

PROBLEM SET 4

Q.1

$$\text{MLE } f(x; \theta_1, \theta_2) = \frac{1}{\sqrt{\theta_2} \cdot \sqrt{2\pi}} \cdot e^{-\frac{(x-\theta_1)^2}{2\theta_2}} \quad \text{where } \theta_1 = \mu \text{ and } \theta_2 = \sigma^2$$

$$L(\theta_1, \theta_2) = \prod_{i=1}^n f(x_i, \theta_1, \theta_2) = \theta_2^{-n/2} \cdot (2\pi)^{-n/2} \cdot e^{-\frac{1}{2\theta_2} \sum (x_i - \theta_1)^2}$$

$$\log L(\theta_1, \theta_2) = -\frac{n}{2} \cdot \log \theta_2 - \frac{n}{2} \log 2\pi - \frac{\sum (x_i - \theta_1)^2}{2\theta_2}$$

$$\frac{\partial \log L(\theta_1, \theta_2)}{\partial \theta_1} = \frac{-2 \sum (x_i - \theta_1) (-1)}{2\theta_2} = 0$$

$$= \sum x_i - n\theta_1 = 0 \Rightarrow \hat{\theta}_1 = \frac{\sum x_i}{n} = \bar{X}$$

$$\frac{\partial \log L(\theta_1, \theta_2)}{\partial \theta_2} = -\frac{n}{2\theta_2} + \frac{\sum (x_i - \theta_1)^2}{2\theta_2^2} = 0$$

$$= -n\theta_2 + \sum (x_i - \theta_1)^2 = 0$$

$$\Rightarrow \hat{\theta}_2 = \frac{\sum (x_i - \bar{X})^2}{n}$$

$$\hat{\mu}_{MLE} = \frac{\sum x_i}{n} = \bar{X} \quad \text{and} \quad \hat{\sigma}_{MLE}^2 = \frac{\sum (x_i - \bar{X})^2}{n}$$

MM

$$\underbrace{\mu_1 = E(X) = \mu}_{\text{pop. moment}} = \underbrace{\frac{1}{n} \cdot \sum x_i}_{\text{sample moment}} \Rightarrow \hat{\mu}_{MM} = \bar{X} = \frac{\sum x_i}{n}$$

$$\underbrace{\mu_2 = E(X^2) = \sigma^2 + \mu^2}_{\text{pop. moment}} = \underbrace{\frac{1}{n} \cdot \sum x_i^2}_{\text{sample moment}} \Rightarrow \hat{\sigma}_{MM}^2 = \frac{\sum x_i^2}{n} - \bar{X}^2$$

$$\hat{\sigma}_{MM}^2 = \frac{\sum (x_i - \bar{X})^2}{n}$$

In general form:

$\mu_k = E(X^k)$: k^{th} pop. moment

$\hat{\mu}_k = \frac{1}{n} \cdot \sum x_i^k$: k^{th} sample moment

b)

$$\hat{\mu} = \bar{X} = \frac{\sum X_i}{n}$$

$$E(\bar{X}) = E\left(\frac{1}{n} \cdot \sum X_i\right) = \frac{1}{n} \cdot E(\sum X_i) = \frac{1}{n} \cdot n \cdot \mu = \mu \rightarrow \bar{X} \text{ is unbiased est for } \mu.$$

$$\hat{\sigma}^2 = \frac{\sum (X_i - \bar{X})^2}{n} = \frac{\sum X_i^2}{n} - \bar{X}^2$$

$$E(\hat{\sigma}^2) = E\left(\frac{\sum X_i^2}{n}\right) - E(\bar{X}^2)$$

$$= \frac{n(\sigma^2 + \mu^2)}{n} - \left(\frac{\sigma^2}{n} + \mu^2\right)$$

$$E(\hat{\sigma}^2) = \frac{(n-1)\sigma^2}{n} \rightarrow \hat{\sigma}^2 \text{ biased est. for } \sigma^2.$$

$$c) \quad E\left(\frac{n\hat{\sigma}^2}{n-1}\right) = \sigma^2 \Rightarrow s^2 = \frac{n}{n-1} \cdot \frac{\sum (X_i - \bar{X})^2}{n}$$

$$s^2 = \frac{\sum (X_i - \bar{X})^2}{n-1} \text{ is unbiased est. for } \sigma^2.$$

Q.2

$$a) \prod_{i=1}^n f(y_i, \theta) = \prod_{i=1}^n \frac{2y_i}{\theta} \cdot e^{-y_i/\theta}$$

$$= \frac{2^n}{\theta^n} \cdot \prod y_i \cdot e^{-\sum y_i/\theta}$$

$$L(\theta|y) = \prod y_i \cdot \frac{2^n}{\theta^n} \cdot e^{-\sum y_i/\theta}$$

$$h(y) \cdot g(u(y), \theta) \text{ where } u(y) = \sum y_i$$

$$b) \ln L(\theta|y) = l(\theta) = \ln \sum y_i + n \ln 2 - n \ln \theta - \frac{\sum y_i}{\theta}$$

$$\frac{\partial l(\theta)}{\partial \theta} = -\frac{n}{\theta} + \frac{\sum y_i}{\theta^2} = 0 \Rightarrow \boxed{\hat{\theta}_{MLE} = \frac{\sum y_i}{n}}$$

Since $\frac{\sum y_i}{n}$ is a one-to-one func of $\sum y_i$ it's also sufficient.

Q.3

$$f(x) = \frac{1}{\Gamma(\alpha) \cdot \theta^\alpha} \cdot x^{\alpha-1} \cdot e^{-x/\theta}$$

$$E(x) = \int_0^{\infty} x \cdot \frac{1}{\Gamma(\alpha) \theta^\alpha} \cdot x^{\alpha-1} \cdot e^{-x/\theta} \cdot dx$$

$$= \frac{1}{\Gamma(\alpha) \cdot \theta^\alpha} \int_0^{\infty} x^\alpha \cdot e^{-x/\theta} \cdot dx$$

since $\int_0^{\infty} \frac{x^\alpha \cdot e^{-x/\theta}}{\Gamma(\alpha+1) \cdot \theta^{\alpha+1}} = 1$

$\int_0^{\infty} x^\alpha \cdot e^{-x/\theta} = \Gamma(\alpha+1) \cdot \theta^{\alpha+1}$

$$= \frac{\Gamma(\alpha+1) \cdot \theta^{\alpha+1}}{\Gamma(\alpha) \cdot \theta^\alpha} = \frac{\alpha \cdot \Gamma(\alpha) \cdot \theta^{\alpha+1}}{\Gamma(\alpha) \cdot \theta^\alpha} = \alpha \theta$$

$E(x) = \alpha \theta \rightarrow 1^{st} \text{ pop. moment}$

$$E(x^2) = \int_0^{\infty} x^2 \cdot \frac{1}{\Gamma(\alpha) \theta^\alpha} \cdot x^{\alpha-1} \cdot e^{-x/\theta} \cdot dx$$

$$= \frac{1}{\Gamma(\alpha) \cdot \theta^\alpha} \int_0^{\infty} x^{\alpha+1} \cdot e^{-x/\theta} \cdot dx$$

$$= \frac{\Gamma(\alpha+2) \theta^{\alpha+2}}{\Gamma(\alpha) \cdot \theta^\alpha} = \frac{(\alpha+1) \alpha \Gamma(\alpha) \cdot \theta^{\alpha+2}}{\Gamma(\alpha) \cdot \theta^\alpha} = \alpha(\alpha+1) \theta^2$$

$E(x^2) = \alpha(\alpha+1) \theta^2 \rightarrow 2^{nd} \text{ pop. moment}$

$$Var(x) = E(x^2) - [E(x)]^2$$

$$= \alpha(\alpha+1) \theta^2 - (\alpha \theta)^2 = \alpha \theta^2$$

$Var(x) = \alpha \theta^2$

Q.3 ...
cont.

$$E(x) = \alpha \theta = \frac{1}{n} \sum_{i=1}^n x_i = \bar{x} \Rightarrow \alpha = \frac{\bar{x}}{\theta}$$

pop. moment sample moment

$$\Rightarrow \alpha = \frac{\bar{x}}{\theta}$$

$$var(x) = \alpha \theta^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$$

$$\frac{\bar{x}}{\theta} \cdot \theta^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$$

$$\theta_{MM} = \frac{1}{n \bar{x}} \sum_{i=1}^n (x_i - \bar{x})^2$$

$$\alpha_{MM} = \frac{n \bar{x}^2}{\sum_{i=1}^n (x_i - \bar{x})^2}$$

Q.4 $f(x; \beta) = \frac{x^2}{2\beta^3} \cdot e^{-x/\beta}, x \geq 0$

a)
$$L(\beta; x) = \prod_{i=1}^n f(x_i; \beta)$$

$$= \prod_{i=1}^n \frac{x_i^2 e^{-x_i/\beta}}{2\beta^3}$$

$$= \frac{1}{(2\beta^3)^n} \cdot \prod_{i=1}^n x_i^2 \cdot e^{-\sum x_i/\beta}$$

$$\ln L(\beta, x) = -n \ln 2 - 3n \ln \beta + \sum \ln x_i^2 - \frac{\sum x_i}{\beta}$$

$$\frac{\partial \ln L(\beta)}{\partial \beta} = -\frac{3n}{\beta} + \frac{\sum x_i}{\beta^2} = 0 \Rightarrow \hat{\beta}_{ML} = \frac{\sum x_i}{3n} = \frac{\bar{x}}{3}$$

b) Since $x \sim \text{gamma}(\alpha=3, \beta) \Rightarrow E(x) = \alpha\beta = 3\beta$
 $\Rightarrow V(x) = \alpha\beta^2 = 3\beta^2$

$E(\frac{\bar{x}}{3}) = \frac{1}{3} \cdot 3\beta = \beta \Rightarrow \hat{\beta} = \frac{\bar{x}}{3}$ is an unbiased estimator of β .

c) $V(\frac{\bar{x}}{3}) = \frac{1}{9} \cdot V(\bar{x}) = \frac{1}{9} \cdot \frac{V(x)}{n} = \frac{1}{9} \cdot \frac{3\beta^2}{n} = \frac{\beta^2}{3n}$

Find CRLB:
$$\frac{1}{-E\left(\frac{\partial^2 \ln L(\beta)}{\partial \beta^2}\right)}$$

$$\frac{\partial^2 \ln L(\beta)}{\partial \beta^2} = \frac{3n}{\beta^2} - \frac{2 \sum x_i}{\beta^3} \Rightarrow E\left(\frac{3n}{\beta^2} - \frac{2 \sum x_i}{\beta^3}\right) = \frac{3n}{\beta^2} - \frac{2}{\beta^3} E(\sum x_i)$$

$$= \frac{3n}{\beta^2} - \frac{2}{\beta^3} \cdot \overbrace{E(\sum x_i)}^{n \cdot 3\beta} = -\frac{3n}{\beta^2}$$

CRLB = $\beta^2/3n$

$\text{Var}(\hat{\beta}) = \beta^2/3n$ is at CRLB. ~~So~~ it's unbiased & efficient
 So, it's MVUE.

d)
$$\lim_{n \rightarrow \infty} \text{var}(\hat{\beta}) = \lim_{n \rightarrow \infty} \frac{\beta^2}{3n} = 0 \Rightarrow \text{it's consistent.}$$

Q.5

$$f(x; \theta) = \theta \cdot e^{-\theta x} \quad x > 0$$

$$\begin{aligned} a) \quad E(x) &= \int_0^{\infty} x \cdot \theta \cdot e^{-\theta x} dx = u \cdot v \Big|_0^{\infty} - \int_0^{\infty} v \cdot du \\ &= x \cdot (e^{-\theta x}) \Big|_0^{\infty} - \int_0^{\infty} -e^{-\theta x} \cdot dx \\ &= -\frac{1}{\theta} \cdot e^{-\theta x} \Big|_0^{\infty} = \frac{1}{\theta} \end{aligned}$$

$$m_1 = \frac{1}{n} \sum x_t \Rightarrow \frac{1}{\hat{\theta}} = \bar{x} \Rightarrow \hat{\theta} = \frac{1}{\bar{x}}$$

$$E(\hat{\theta}) = E\left(\frac{n}{\sum x}\right) \quad T = \sum x_t \sim \text{Gamma}(n, \theta)$$

$$\begin{aligned} E\left(\frac{n}{T}\right) &= n \int_0^{\infty} \frac{1}{t} \cdot \frac{t^{n-1} \cdot e^{-\theta t} \cdot \theta^n}{\Gamma(n)} dt \\ &= \frac{n \cdot \theta^n}{\Gamma(n)} \int_0^{\infty} t^{n-2} \cdot e^{-\theta t} dt \end{aligned} \quad \left\{ \begin{aligned} \int_0^{\infty} \frac{\theta^n}{\Gamma(n)} \cdot t^{n-1} \cdot e^{-\theta t} dt &= 1 \\ \Gamma(n) &= (n-1) \Gamma(n-1) \end{aligned} \right.$$

$$= \frac{n \cdot \Gamma(n-1) \theta^n}{\Gamma(n) \theta^{n-1}} = \frac{n \cdot \Gamma(n-1)}{(n-1) \Gamma(n-1)} \cdot \frac{\theta^n}{\theta^{n-1}} = \frac{n}{n-1} \cdot \theta$$

$$E(\hat{\theta}_{MLE}) = \frac{n}{n-1} \cdot \theta \quad \text{is biased} \Rightarrow \frac{n-1}{n} \cdot \hat{\theta} \text{ is unbiased}$$

$$\frac{n-1}{n} \cdot \hat{\theta} = \frac{n-1}{n} \cdot \frac{1}{\bar{x}} = \frac{n-1}{n \bar{x}} = \frac{n-1}{\sum x_i}$$

c)

$$\text{Bias}(\hat{\theta}_{MM}) = E(\hat{\theta}) - \theta = \frac{n}{n-1} \cdot \theta - \theta = \frac{\theta}{n-1}$$

d) $\text{MSE}(\hat{\theta}) = \text{var}(\hat{\theta}) + [B(\hat{\theta})]^2$

$$\text{var}(\hat{\theta}) = E\left(\frac{1}{T^2}\right) - \left[E\left(\frac{1}{T}\right)\right]^2$$

$$E\left(\frac{1}{T^2}\right) = \frac{\theta^n}{\Gamma(n)} \int_0^{\infty} t^{n-3} \cdot e^{-\theta t} \cdot dx$$

$$= \frac{\theta^n}{\Gamma(n)} \cdot \frac{\Gamma(n-2)}{\theta^{n-2}} = \frac{\theta^2}{(n-1)(n-2)}$$

$$\text{var}\left(\frac{1}{T}\right) = E\left(\frac{1}{T^2}\right) - \left[E\left(\frac{1}{T}\right)\right]^2 = \frac{\theta^2}{(n-1)(n-2)} - \left(\frac{\theta}{n-1}\right)^2 = \frac{\theta^2}{(n-1)^2(n-2)}$$

$$\text{var}(\hat{\theta}) = \text{var}\left(\frac{1}{\bar{X}_t}\right) = \text{var}\left(\frac{n}{T}\right) = n^2 \cdot \text{var}\left(\frac{1}{T}\right) = \frac{n^2 \theta^2}{(n-1)^2(n-2)}$$

$$\text{MSE} = \frac{\theta^2}{(n-1)^2} + \frac{n^2 \theta^2}{(n-1)^2(n-2)}$$

e) $\lim_{n \rightarrow \infty} \text{MSE} = 0$ then it's consistent

$$\lim_{n \rightarrow \infty} \frac{\theta^2}{(n-1)^2} + \frac{n^2 \theta^2}{(n-1)^2(n-2)} = 0 \Rightarrow \text{it's consistent.}$$

Q.6

$$f(x; \beta, \sigma) = \beta \sigma x^{\sigma-1} \cdot e^{-\beta x^\sigma}$$

$$L(\beta, \sigma) = \prod_{i=1}^n f(x_i) = \prod_{i=1}^n (\beta \sigma x_i^{\sigma-1} e^{-\beta x_i^\sigma})$$

$$= \beta^n \cdot \sigma^n \cdot \prod_{i=1}^n x_i^{\sigma-1} \cdot e^{-\beta \sum x_i^\sigma}$$

$$\ln L(\beta, \sigma | x) = n \ln \beta + n \ln \sigma + (\sigma-1) \ln \sum x_i - \beta \sum x_i^\sigma$$

$$\frac{\partial \ln L(\beta)}{\partial \beta} = \frac{n}{\beta} - \sum x_i^\sigma = 0 \Rightarrow \boxed{\hat{\beta}_{MLE} = \frac{n}{\sum x_i^\sigma}}$$

Check for 2nd derivative < 0