$$

Completing the square in the exponent, we have

$$M_Z(t) = e^{t^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(z^2 - 2tz + t^2)/2} dz = e^{t^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(z-t)^2/2} dz$$

The last integral is the area under a normal density with mean t and standard deviation 1, so the value of the integral is 1. Therefore, $M_Z(t) = e^{t^2/2}$.

Now let X be any normal rv with mean μ and standard deviation σ . Then, by the first proposition in this section, $(X - \mu)/\sigma = Z$, where Z is standard normal. That is, $X = \mu + \sigma Z$. Now use the property $M_{aY+b}(t) = e^{bt} M_Y(at)$:

$$M_X(t) = M_{\mu + \sigma Z}(t) = e^{\mu t} M_Z(\sigma t) = e^{\mu t} e^{\sigma^2 t^2 / 2} = e^{\mu t + \sigma^2 t^2 / 2} \quad \blacksquare$$

EXAMPLES

Example 1: Pdfs and Uniform Distribution

The direction of an imperfection with respect to a reference line on a circular object such as a tire, brake rotor, or flywheel is, in general, subject to uncertainty. Consider the reference line connecting the valve stem on a tire to the center point, and let X be the angle measured clockwise to the location of an imperfection. One possible pdf for X is

$$f(x) = \begin{cases} \frac{1}{360} & 0 \leq x < 360 \\ 0 & \text{otherwise} \end{cases}$$

The pdf is graphed in Figure 4.3. Clearly $f(x) \geq 0$. The area under the density curve is just the area of a rectangle: $(\text{height})(\text{base}) = (\frac{1}{360})(360) = 1$. The probability that the angle is between 90° and 180° is

$$P(90 \leq X \leq 180) = \int_{90}^{180} \frac{1}{360} dx = \left. \frac{x}{360} \right|_{x=90}^{x=180} = \frac{1}{4} = .25$$

The probability that the angle of occurrence is within 90° of the reference line is

$$P(0 \leq X \leq 90) + P(270 \leq X < 360) = .25 + .25 = .50$$

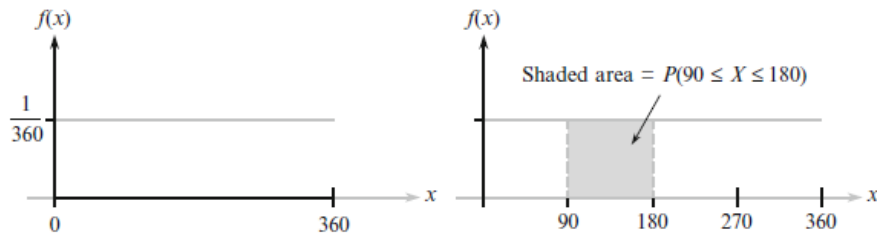


Figure 4.3 The pdf and probability for Example



Example 2: Cdf F(x)

EXAMPLE Let Y be a continuous random variable with probability density function given by

$$f(y) = \begin{cases} 3y^2, & 0 \leq y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find $F(y)$. Graph both $f(y)$ and $F(y)$.

Solution The graph of $f(y)$ appears in Figure 4.6. Because

$$F(y) = \int_{-\infty}^y f(t) dt,$$

we have, for this example,

$$F(y) = \begin{cases} \int_{-\infty}^y 0 dt = 0, & \text{for } y < 0, \\ \int_{-\infty}^0 0 dt + \int_0^y 3t^2 dt = 0 + t^3 \Big|_0^y = y^3, & \text{for } 0 \leq y \leq 1, \\ \int_{-\infty}^0 0 dt + \int_0^1 3t^2 dt + \int_1^y 0 dt = 0 + t^3 \Big|_0^1 + 0 = 1, & \text{for } 1 < y. \end{cases}$$

Notice that some of the integrals that we evaluated yield a value of 0. These are included for completeness in this initial example. In future calculations, we will not explicitly display any integral that has value 0. The graph of $F(y)$ is given in Figure 4.7.

FIGURE 4.6
Density function
for Example

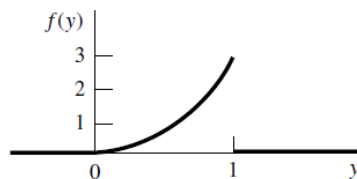
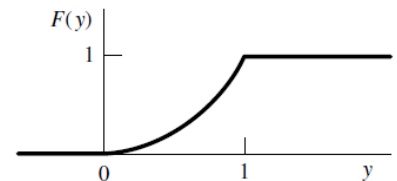


FIGURE 4.7
Distribution function
for Example



Example 3: Cdf $F(x)$ and using $F(x)$ to compute probabilities

Suppose the pdf of the magnitude X of a dynamic load on a bridge (in newtons) is given by

$$f(x) = \begin{cases} \frac{1}{8} + \frac{3}{8}x & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

For any number x between 0 and 2,

$$F(x) = \int_{-\infty}^x f(y) dy = \int_0^x \left(\frac{1}{8} + \frac{3}{8}y \right) dy = \frac{x}{8} + \frac{3x^2}{16}$$

Thus

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{8} + \frac{3x^2}{16} & 0 \leq x \leq 2 \\ 1 & 2 < x \end{cases}$$

The graphs of $f(x)$ and $F(x)$ are shown in Figure 4.9. The probability that the load is between 1 and 1.5 is

$$\begin{aligned} P(1 \leq X \leq 1.5) &= F(1.5) - F(1) = \left[\frac{1}{8}(1.5) + \frac{3}{16}(1.5)^2 \right] - \left[\frac{1}{8}(1) + \frac{3}{16}(1)^2 \right] \\ &= \frac{19}{64} = .297 \end{aligned}$$

The probability that the load exceeds 1 is

$$P(X > 1) = 1 - P(X \leq 1) = 1 - F(1) = 1 - \left[\frac{1}{8}(1) + \frac{3}{16}(1)^2 \right] = \frac{11}{16} = .688$$

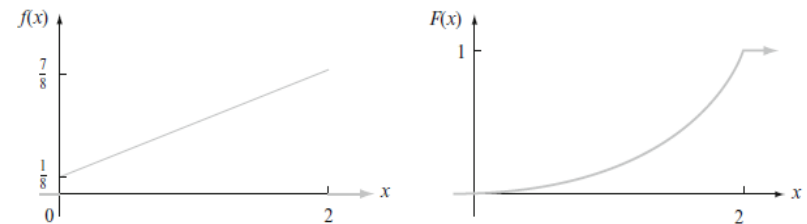


Figure 4.9 The pdf and cdf for Example

Example 4: Obtaining $f(x)$ from $F(x)$

Suppose that

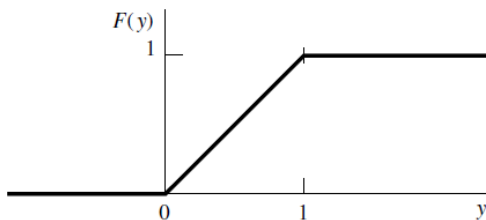
$$F(y) = \begin{cases} 0, & \text{for } y < 0, \\ y, & \text{for } 0 \leq y \leq 1, \\ 1, & \text{for } y > 1. \end{cases}$$

Find the probability density function for Y and graph it.

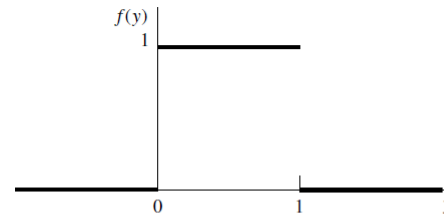
Solution Because the density function $f(y)$ is the derivative of the distribution function $F(y)$, when the derivative exists,

$$f(y) = \frac{dF(y)}{dy} = \begin{cases} \frac{d(0)}{dy} = 0, & \text{for } y < 0, \\ \frac{d(y)}{dy} = 1, & \text{for } 0 < y < 1, \\ \frac{d(1)}{dy} = 0, & \text{for } y > 1, \end{cases}$$

and $f(y)$ is undefined at $y = 0$ and $y = 1$. A graph of $F(y)$ is shown in Figure



Remark: In addition to what was asked in the question, the graph of the pdf $f(y)$ can be drawn and it is as the following:



$F(y)$ is a continuous function of y , but $f(y)$ is discontinuous at the points $y = 0, 1$. In general, the distribution function for a continuous random variable must be continuous, but the density function need not be everywhere continuous.

Example 5: Percentiles of a continuous distribution

The distribution of the amount of gravel (in tons) sold by a construction supply company in a given week is a continuous rv X with pdf that is,

$$f(x) = \begin{cases} \frac{3}{2}(1-x^2) & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

The cdf of sales for any x between 0 and 1 is

$$F(x) = \int_0^x \frac{3}{2}(1-y^2)dy = \frac{3}{2} \left(y - \frac{y^3}{3} \right) \Big|_{y=0}^{y=x} = \frac{3}{2} \left(x - \frac{x^3}{3} \right)$$

The graphs of both $f(x)$ and $F(x)$ appear in Figure 4.11. The $(100p)$ th percentile of this distribution satisfies the equation

$$p = F[\eta(p)] = \frac{3}{2} \left[\eta(p) - \frac{[\eta(p)]^3}{3} \right]$$

$$[\eta(p)]^3 - 3\eta(p) + 2p = 0$$

For the 50th percentile, $p = .5$, and the equation to be solved is $\eta^3 - 3\eta + 1 = 0$; the solution is $\eta = \eta(.5) = .347$. If the distribution remains the same from week to week, then in the long run 50% of all weeks will result in sales of less than .347 tons and 50% in more than .347 tons.

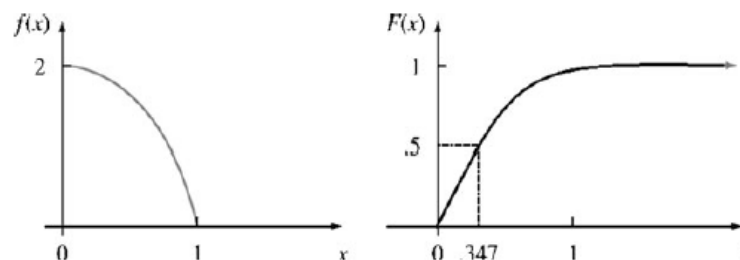


Figure 4.11 The pdf and cdf for Example

Recall: The 50th percentile is called the **median**.

Example 6: Expected Value and Variance of a Continuous Random Variable

EXAMPLE In Example 4.4 we determined that $f(y) = (3/8)y^2$ for $0 \leq y \leq 2$, $f(y) = 0$ elsewhere, is a valid density function. If the random variable Y has this density function, find $\mu = E(Y)$ and $\sigma^2 = V(Y)$.

Solution According to Definition

$$\begin{aligned} E(Y) &= \int_{-\infty}^{\infty} yf(y) \, dy \\ &= \int_0^2 y \left(\frac{3}{8} \right) y^2 \, dy \\ &= \left(\frac{3}{8} \right) \left(\frac{1}{4} \right) y^4 \Big|_0^2 = 1.5. \end{aligned}$$

The variance of Y can be found once we determine $E(Y^2)$. In this case,

$$\begin{aligned} E(Y^2) &= \int_{-\infty}^{\infty} y^2 f(y) \, dy \\ &= \int_0^2 y^2 \left(\frac{3}{8} \right) y^2 \, dy \\ &= \left(\frac{3}{8} \right) \left(\frac{1}{5} \right) y^5 \Big|_0^2 = 2.4. \end{aligned}$$

Thus, $\sigma^2 = V(Y) = E(Y^2) - [E(Y)]^2 = 2.4 - (1.5)^2 = 0.15$.

Examples 7-8: Expected Values, Uniform Distribution

Example 7: Prove the following:

If $\theta_1 < \theta_2$ and Y is a random variable uniformly distributed on the interval (θ_1, θ_2) , then

$$\mu = E(Y) = \frac{\theta_1 + \theta_2}{2}$$

Proof:

$$\begin{aligned} E(Y) &= \int_{-\infty}^{\infty} y f(y) dy \\ &= \int_{\theta_1}^{\theta_2} y \left(\frac{1}{\theta_2 - \theta_1} \right) dy \\ &= \left(\frac{1}{\theta_2 - \theta_1} \right) \frac{y^2}{2} \Big|_{\theta_1}^{\theta_2} = \frac{\theta_2^2 - \theta_1^2}{2(\theta_2 - \theta_1)} \\ &= \frac{\theta_2 + \theta_1}{2}. \end{aligned}$$

Example 8: Expected Value of a function of a random variable

Two species are competing in a region for control of a limited amount of a resource. Let X = the proportion of the resource controlled by species 1 and suppose X has pdf

$$f(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

which is a uniform distribution on $[0, 1]$. (In her book *Ecological Diversity*, E. C. Pielou calls this the “broken-stick” model for resource allocation, since it is analogous to breaking a stick at a randomly chosen point.) Then the species that controls the majority of this resource controls the amount

$$h(X) = \max(X, 1 - X) = \begin{cases} 1 - X & \text{if } 0 \leq X < \frac{1}{2} \\ X & \text{if } \frac{1}{2} \leq X \leq 1 \end{cases}$$

The expected amount controlled by the species having majority control is then

$$\begin{aligned} E[h(X)] &= \int_{-\infty}^{\infty} \max(x, 1 - x) \cdot f(x) dx = \int_0^1 \max(x, 1 - x) \cdot 1 dx \\ &= \int_0^{1/2} (1 - x) \cdot 1 dx + \int_{1/2}^1 x \cdot 1 dx = \frac{3}{4} \end{aligned}$$

■

Examples 9-10: Expected Value, Variance, Mgf

Example 9: Expected Value and Variance

When a dart is thrown at a circular target, consider the location of the landing point relative to the bull's eye. Let X be the angle in degrees measured from the horizontal, and assume that X is uniformly distributed on $[0, 360]$. $E(X) = 180$ and $\sigma_X = 360/\sqrt{12}$. Define Y to be the transformed variable $Y = h(X) = (2\pi/360)X - \pi$, so Y is the angle measured in radians and Y is between $-\pi$ and π . Then

$$E(Y) = \frac{2\pi}{360}E(X) - \pi = \frac{2\pi}{360}180 - \pi = 0.$$

and

$$\sigma_Y = \frac{2\pi}{360}\sigma_X = \frac{2\pi}{360} \frac{360}{\sqrt{12}} = \frac{2\pi}{\sqrt{12}}$$

Recall:

$$E(aX + b) = aE(X) + b \qquad V(aX + b) = a^2\sigma_X^2 \qquad \sigma_{aX+b} = |a|\sigma_X$$

Example 10: Find the mgf, mean and variance:

At a store the checkout time X in minutes has the pdf $f(x) = 2e^{-2x}, x \geq 0; f(x) = 0$ otherwise. Then

$$\begin{aligned} M_X(t) &= \int_{-\infty}^{\infty} e^{tx}f(x)dx = \int_0^{\infty} e^{tx}(2e^{-2x})dx = \int_0^{\infty} 2e^{-(2-t)x}dx \\ &= \frac{-2}{2-t}e^{-(2-t)x} \Big|_0^{\infty} = \frac{2}{2-t} \quad \text{if } t < 2. \end{aligned}$$

This mgf exists because it is defined for an interval of values including 0 in its interior.

Notice that $M_X(0) = 2/(2-0) = 1$

To find the mean and variance, first

compute the derivatives.

$$\begin{aligned} M'_X(t) &= -2(2-t)^{-2}(-1) = \frac{2}{(2-t)^2} \\ M''_X(t) &= (-2)(-2)(2-t)^{-3}(-1)(-1) = \frac{4}{(2-t)^3} \end{aligned}$$

Setting t to 0 in the first derivative gives the expected checkout time as

$$E(X) = M'_X(0) = M_X^{(1)}(0) = .5.$$

Setting t to 0 in the second derivative gives the second moment

$$E(X^2) = M''_X(0) = M_X^{(2)}(0) = .5.$$

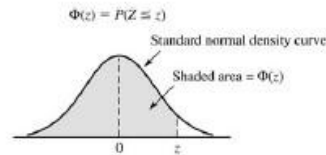
The variance of the checkout time is then:

$$V(X) = \sigma^2 = E(X^2) - [E(X)]^2 = .5 - .5^2 = .25$$



Table A3 Devore & Berk

Table A.3 Standard Normal Curve Areas



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
-3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
-3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
-3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
-3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
-3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
-2.9	.0019	.0018	.0017	.0017	.0016	.0016	.0015	.0015	.0014	.0014
-2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
-2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
-2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
-2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
-2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
-2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
-2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
-2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
-2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
-1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
-1.8	.0359	.0352	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
-1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
-1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
-1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
-1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0722	.0708	.0694	.0681
-1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
-1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
-1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
-1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
-0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
-0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
-0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
-0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
-0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
-0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
-0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3482
-0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
-0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
-0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

(continued)

Table A.3 Standard Normal Curve Areas (cont.)

$$\Phi(z) = P(Z \leq z)$$

[illegible]

Examples 11-12: The Normal Distribution, The Standard Normal Distribution

Example 11: Let $X \sim N(\mu, \sigma^2)$. Show that

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

Proof:

Consider the special case where $\mu = 0$ and $\sigma = 1$, so $f(x) = (1/\sqrt{2\pi})e^{-x^2/2}$, and define $\int_{-\infty}^{\infty} (1/\sqrt{2\pi})e^{-x^2/2}dx = A$. Let $g(x, y)$ be the function of two variables

$$g(x, y) = f(x) \cdot f(y) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2} \frac{1}{\sqrt{2\pi}}e^{-y^2/2} = \frac{1}{2\pi}e^{-(x^2+y^2)/2}$$

Using the rotational symmetry of $g(x, y)$, let's evaluate the volume under it by the shell method, which adds up the volumes of shells from rotation about the y -axis:

$$V = \int_0^{\infty} 2\pi x \frac{1}{2\pi}e^{-x^2/2}dx = \left[-e^{-x^2/2}\right]_0^{\infty} = 1$$

Now evaluate V by the usual double integral

$$V = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)f(y)dxdy = \int_{-\infty}^{\infty} f(x)dx \cdot \int_{-\infty}^{\infty} f(y)dy = \left[\int_{-\infty}^{\infty} f(x)dx\right]^2 = A^2$$

Because $1 = V = A^2$, we have $A = 1$ in this special case where $\mu = 0$ and $\sigma = 1$. How about the general case? Using a change of variables, $z = (x - \mu)/\sigma$,

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma}e^{-(x-\mu)^2/(2\sigma^2)}dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz = 1. \quad \blacksquare$$

Example 12: The Standard Normal Distribution

Compute the following standard normal probabilities: (a) $P(Z \leq 1.25)$, (b) $P(Z > 1.25)$, (c) $P(Z \leq -1.25)$, and (d) $P(-.38 \leq Z \leq 1.25)$.

a. $P(Z \leq 1.25) = \Phi(1.25)$, a probability that is tabulated in Appendix Table A.3 at the intersection of the row marked 1.2 and the column marked .05. The number there is .8944, so $P(Z \leq 1.25) = .8944$. See Figure 4.15(a).

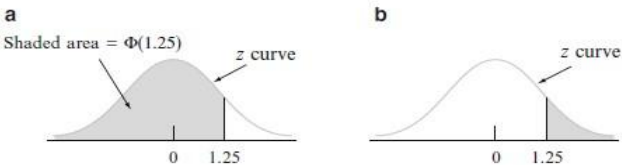


Figure 4.15 Normal curve areas (probabilities) for Example 4.19

- b. $P(Z > 1.25) = 1 - P(Z \leq 1.25) = 1 - \Phi(1.25)$, the area under the standard normal curve to the right of 1.25 (an upper-tail area). Since $\Phi(1.25) = .8944$, it follows that $P(Z > 1.25) = .1056$. Since Z is a continuous rv, $P(Z \geq 1.25)$ also equals .1056. See Figure 4.15(b).
- c. $P(Z \leq -1.25) = \Phi(-1.25)$, a lower-tail area. Directly from Appendix Table A.3, $\Phi(-1.25) = .1056$. By symmetry of the normal curve, this is the same answer as in part (b).
- d. $P(-.38 \leq Z \leq 1.25)$ is the area under the standard normal curve above the interval whose left endpoint is $-.38$ and whose right endpoint is 1.25. From Section 4.1, if X is a continuous rv with cdf $F(x)$, then $P(a \leq X \leq b) = F(b) - F(a)$. This gives $P(-.38 \leq Z \leq 1.25) = \Phi(1.25) - \Phi(-.38) = .8944 - .3520 = .5424$. (See Figure 4.16.)

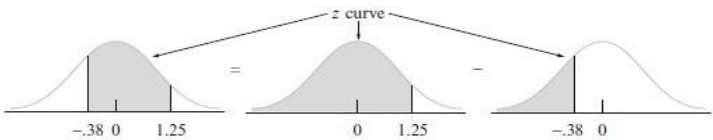


Figure 4.16 $P(-.38 \leq Z \leq 1.25)$ as the difference between two cumulative areas ■

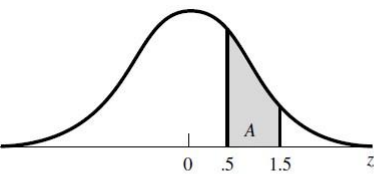
Examples 13-14: The (Nonstandard) Normal Distribution, Percentiles of the Standard Normal Distribution

Example 13: The (Nonstandard) Normal distribution

The achievement scores for a college entrance examination are normally distributed with mean 75 and standard deviation 10. What fraction of the scores lies between 80 and 90?

Recall that z is the distance from the mean of a normal distribution expressed in units of standard deviation. Thus,

$$z = \frac{y - \mu}{\sigma}.$$



Then the desired fraction of the population is given by the area between

$$z_1 = \frac{80 - 75}{10} = .5 \quad \text{and} \quad z_2 = \frac{90 - 75}{10} = 1.5.$$

This area is shaded in Figure.

$$\begin{aligned} A &= 0.9332 - 0.6915 \text{ (from Table A3)} \\ &= 0.2417 \end{aligned}$$

Example 14: Percentiles of the Standard Normal Distribution

Find the 95th percentile of the standard normal distribution.

Solution:

to find the 95th percentile, we look for .9500 inside the table. Although .9500 does not appear, both .9495 and .9505 do, corresponding to $z = 1.64$ and 1.65 , respectively. Since .9500 is halfway between the two probabilities that do appear, we will use 1.645 as the 95th percentile and -1.645 as the 5th percentile.

Using the z_α notation:

The $100(1 - .05)\text{th} = 95\text{th}$ percentile of the standard normal distribution is $z_{.05}$, so $z_{.05} = 1.645$. The area under the standard normal curve to the left of $-z_{.05}$ is also .05. (See Figure 4.20.)

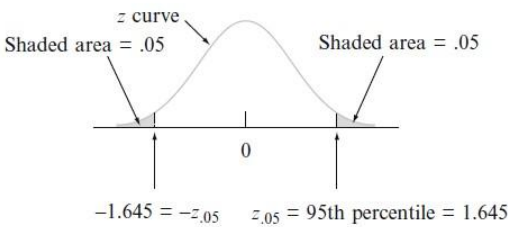


Figure 4.20 Finding $z_{.05}$

Examples 15-16: Percentiles of the Normal Distribution, Approximating the Binomial Distribution

Example 15: Percentiles of the (Nonstandard) Normal Distribution

The amount of distilled water dispensed by a machine is normally distributed with mean value 64 oz and standard deviation .78 oz. What container size c will ensure that overflow occurs only .5% of the time? If X denotes the amount dispensed, the desired condition is that $P(X > c) = .005$, or, equivalently, that $P(X \leq c) = .995$. Thus c is the 99.5th percentile of the normal distribution with $\mu = 64$ and $\sigma = .78$. The 99.5th percentile of the standard normal distribution is 2.58, so

$$c = \eta(.995) = 64 + (2.58)(.78) = 64 + 2.0 = 66 \text{ oz}$$

This is illustrated in Figure 4.23.

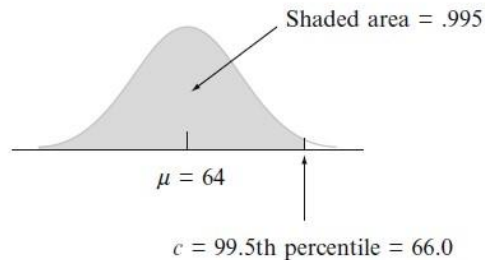


Figure 4.23 Distribution of amount dispensed for Example

Recall:
$$\begin{matrix} (100p)\text{th percentile} \\ \text{for normal}(\mu, \sigma) \end{matrix} = \mu + \left[\begin{matrix} (100p)\text{th percentile} \\ \text{for standard normal} \end{matrix} \right] \cdot \sigma$$

Example 16: Approximating the Binomial Distribution

Suppose that 25% of all licensed drivers in a state do not have insurance. Let X be the number of uninsured drivers in a random sample of size 50 (somewhat perversely, a success is an uninsured driver), so that $p = .25$. Then $\mu = 12.5$ and $\sigma = 3.062$. Since $np = 50(.25) = 12.5 \geq 10$ and $nq = 37.5 \geq 10$, the approximation can safely be applied:

$$\begin{aligned} P(X \leq 10) &= B(10; 50, .25) \approx \Phi\left(\frac{10 + .5 - 12.5}{3.062}\right) \\ &= \Phi(-.65) = .2578 \end{aligned}$$

Similarly, the probability that between 5 and 15 (inclusive) of the selected drivers are uninsured is

$$\begin{aligned} P(5 \leq X \leq 15) &= B(15; 50, .25) - B(4; 50, .25) \\ &\approx \Phi\left(\frac{15.5 - 12.5}{3.062}\right) - \Phi\left(\frac{4.5 - 12.5}{3.062}\right) = .8320 \end{aligned}$$

The exact probabilities are .2622 and .8348, respectively, so the approximations are quite good. In the last calculation, the probability $P(5 \leq X \leq 15)$ is being approximated by the area under the normal curve between 4.5 and 15.5—the continuity correction is used for both the upper and lower limits. ■

Recall: Approximating the Binomial Distribution

Proposition:

Let X be a binomial rv based on n trials with success probability p . Then if the binomial probability histogram is not too skewed, X has approximately a normal distribution with $\mu = np$ and $\sigma = \sqrt{npq}$. In particular, for $x =$ a possible value of X ,

$$P(X \leq x) = B(x; n, p) \approx (\text{area under the normal curve to the left of } x + .5) \\ = \Phi\left(\frac{x + .5 - np}{\sqrt{npq}}\right)$$

In practice, the approximation is adequate provided that both $np \geq 10$ and $nq \geq 10$.

If either $np < 10$ or $nq < 10$, the binomial distribution may be too skewed for the (symmetric) normal curve to give accurate approximations.

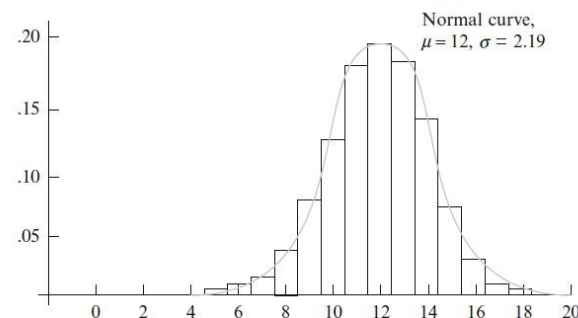


Figure 4.25 Binomial probability histogram for $n = 20$, $p = .6$ with normal approximation curve superimposed

References

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- Wackerly, D.D., Mendenhall, W. and Scheaffer, R.L., (2008) Mathematical Statistics With Applications, 7th Edition, Pacific Grove: Duxbury.