

Problem Set 4 – Solutions

Department of Economics

Q1. We solve

$$\max_{x,y \geq 0} u(x,y) = \ln x + \ln y \quad \text{s.t.} \quad p_x x + p_y y \leq M, \quad p_x, p_y, M > 0.$$

(a) *Marshallian demands.*

Since $\ln x$ and $\ln y$ are strictly increasing in $x, y > 0$, the consumer does not leave income unspent. Hence at the optimum

$$p_x x + p_y y = M.$$

Lagrangian:

$$\mathcal{L} = \ln x + \ln y + \lambda(M - p_x x - p_y y).$$

FOCs for an interior optimum:

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{1}{x} - \lambda p_x = 0 \Rightarrow \frac{1}{x} = \lambda p_x,$$

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{1}{y} - \lambda p_y = 0 \Rightarrow \frac{1}{y} = \lambda p_y,$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = M - p_x x - p_y y = 0.$$

From the first two:

$$\lambda = \frac{1}{p_x x} = \frac{1}{p_y y} \Rightarrow p_x x = p_y y \Rightarrow y = \frac{p_x}{p_y} x.$$

Plug into the budget:

$$p_x x + p_y \left(\frac{p_x}{p_y} x \right) = 2p_x x = M \Rightarrow x^* = \frac{M}{2p_x}.$$

Then

$$y^* = \frac{p_x}{p_y} x^* = \frac{M}{2p_y}.$$

So Marshallian demands:

$$x^*(p_x, p_y, M) = \frac{M}{2p_x}, \quad y^*(p_x, p_y, M) = \frac{M}{2p_y}.$$

(b) *Sufficiency of FOCs.*

The Hessian of u is

$$H = \begin{pmatrix} -1/x^2 & 0 \\ 0 & -1/y^2 \end{pmatrix},$$

which is negative definite on $\mathbb{R}_{++}^2$. Hence u is strictly concave.

The budget set $\{(x, y) \geq 0 : p_x x + p_y y \leq M\}$ is non-empty, compact and convex. Maximizing a strictly concave function on a convex set yields a unique global maximizer, and any interior point satisfying the FOCs is that maximizer. Therefore our FOCs are both necessary and sufficient, and (x^*, y^*) is unique.

Q2. $u(x, y) = \sqrt{x} + 2\sqrt{y}$, $p_x x + p_y y \leq M$, $p_x, p_y, M > 0$.

(a) *Demand.*

$u_x = \frac{1}{2\sqrt{x}} > 0$ (for $x > 0$), $u_y = \frac{2}{2\sqrt{y}} = \frac{1}{\sqrt{y}} > 0$ (for $y > 0$), so utility is strictly increasing in both goods. Hence the budget binds:

$$p_x x + p_y y = M.$$

Lagrangian:

$$\mathcal{L} = \sqrt{x} + 2\sqrt{y} + \lambda(M - p_x x - p_y y).$$

FOCs (interior):

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= \frac{1}{2\sqrt{x}} - \lambda p_x = 0 \Rightarrow \frac{1}{2\sqrt{x}} = \lambda p_x, \\ \frac{\partial \mathcal{L}}{\partial y} &= \frac{1}{\sqrt{y}} - \lambda p_y = 0 \Rightarrow \frac{1}{\sqrt{y}} = \lambda p_y, \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= M - p_x x - p_y y = 0. \end{aligned}$$

From the first two:

$$\lambda = \frac{1}{2p_x \sqrt{x}} = \frac{1}{p_y \sqrt{y}} \Rightarrow p_y \sqrt{y} = 2p_x \sqrt{x} \Rightarrow \sqrt{y} = 2\frac{p_x}{p_y} \sqrt{x} \Rightarrow y = 4\frac{p_x^2}{p_y^2} x.$$

Plug into budget:

$$p_x x + p_y \left(4\frac{p_x^2}{p_y^2} x\right) = p_x x + 4\frac{p_x^2}{p_y} x = x \left(p_x + \frac{4p_x^2}{p_y}\right) = M.$$

Thus

$$x^* = \frac{M}{p_x + 4p_x^2/p_y} = \frac{Mp_y}{p_x p_y + 4p_x^2} = \frac{Mp_y}{p_x(p_y + 4p_x)}.$$

Then

$$y^* = 4\frac{p_x^2}{p_y^2} x^* = 4\frac{p_x^2}{p_y^2} \cdot \frac{Mp_y}{p_x(p_y + 4p_x)} = \frac{4Mp_x}{p_y(p_y + 4p_x)}.$$

So

$$x^*(p_x, p_y, M) = \frac{Mp_y}{p_x(p_y + 4p_x)}, \quad y^*(p_x, p_y, M) = \frac{4Mp_x}{p_y(p_y + 4p_x)}.$$

(b) *KT sufficiency.*

$u(x, y)$ is the sum of concave functions (square roots), so it is ****strictly concave**** on $\mathbb{R}_{++}^2$. The budget set is non-empty, convex, compact, and there exists a strictly feasible point (Slater condition). Hence this is a concave maximization problem; Kuhn–Tucker conditions (which here reduce to the Lagrange FOCs with binding budget) are *necessary and sufficient*, and the solution is unique.

Q3. $U(x, y) = xy + 20y$ subject to $4x + y \leq 40$, $x, y \geq 0$.

Demand and KT sufficiency.

Marginal utilities:

$$U_x = y, \quad U_y = x + 20.$$

For any feasible bundle (x, y) , $U_y = x + 20 \geq 20 > 0$, so the consumer always strictly prefers more y . Thus the budget constraint $4x + y \leq 40$ must bind at an optimum:

$$4x + y = 40, \quad x \geq 0, \quad y \geq 0.$$

From $y = 40 - 4x$ with $0 \leq x \leq 10$, write U as a single-variable function:

$$U(x) = x(40 - 4x) + 20(40 - 4x) = 40x - 4x^2 + 800 - 80x = -4x^2 - 40x + 800.$$

This is a concave quadratic in x (coefficient $-4 < 0$). Its derivative:

$$U'(x) = -8x - 40 < 0 \quad \forall x \in [0, 10].$$

So U is strictly decreasing in x on the feasible interval, and is maximized at the smallest feasible x :

$$x^* = 0, \quad y^* = 40.$$

So the (price/income-specific) Marshallian demand:

$$x^* = 0, \quad y^* = 40.$$

Sufficiency:

Although $U(x, y)$ is not globally concave in (x, y) , after using monotonicity to impose $4x + y = 40$, the problem reduces to maximizing a concave $U(x)$ over a convex interval. In that reduced one-dimensional problem, any FOC or endpoint check is sufficient. The global maximum is at the boundary. In terms of KKT, sufficiency follows from the concavity of $U(x)$ on the feasible set and the fact that we checked the boundary; but because the original $U(x, y)$ is not concave, one cannot rely solely on general KKT sufficiency conditions without this extra step.

Q4. Three separate KKT problems.

(a) $u(x_1, x_2) = \ln(x_1 + 1) + \ln(x_2 + 1)$, $p_1x_1 + p_2x_2 \leq m$, $x_1, x_2 \geq 0$.

Concavity and monotonicity.

Each $\ln(x_i + 1)$ is concave on $[0, \infty)$ and strictly increasing. So u is strictly concave and strictly increasing in both x_1, x_2 . The budget set is convex, closed and non-empty. This is

a concave maximization problem over a convex set, so KKT conditions are necessary and sufficient, and the maximizer is unique.

Solution.

Budget binds at an interior optimum (due to monotonicity):

$$p_1x_1 + p_2x_2 = m.$$

Lagrangian:

$$\mathcal{L} = \ln(x_1 + 1) + \ln(x_2 + 1) + \lambda(m - p_1x_1 - p_2x_2).$$

FOCs (interior):

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x_1} &= \frac{1}{x_1 + 1} - \lambda p_1 = 0 \Rightarrow \frac{1}{x_1 + 1} = \lambda p_1, \\ \frac{\partial \mathcal{L}}{\partial x_2} &= \frac{1}{x_2 + 1} - \lambda p_2 = 0 \Rightarrow \frac{1}{x_2 + 1} = \lambda p_2,\end{aligned}$$

$$m - p_1x_1 - p_2x_2 = 0.$$

From the first two:

$$\lambda = \frac{1}{p_1(x_1 + 1)} = \frac{1}{p_2(x_2 + 1)} \Rightarrow p_1(x_1 + 1) = p_2(x_2 + 1).$$

Let $a = x_1 + 1$, $b = x_2 + 1$. Then $p_1a = p_2b$ so $b = \frac{p_1}{p_2}a$. The budget:

$$p_1(a - 1) + p_2(b - 1) = m \Rightarrow p_1a - p_1 + p_2b - p_2 = m.$$

Using $p_2b = p_1a$:

$$p_1a - p_1 + p_1a - p_2 = m \Rightarrow 2p_1a = m + p_1 + p_2 \Rightarrow a = \frac{m + p_1 + p_2}{2p_1}.$$

Then

$$b = \frac{p_1}{p_2}a = \frac{m + p_1 + p_2}{2p_2}.$$

Thus

$$x_1^* = a - 1 = \frac{m + p_1 + p_2}{2p_1} - 1, \quad x_2^* = b - 1 = \frac{m + p_1 + p_2}{2p_2} - 1,$$

provided these are non-negative; otherwise a corner with one of $x_i = 0$ solves the KKT system.

Given strict concavity and convex budget set, the KKT solution (interior or corner) is the unique global maximizer; hence KT sufficiency holds.

- (b) $u(x_1, x_2) = x_1^2 + (x_2 - 5)^2$, $x_1, x_2 \geq 0$, $2x_1 + x_2 \leq 4$.

Here u is a sum of squares, hence **convex**, not concave. We are maximizing a convex function on a convex polytope. Any interior KKT point (if it exists) is a minimum, not maximum; so any maximum must lie on the boundary (extreme points).

Feasible set: $\{x_1, x_2 \geq 0 : 2x_1 + x_2 \leq 4\}$ is a triangle with vertices:

$$(0, 0), (2, 0), (0, 4).$$

Evaluate:

$$u(0,0) = 0^2 + (0-5)^2 = 25, \quad u(2,0) = 2^2 + (0-5)^2 = 4 + 25 = 29, \quad u(0,4) = 0^2 + (4-5)^2 = 1.$$

Max is at $(x_1^*, x_2^*) = (2, 0)$.

KT sufficiency *fails* in the usual concavity sense: because u is convex, KKT conditions (if applied) would not guarantee a maximum. We found the optimum by comparing boundary values.

(c) $u(x_1, x_2) = x_1 x_2$, constraints $x_1 + x_2 \leq 100$, $x_1 \leq 40$, $x_1, x_2 \geq 0$.

Without the cap $x_1 \leq 40$, the product $x_1 x_2$ under $x_1 + x_2 = 100$ is maximized at $x_1 = x_2 = 50$, but this violates $x_1 \leq 40$.

Given monotonicity in both goods for positive x_1, x_2 , the optimum under all constraints must satisfy:

$$x_1 + x_2 = 100, \quad x_1 \leq 40,$$

and it is best to push x_1 up to the maximum allowed by the cap:

$$x_1^* = 40, \quad x_2^* = 60.$$

Again u is not globally concave (though it is quasi-concave on $\mathbb{R}_{++}^2$). KKT conditions are not sufficient by concavity alone; however, given the linear constraints and monotonicity, once we identify $(40, 60)$ as the only feasible point with maximal x_1 on the budget line, and compare with corners, we can see it is the global maximizer.

Q5. $u(x, y) = x + y + xy$, $p_x x + p_y y \leq M$, $x, y \geq 0$.

Monotonicity.

$u_x = 1 + y > 0$, $u_y = 1 + x > 0$, so the consumer always wants more of both goods. Thus the budget binds: $p_x x + p_y y = M$.

Express y :

$$y = \frac{M - p_x x}{p_y},$$

for $0 \leq x \leq M/p_x$.

Substitute:

$$u(x) = x + \frac{M - p_x x}{p_y} + x \cdot \frac{M - p_x x}{p_y} = x^2 \left(-\frac{p_x}{p_y} \right) + x \left(1 + \frac{M}{p_y} \right) + \frac{M}{p_y}.$$

Compute $u''(x)$:

$$u''(x) = -2 \frac{p_x}{p_y} < 0$$

if we include the term $x^2(M/p_y - p_x x/p_y)$ carefully; a simpler route is: write

$$u(x) = x + \frac{M - p_x x}{p_y} + \frac{x(M - p_x x)}{p_y} = x + \frac{M}{p_y} - \frac{p_x}{p_y} x + \frac{Mx}{p_y} - \frac{p_x}{p_y} x^2.$$

Then

$$u''(x) = -2 \frac{p_x}{p_y} < 0,$$

so $u(x)$ is concave in x along the budget line, and the FOC gives the unique interior maximizer if it lies in the interval. Alternatively, we can simply note that u is not concave in (x, y) overall (Hessian is indefinite), so global KT sufficiency does not follow from standard concavity arguments; we must still check boundaries. The safe approach is to compare utility at endpoints $x = 0$ and $x = M/p_x$:

$$\begin{aligned} u(0, M/p_y) &= 0 + M/p_y + 0 = M/p_y, \\ u(M/p_x, 0) &= (M/p_x) + 0 + 0 = M/p_x. \end{aligned}$$

Global optimum:

$$(x^*, y^*) = \begin{cases} (0, M/p_y) & \text{if } M/p_y \geq M/p_x, \\ (M/p_x, 0) & \text{otherwise.} \end{cases}$$

Because u is not concave, KKT are not sufficient in the usual sense; we must check the boundary explicitly as above.

Q6. $u(x, y) = \min\{x, 2y\}$, $p_x x + p_y y \leq M$, $x, y \geq 0$.

(a) *Demand.*

For a given bundle, utility is $\min\{x, 2y\}$. At optimum we want x and $2y$ to be equal; if $x > 2y$, reducing x and increasing y (keeping expenditure constant) raises the minimum, and if $x < 2y$, increasing x is better. Therefore, at an interior optimum:

$$x = 2y,$$

and the budget binds (monotonicity):

$$p_x x + p_y y = M.$$

Substitute $x = 2y$:

$$p_x(2y) + p_y y = (2p_x + p_y)y = M \Rightarrow y^* = \frac{M}{2p_x + p_y}, \quad x^* = 2y^* = \frac{2M}{2p_x + p_y}.$$

(b) *Why Lagrangian derivatives fail; sufficiency.*

$u(x, y) = \min\{x, 2y\}$ is not differentiable along the kink line $x = 2y$, and the optimum lies on this kink. The usual Lagrange FOCs rely on differentiability of the objective and do not apply directly here. We must instead use geometry / kink conditions.

Upper contour sets of Leontief preferences are convex, and the budget set is convex, so we still maximize a quasi-concave function over a convex set. The bundle (x^*, y^*) on $x = 2y$ and the budget line is the unique global maximizer (once we check feasibility and that no corner gives higher utility).

Q7. $u(x, y) = x^2 + y$, $p_x x + p_y y = M$, $x, y \geq 0$, $p_x, p_y, M > 0$.

(a) *Demand via 1D reduction.*

From the budget:

$$y = \frac{M - p_x x}{p_y}, \quad 0 \leq x \leq \frac{M}{p_x}.$$

Plug into u :

$$u(x) = x^2 + \frac{M - p_x x}{p_y} = x^2 - \frac{p_x}{p_y}x + \frac{M}{p_y}.$$

Derivative:

$$u'(x) = 2x - \frac{p_x}{p_y}, \quad u''(x) = 2 > 0.$$

Stationary point:

$$2x - \frac{p_x}{p_y} = 0 \Rightarrow x^{\text{FOC}} = \frac{p_x}{2p_y}.$$

Since $u''(x) = 2 > 0$, the stationary point is a *minimum*, not a maximum. Therefore the maximum occurs at one of the endpoints $x = 0$ or $x = M/p_x$.

Evaluate:

$$u(0, M/p_y) = 0^2 + M/p_y = \frac{M}{p_y},$$

$$u(M/p_x, 0) = (M/p_x)^2.$$

Thus

$$(x^*, y^*) = \begin{cases} (0, M/p_y), & \text{if } M/p_y \geq (M/p_x)^2, \\ (M/p_x, 0), & \text{if } (M/p_x)^2 \geq M/p_y. \end{cases}$$

So the optimum is always a **corner solution**.

(b) *Why FOCs/KKT alone are not sufficient.*

The Hessian of u in (x, y) is

$$\nabla^2 u = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix},$$

which is positive semidefinite (convex), not negative semidefinite. Hence u is not concave. This is a convex objective maximized on a convex set. Lagrange FOCs (or KKT) can produce stationary points that are minima, not maxima (as we saw). Thus first-order conditions alone are not sufficient to identify the true optimum; we must analyze the behavior along the budget line and check the boundaries explicitly.