

METU, Department of Economics

ECON 201 – Problem Set 4

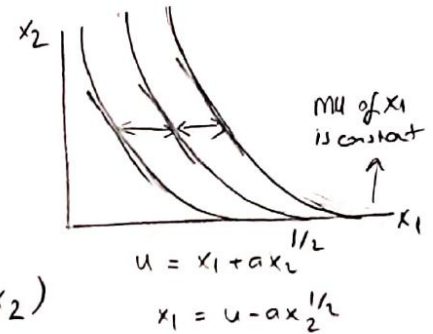
Q.1 $u(x_1, x_2) = x_1 + \alpha x_2^{1/2}$ $P = (1, 1)$ $I = 64$.
 Quasi-Linear Function.

$$\max u = x_1 + \alpha x_2^{1/2}$$

subject to

$$P_{x_1} x_1 + P_{x_2} x_2 \leq I$$

$$\mathcal{L} = x_1 + \alpha x_2^{1/2} + \lambda (I - P_{x_1} x_1 - P_{x_2} x_2)$$



$$(1) \quad \frac{\partial \mathcal{L}}{\partial x_1} = 1 - \lambda P_{x_1} \leq 0, \quad \bar{z}_1 \cdot x_1 = 0, \quad x_1 \geq 0$$

$$(2) \quad \frac{\partial \mathcal{L}}{\partial x_2} = \frac{\alpha}{2} \cdot x_2^{-1/2} - \lambda P_{x_2} \leq 0, \quad \bar{z}_2 \cdot x_2 = 0, \quad x_2 \geq 0$$

$$(3) \quad \frac{\partial \mathcal{L}}{\partial \lambda} = I - P_{x_1} x_1 - P_{x_2} x_2 \geq 0, \quad \bar{z}_3 \cdot \lambda = 0, \quad \lambda \geq 0$$

Note that, $\lim_{x_2 \rightarrow 0} \frac{1}{\sqrt{x_2}} = +\infty \Rightarrow x_2 > 0$

Since utility is increasing in both of the goods (MU_{x_1} and $MU_{x_2} > 0$)
 reasonably guess that budget constraint is binding and $\lambda > 0$ then,
 we have two cases. $x_1 > 0$ or $x_1 = 0$

a) Case 1: $x_1 > 0$

$$\text{From (1)} \quad 1 - \lambda P_{x_1} = 0 \Rightarrow \lambda = 1 > 0$$

$$(2) \quad \frac{\alpha}{2} \cdot x_2^{-1/2} - \lambda P_{x_2} = 0 \Rightarrow x_2 = 100$$

$$(3) \quad I - x_1 - x_2 < 0 \Rightarrow x_1 = I - x_2$$

$$x_1 = -36 < 0$$

(contradicts $x_1 > 0$)

Case 2: $x_1 = 0$

$$(1) \quad (1-\lambda) \leq 0 \Rightarrow \lambda \geq 1$$

$$(2) \quad \frac{1}{2} \cdot 20 x_2^{-1/2} - \lambda = 0 \Rightarrow x_2 = \left(\frac{10}{\lambda}\right)^2$$

$$(3) \quad \text{Since } \lambda > 0 \Rightarrow \underbrace{64 - x_1 - x_2}_0 = 0 \Rightarrow x_2^* = 64$$

$$\text{At } a=20 \Rightarrow \{x_1^*, x_2^*\} = \{0, 64\}$$

(b) Assume $a = 8$

Case 1: $x_1 > 0$

$$(1) \quad 1-\lambda = 0 \Rightarrow \lambda = 1 > 0$$

$$(2) \quad \text{Since } x_2 > 0 \Rightarrow \frac{4}{\sqrt{x_2}} = \lambda \Rightarrow x_2^* = 16$$

$$(3) \quad \text{Since } \lambda > 0 \Rightarrow \underbrace{64 - x_1 - x_2}_{16} = 0 \Rightarrow x_1^* = 48$$

Case 2: $x_1 = 0$

$$(1) \quad 1-\lambda \leq 0 \quad \lambda \geq 1$$

$$(2) \quad \frac{4}{\sqrt{x_2}} = \lambda \Rightarrow x_2 = \left(\frac{4}{\lambda}\right)^2$$

$$(3) \quad \text{Since } \lambda > 0 \Rightarrow \underbrace{64 - x_1 - x_2}_0 = 0 \Rightarrow x_2 = 64$$

$$\text{At (2)} \Rightarrow x_2 = \left(\frac{4}{\lambda}\right)^2 \Rightarrow \frac{64}{16} = \left(\frac{1}{\lambda}\right)^2 \Rightarrow \lambda = \frac{1}{2}$$

contradicts with (1) $\lambda \geq 1$

$$\text{Then, at } a=8 \quad \{x_1^*, x_2^*\} = \{48, 16\}$$

(c) We need conditions for interior solution.

So, we will look at the case where $x_1 > 0$ & $x_2 > 0$ and find conditions for a , that does not contradict with our "Kuhn-Tucker" conditions.

Case: $x_1 > 0$ & $x_2 > 0$

$$(1) \quad 1 - \lambda = 0 \Rightarrow \lambda = 1$$

$$(2) \quad \frac{a}{2} \cdot \frac{1}{\sqrt{x_2}} - \lambda = 0 \Rightarrow x_2 = \left(\frac{a}{2}\right)^2$$

$$(3) \quad \text{Since } \lambda > 0 \Rightarrow I - x_1 - x_2 = 0$$

$$64 - x_1 - \left(\frac{a}{2}\right)^2 = 0 \Rightarrow x_1 = 64 - \left(\frac{a}{2}\right)^2 > 0$$

$$\Rightarrow a < 16$$

$$\text{Check (2)} \quad \frac{a}{2} \cdot \frac{1}{\sqrt{\left(\frac{a}{2}\right)^2}} = 1 \Rightarrow a > 0 \Rightarrow \boxed{0 < a < 16}$$

(d) $a = 12$, conditions for I to have a corner solution?

So, we need conditions for the case $x_1 > 0$ & $x_2 > 0$.

$$(1) \quad \text{Since } x_1 > 0 \Rightarrow (1 - \lambda) = 0 \Rightarrow \lambda = 1$$

$$(2) \quad \text{Since } x_2 > 0 \Rightarrow \frac{1}{2} \cdot 12 \cdot x_2^{-1/2} - \lambda = 0 \Rightarrow x_2 = 36$$

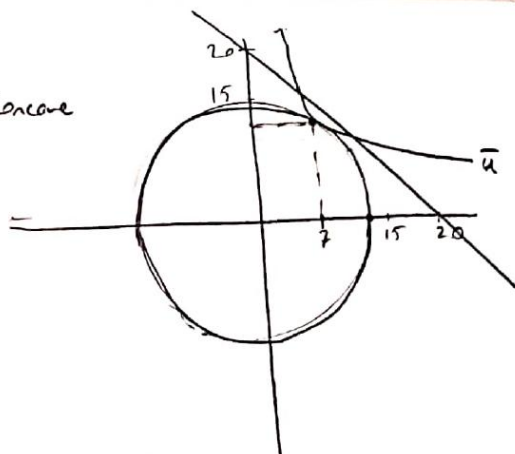
$$(3) \quad \text{Since } \lambda > 0 \Rightarrow I - x_1 - x_2 = 0 \Rightarrow I = 36$$

Q.2 $\max_{x,y} xy^3 \rightarrow$ Strictly Quasi-Concave fnc.

$$200 - x^2 - y^2 \geq 0$$

and

$$20 - x - y \geq 0$$



(a)

$$L = xy^3 + \lambda_1(200 - x^2 - y^2) + \lambda_2(20 - x - y)$$

Kuhn-Tucker FOCs

$$(1) \frac{\partial L}{\partial x} = \underbrace{y^3 - 2\lambda_1 x - \lambda_2}_{z_x} \leq 0, \quad z_x x = 0, \quad x \geq 0$$

$$(2) \frac{\partial L}{\partial y} = \underbrace{3xy^2 - 2\lambda_1 y - \lambda_2}_{z_y} \leq 0, \quad z_y y = 0, \quad y \geq 0$$

$$(3) \frac{\partial L}{\partial \lambda_1} = \underbrace{200 - x^2 - y^2}_{z_{\lambda_1}} \geq 0, \quad z_{\lambda_1} \lambda_1 = 0, \quad \lambda_1 \geq 0$$

$$(4) \frac{\partial L}{\partial \lambda_2} = \underbrace{20 - x - y}_{z_{\lambda_2}} \geq 0, \quad z_{\lambda_2} \lambda_2 = 0, \quad \lambda_2 \geq 0$$

$$\text{Since } u = xy^3 \Rightarrow \left. \begin{aligned} y &= \sqrt[3]{\frac{u}{x}} & \lim_{x \rightarrow 0} y &= +\infty \\ x &= \frac{u}{y^3} & \lim_{y \rightarrow 0} x &= +\infty \end{aligned} \right\} x \text{ and } y > 0$$

So, we will investigate whether the constraints are binding or not.

Case 1: $\lambda_1 = 0, \lambda_2 > 0$

$$\left. \begin{aligned} (1) \quad y^3 - \lambda_2 &= 0 \\ (2) \quad 3xy^2 - \lambda_2 &= 0 \\ (4) \quad 20 - x - y &= 0 \end{aligned} \right\} \frac{y}{3x} = 1 \quad 20 - x - 3x = 0 \Rightarrow \begin{aligned} x &= 5 \\ y &= 15 \end{aligned}$$

check for $\lambda_1 = 0$

$$(3) \quad 200 - x^2 - y^2 > 0 \Rightarrow 200 - (5)^2 - (15)^2 < 0 \rightarrow \text{contradicts}$$

Then, production possibility frontier is binding.

Case 2 $\lambda_1 > 0$, $\lambda_2 = 0$

$$\begin{cases} (1) & y^3 - 2\lambda_1 \cdot x = 0 \\ (2) & 3xy^2 - 2\lambda_1 y = 0 \end{cases} \Rightarrow \frac{y}{x} = \frac{1}{y} \Rightarrow y^2 = 3x^2$$

$$(3) \quad 200 - x^2 - y^2 = 0 \Rightarrow 200 - 4x^2 = 0 \Rightarrow \begin{matrix} x = 5\sqrt{2} \\ y = 5\sqrt{6} \end{matrix}$$

$$(4) \quad 20 - x - y \geq 0 \Rightarrow \underbrace{20 - 5\sqrt{2} - 5\sqrt{6}}_{19,3} > 0 \Rightarrow \text{resource constraint is not binding. } \lambda_2 = 0.$$

Our optimal bundle $\{x^*, y^*\} = \{5\sqrt{2}, 5\sqrt{6}\}$. PPF is binding but resource constraint is not binding.

Q.3 We have quasi-linear utility function. So, we should check for corner solution. Also, we have two inequality constraints.

In the case of quasi-linear objective function and inequality constraint, we should solve for the Kuhn-Tucker conditions.

$$\text{Max } u = c + 2 \ln m \quad \rightarrow \text{Quasi-linear fnc.}$$

s.t

$$I - 10c - 5m \geq 0$$

and

$$m \geq 8$$

$$\mathcal{L} = c + 2 \ln m + \lambda_1 \underbrace{(120 - 10c - 5m)}_{\geq 0} + \lambda_2 \underbrace{(m - 8)}_{\geq 0}$$

Kuhn-Tucker Conditions

$$(1) \quad z_c = \frac{\partial \mathcal{L}}{\partial c} = 1 - 10\lambda_1 \leq 0, \quad (1 - 10\lambda_1)c = 0, \quad c \geq 0$$

$$(2) \quad z_m = \frac{\partial \mathcal{L}}{\partial m} = \frac{2}{m} - 5\lambda_1 + \lambda_2 \leq 0, \quad \underbrace{z_m \cdot m}_{=0} = 0, \quad m \geq 0$$

$$(3) \quad z_{\lambda_1} = \frac{\partial \mathcal{L}}{\partial \lambda_1} = 120 - 10c - 5m \leq 0, \quad z_{\lambda_1} \cdot \lambda_1 = 0, \quad \lambda_1 \geq 0$$

$$(4) \quad z_{\lambda_2} = \frac{\partial \mathcal{L}}{\partial \lambda_2} = m - 8 \geq 0, \quad z_{\lambda_2} \cdot \lambda_2 = 0, \quad \lambda_2 \geq 0$$

$\lim_{m \rightarrow 0} \frac{2}{m} = +\infty$ implies that $m > 0$

We can make some reasonable guesses to simplify our work. Budget constraint should be binding because objective function is strictly increasing in both goods. However, when doing this

kind of reasoning, you should be careful because other constraints might make budget constraint non-binding.

For example, here if the 2nd constraint was $m + c \leq 8$, we'd have non-binding budget constraint.

Now, return to the question:

Case 1 $c = 0$, $\lambda_1 > 0$, $\lambda_2 > 0$

$$(1) \quad 1 - 10\lambda_1 \leq 0 \quad \rightarrow \text{Since } c = 0$$

$$(2) \quad \frac{2}{m} - 5\lambda_1 + \lambda_2 = 0 \quad \rightarrow \text{Since } m > 0$$

$$(3) \quad 120 - 10c - 5m = 0 \quad \rightarrow \text{Since } \lambda_1 > 0$$

$$(4) \quad m - 8 = 0 \quad \rightarrow \text{Since } \lambda_2 > 0$$

$$\text{From (4)} \Rightarrow m = 8$$

Plug (4) into (3)

$$(3) \quad 120 - 10c - 5m = 0; c = 0, m = 8 \text{ then } 120 - 10c - 5m > 0$$

contradicts with (3)

Case 2 $c > 0$, $\lambda_1 > 0$, $\lambda_2 > 0$

$$(1) \quad 1 - 10\lambda_1 = 0$$

$$(2) \quad \frac{2}{m} - 5\lambda_1 + \lambda_2 = 0$$

$$(3) \quad 120 - 10c - 5m = 0$$

$$(4) \quad m - 8 = 0$$

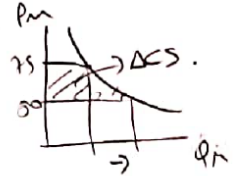
$$\text{From (4)} \quad m = 8 \Rightarrow \text{Plug into (3)} \quad c = 8$$

$$\text{From (1)} \quad \lambda_1 = 1/10 \quad \text{Plug into (2)} \quad \lambda_2 = \frac{1}{4} > 0$$

No contradiction optimum bundle is $\boxed{\{c, m\} = \{8, 8\}}$

$\Rightarrow$ Both constraints are funded binding. That means, Patrick would choose to drink less milk and eat more chocolate if his mother does not make him drink at least 8 containers of milk.

Q.64 $U(V, M) = V^2 M$ $P_V = 200$ $P_M = 50$ $P_M' = 75$ $I = 1500$
 $(200, 50)$ $(200, 75)$



$$\mathcal{L} = V^2 M + \lambda (I - P_V V - P_M M)$$

$$\frac{\partial \mathcal{L}}{\partial V} = 2VM - \lambda P_V = 0$$

$$\frac{2M}{V} = \frac{P_V}{P_M} \Rightarrow V = 2M \cdot \frac{P_M}{P_V}$$

$$\frac{\partial \mathcal{L}}{\partial M} = V^2 - \lambda P_M = 0$$

$$\frac{\partial \mathcal{L}}{\partial I} = 1 = P_V V + P_M M$$

$$I = 3P_M M \Rightarrow M = \frac{I}{3P_M}, V = \frac{2I}{3P_V}$$

$$M^0 = 10$$

$$V^0 = 5$$

$$M^1 = 20/3$$

$$V^1 = 5$$

$$\begin{aligned} \text{(a)} \quad \Delta CS &= - \int_{50}^{75} \frac{I}{3P_M} \cdot dP_M \\ &= -500 \int_{50}^{75} \frac{1}{P_M} \cdot dP_M = -500 \ln P_M \Big|_{50}^{75} \\ &= -500 (\ln 75 - \ln 50) = -202.7 \end{aligned}$$

(b) CV for the price change in meat?

$$V(P_M, P_V, I) = \left(\frac{2I}{3P_V} \right)^2 \left(\frac{I}{3P_M} \right) = \frac{4}{27} \cdot \frac{I^3}{P_V^2 P_M}$$

$$P_M = 50 \quad P_V = 200 \quad I = 1500 \rightarrow \bar{U}_0 = 250$$

$$P_M' = 75 \quad P_V = 200 \quad I = 1500 \rightarrow \bar{U}_1 = 167$$

$$CV \Rightarrow 250 = V_{\text{new}}^2 M_{\text{new}} = \left(\frac{2I_{\text{new}}}{3P_V} \right)^2 \cdot \frac{I_{\text{new}}}{3P_M'} \Rightarrow I_{\text{new}} = 1717$$

$$CV = 1717 - 1500 = 217$$

For CV, we use new prices and initial utility level

c) EV for the price change.

$$EV \Rightarrow 167 = \left(\frac{2I_{\text{new}}}{3P_V} \right)^2 \cdot \frac{I_{\text{new}}}{3P_M} \Rightarrow I_{\text{new}} = 1310$$

$$EV = 1310 - 1500 = -190$$

For EV, we use initial prices and new utility level.

Q.5

a) $u(x_1, x_2) = \min\{x_1, 2x_2\}$

$$x_1 = 2x_2 \Rightarrow x_1 = 2I/2P_1 + P_2$$

$$x_2 = I/2P_1 + P_2$$

(i) $P_1 = 3, P_2 = 4, I = 160$

$P_1' = 2, P_2' = 6, I = 160$

Initially $\Rightarrow x_1 = 32, x_2 = 16 \Rightarrow \bar{u}_0 = 32$

Now $\Rightarrow x_1 = 32, x_2 = 16 \Rightarrow \bar{u}_1 = 32$

$$\left. \begin{array}{l} CV = 0 \\ EV = 0 \end{array} \right\}$$

(ii) $P_1 = 3, P_2 = 4, I = 160$

$P_1' = 6, P_2' = 4, I = 160$

Initially $\Rightarrow x_1 = 32, x_2 = 16 \Rightarrow \bar{u}_0 = 32$

With new prices $\Rightarrow \bar{u}_0 = \frac{2I_{new}^{CV}}{2P_1' + P_2'} \Rightarrow 32 = \frac{2I_{new}^{CV}}{16} \Rightarrow I_{new}^{CV} = 256$

$\Rightarrow CV = 256 - 160 = 96$

New bundle $\Rightarrow x_1' = 20, x_2' = 10 \Rightarrow \bar{u}_1 = 20$

To achieve this utility with initial prices, income should be reduced. The amount of this change gives us EV.

$\bar{u}_1 = 20, \bar{u}_1 = \frac{2I_{new}^{EV}}{2P_1 + P_2} \Rightarrow I_{new}^{EV} = 100$

$EV = 100 - 160 = -60$

b) $u(x_1, x_2) = 2x_1 + 3x_2 \rightarrow$ Perfect Substitutes

If $\frac{P_1}{P_2} < \frac{MU_1}{MU_2} = \frac{2}{3} \Rightarrow x_1 = I/P_1, x_2 = 0, \Rightarrow u = 2I/P_1$

If $\frac{P_1}{P_2} > \frac{MU_1}{MU_2} = \frac{2}{3} \Rightarrow x_1 = 0, x_2 = I/P_2, \Rightarrow u = 3I/P_2$

If $\frac{P_1}{P_2} = \frac{MU_1}{MU_2} \Rightarrow \{x_1, x_2\} = \{0, \frac{I}{P_1}\}, \{(\frac{I}{P_2}, 0)\}$

$$i) P_1 = 2 \quad P_2 = 4 \quad I = 300, \quad P_1' = 4 \quad P_2' = 4$$

$$\frac{P_1}{P_2} = \frac{1}{2} < \frac{2}{3} \Rightarrow x_1 = I/P_1, \quad x_2 = 0 \Rightarrow x_1 = 150$$

$$\bar{u}_0 = 2 \cdot x_1 = 300$$

$$\frac{P_1'}{P_2'} = 1 > \frac{2}{3} \Rightarrow x_1 = 0, \quad x_2 = I/P_2' \Rightarrow x_2 = 75$$

$$\bar{u}_1 = 3 \cdot x_2 = 225$$

In order to compute compensating variation, we need the income level that keeps initial utility with new prices.
(new bundle)

$$\bar{u}_0 = 3 \frac{I_{\text{new}}}{P_2'} \Rightarrow 300 = 3 \frac{I_{\text{new}}}{4} \Rightarrow I_{\text{new}} = 400$$

$$\Rightarrow CV = 400 - 300 = 100$$

In order to compute equivalent variation, we need find income level that provides new utility level with initial prices.
(initial bundle)

$$\bar{u}_1 = 2 \cdot \frac{I_{\text{new}}}{P_1} \Rightarrow 225 = 2 \cdot \frac{I_{\text{new}}}{2} \Rightarrow I_{\text{new}} = 225$$

$$\Rightarrow EV = 225 - 300 = -75$$

$$(ii) \quad P_1' = 2, \quad P_2' = 3, \quad I = 300$$

$$\frac{P_1}{P_2} = \frac{2}{3} = MRS \Rightarrow \text{Any bundle satisfying } 2x_1 + 3x_2 = 300 \text{ is optimal, yielding } \bar{u}_0 = 300$$

$$\Rightarrow CV = 0, \quad EV = 0$$

Q.6

$$\max u = x + 10y - y^2/2$$

s.t

$$I - P_x \cdot x - P_y \cdot y \geq 0$$

$$L = x + 10y - y^2/2 + \lambda(I - P_x \cdot x - P_y \cdot y)$$

$$\left. \begin{aligned} \frac{\partial L}{\partial x} &= 1 - \lambda P_x = 0 \\ \frac{\partial L}{\partial y} &= 10 - y - \lambda P_y = 0 \end{aligned} \right\} \frac{1}{10-y} = \frac{P_x}{P_y} \Rightarrow y = \begin{cases} 10 \cdot \frac{P_y}{P_x} & \text{if } \frac{P_y}{P_x} < 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\partial L}{\partial \lambda} = I - P_x \cdot x - P_y \cdot y = 0$$

With initial prices : $P_x = 1$ $P_y = 5$ $\frac{P_y}{P_x} = 5 \Rightarrow y = 10 \cdot \frac{P_y}{P_x}$

$$I - P_x \cdot x - P_y \cdot \left(10 \cdot \frac{P_y}{P_x}\right) = 0 \Rightarrow x = \frac{I - P_y \left(10 \cdot \frac{P_y}{P_x}\right)}{P_x} = 17.5$$

$$y = 10 - \frac{P_y}{P_x} = 5$$

$$u_0 = 17.5 + 10 \cdot 5 - \frac{25}{2} = 212.5$$

He can pay to the club as much as it keeps his initial utility level with the new prices. $\Rightarrow CV$

$$212.5 = \frac{I_{\text{new}} - P_y' \left(10 \cdot \frac{P_y'}{P_x'}\right)}{P_x'} + \underbrace{10 \cdot \left(10 - \frac{P_y'}{P_x'}\right)}_{y'} - \frac{\left(10 \cdot \frac{P_y'}{P_x'}\right)^2}{2}$$

$$\frac{P_y'}{P_x'} = 1 \Rightarrow 212.5 = I_{\text{new}} - 9 + 10 \cdot 9 - \frac{(9)^2}{2} \Rightarrow I_{\text{new}} = 172$$

$$CV = 172 - 200 = -28$$

Q.7 $u(x, y) = \min \{x, y\}$

At optimum $x = y$

$$I = P_x \cdot x + P_y \cdot y$$

$$\Rightarrow x = y = \frac{I}{P_x + P_y}$$

$$I = 150, \quad P_x = 1 \\ P_y = 1$$

$$x = y = 75 \\ \bar{u}_0 = 75$$

For the CV :

$$\bar{u}_0 = 75 = \frac{I_{\text{new}}}{P'_x + P'_y}$$

$$75 = \frac{I_{\text{new}}}{1 + 2}$$

$$\Rightarrow I_{\text{new}} = 225$$

$$CV = 225 - 150 = 75 = B$$

$\Rightarrow$ She would mind if she got a raise of \$75.

It would compensate the price increase.

$$I = 150, \quad P'_x = 1 \\ P'_y = 2$$

$$x = y = 50 \\ \bar{u}_1 = 50$$

For the EV

$$\bar{u}_1 = 50 = \frac{I_{\text{new}}}{P_x + P_y}$$

$$50 = \frac{I_{\text{new}}}{1 + 1}$$

$$\Rightarrow I_{\text{new}} = 100$$

$$EV = 100 - 150 = -50 = A$$

$\Rightarrow$ It's as bad as a cut in pay of \$50.

This price increase is equivalent to reduction in her income as much as \$50.