

PS#3 Suggested Solutions

① (a) $u(x_1, x_2) = x_1^{1/4} x_2^{3/4}$ (Cobb-Douglas)

It is a well-behaved, strictly concave utility function. By the way, we assume that $p_1 x_1 + p_2 x_2 \leq m$ and we will consider $p_1 x_1 + p_2 x_2 = m$ while solving the maximization problem.

Lagrangian:

$$\mathcal{L} = x_1^{1/4} x_2^{3/4} + \lambda (m - p_1 x_1 - p_2 x_2)$$

$$\text{FOCs: } \left. \begin{aligned} \text{(i) } \partial \mathcal{L} / \partial x_1 &: \frac{1}{4} x_1^{-3/4} x_2^{3/4} - \lambda p_1 = 0 \\ \text{(ii) } \partial \mathcal{L} / \partial x_2 &: \frac{3}{4} x_1^{1/4} x_2^{-1/4} - \lambda p_2 = 0 \end{aligned} \right\} \frac{x_2}{x_1} = \frac{3p_1}{p_2} \Rightarrow x_2 = \frac{3p_1 x_1}{p_2}$$

$$\text{(iii) } \partial \mathcal{L} / \partial \lambda: m - p_1 x_1 - p_2 x_2 = 0$$

Let's insert x_2 found in (i) and (ii) into (iii) then:

$$\begin{aligned} m &= p_1 x_1 + p_2 \frac{3p_1 x_1}{p_2} \Rightarrow p_1 x_1 + 3p_1 x_1 = m \\ &4p_1 x_1 = m \\ x_1^* &= \frac{m}{4p_1} \quad \text{and} \quad x_2^* = \frac{3p_1}{p_2} \cdot \frac{m}{4p_1} = \frac{3m}{4p_2} \end{aligned}$$

(b) $u(x_1, x_2) = 3x_1 + x_2$

Note that for perfect substitute case, we cannot use the Lagrangian method, since it only gives interior solution and ignores corner solutions.

Thus, to find the optimal point, we interpret indifference curve's slope & budget line's slope.

slope of the indifference curve is, $MRS = -3$

slope of the budget = $-p_1/p_2$

• First suppose $3 > p_1/p_2$ $MRS_{12} = \frac{mu_1}{mu_2} > \frac{p_1}{p_2}$ or $\frac{mu_1}{p_1} > \frac{mu_2}{p_2}$

so, we should $\downarrow$ mu_1 by consuming more of x_1 .

In this case, slope of indifference curve is steeper than slope of the budget line.

It is clear that utility is maximized when $x_2^* = 0$ and

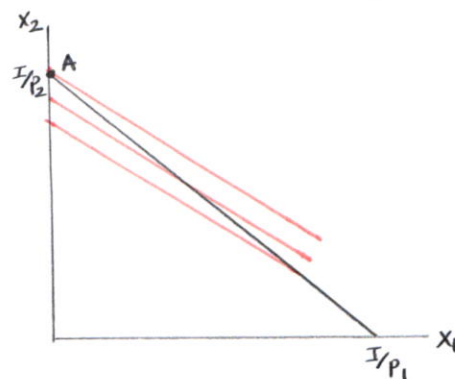
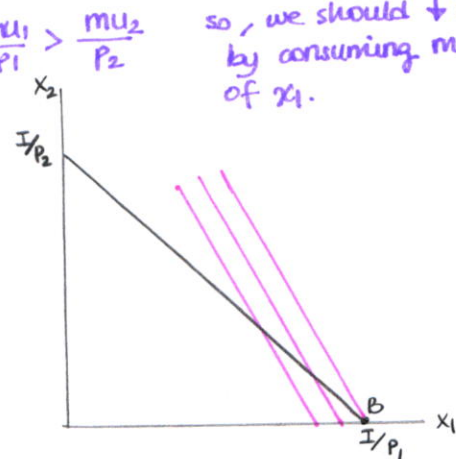
$$x_1^* = \frac{I}{p_1} \quad (\text{point B})$$

• similarly if $3 < p_1/p_2$

slope of indifference curve is flatter than slope of the budget line. It is clear that utility is maximized

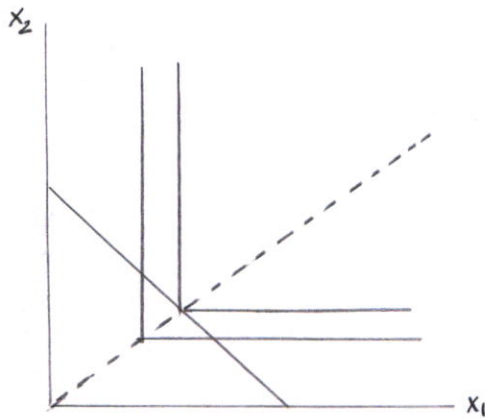
when $x_2^* = \frac{I}{p_2}$ and $x_1^* = 0$ (Point A).

• If $3 = \frac{p_1}{p_2} \Rightarrow$ any point on the budget line becomes utility maximizer.



$$x_1^*(I, p_1, p_2) = \begin{cases} I/p_1 & \text{if } p_1/p_2 < 3 \\ t \in [0, I/p_1] & \text{if } p_1/p_2 = 3 \\ 0 & \text{if } p_1/p_2 > 3 \end{cases} \quad x_2^*(I, p_1, p_2) = \begin{cases} 0 & \text{if } p_1/p_2 < 3 \\ I - 3t & \text{if } p_1/p_2 = 3 \\ I/p_2 & \text{if } p_1/p_2 > 3 \end{cases}$$

(c) $u(x_1, x_2) = \min \{2x_1, 3x_2\}$



The utility function is not differentiable at the kink so we cannot use Lagrangian method.

From the utility function, we know that, at the optimum,

$$2x_1 = 3x_2 \Rightarrow x_1 = \frac{3}{2}x_2$$

$$\text{Budget constraint: } p_1x_1 + p_2x_2 = I$$

$$p_1 \cdot \frac{3}{2}x_2 + p_2x_2 = I \Rightarrow x_2 \left[\frac{3p_1 + 2p_2}{2} \right] = I$$

$$x_2^* = \frac{2I}{3p_1 + 2p_2} \quad \text{and} \quad x_1^* = \frac{3I}{3p_1 + 2p_2}$$

(d) $u(x_1, x_2) = x_1 + x_2 + \sqrt{x_2}$

$$MRS = -\frac{MU_1}{MU_2} = -\frac{1}{1 + \frac{1}{2\sqrt{x_2}}} = -\frac{p_1}{p_2} \quad \text{at the interior solution}$$

$$\text{Hence, } p_2 = p_1 \left[1 + \frac{1}{2\sqrt{x_2}} \right] \Rightarrow x_2 = \frac{1}{4} \left(\frac{p_1}{p_2 - p_1} \right)^2$$

$$\text{By budget constraint, } p_1x_1 + p_2 \frac{1}{4} \left(\frac{p_1}{p_2 - p_1} \right)^2 = I$$

$$\left. \begin{aligned} x_2^*(p_1, p_2, I) &= \frac{1}{4} \left(\frac{p_1}{p_2 - p_1} \right)^2 \\ x_1^*(p_1, p_2, I) &= \frac{I - \frac{1}{4} p_2 \left(\frac{p_1}{p_2 - p_1} \right)^2}{p_1} \end{aligned} \right\} \text{Interior Solution}$$

We should also check for the corner solution.

• It is clear that if $I - \frac{1}{4} p_2 \left(\frac{p_1}{p_2 - p_1} \right)^2 < 0 \Rightarrow x_1^* = 0$

Therefore $x_2^* = I/p_2$

$$x_1^* = \begin{cases} \frac{I - \frac{1}{4} p_2 \left(\frac{p_1}{p_2 - p_1} \right)^2}{p_1} & \text{if } I \geq \frac{1}{4} p_2 \left(\frac{p_1}{p_2 - p_1} \right)^2 \\ 0 & \text{otherwise} \end{cases}$$

$$x_2^* = \begin{cases} \frac{1}{4} p_2 \left(\frac{p_1}{p_2 - p_1} \right)^2 & \text{if } I \geq \frac{1}{4} p_2 \left(\frac{p_1}{p_2 - p_1} \right)^2 \\ I/p_2 & \text{otherwise} \end{cases}$$

(e) $u(x_1, x_2, x_3, x_4) = \sqrt{x_1 x_2} + \sqrt{x_3 x_4}$ s.t. $p_1 x_1 + p_2 x_2 + p_3 x_3 + p_4 x_4 \leq I$

Suppose the agent has total income I and spends m on good 1 and good 2 and $I-m$ on good 3 and good 4. Then, agent will choose x_1 and x_2 to maximize $\sqrt{x_1 x_2}$ subject to $p_1 x_1 + p_2 x_2 \leq m$ and the agent will choose x_3 and x_4 to maximize $\sqrt{x_3 x_4}$ subject to $p_3 x_3 + p_4 x_4 \leq I-m$.

From these maximization problems which are in classical Cobb-Douglas form (left as an exercise; similar to part (a)).

$$x_1^* = \frac{1}{2} \cdot \frac{m}{p_1}, \quad x_2^* = \frac{1}{2} \cdot \frac{m}{p_2}, \quad x_3^* = \frac{1}{2} \cdot \frac{I-m}{p_3}, \quad x_4^* = \frac{1}{2} \cdot \frac{I-m}{p_4}$$

Thus, our agent's utility becomes:

$$u(x_1, x_2, x_3, x_4) = \frac{m}{2\sqrt{p_1 p_2}} + \frac{I-m}{2\sqrt{p_3 p_4}}$$

Now, at this point, agent must know the optimal m^* to maximize his/her utility.

$$\begin{aligned} \max_{\{m\}} \frac{m}{2\sqrt{p_1 p_2}} + \frac{I-m}{2\sqrt{p_3 p_4}} &\Rightarrow \max_{\{m\}} \frac{2m\sqrt{p_3 p_4} + 2(I-m)\sqrt{p_1 p_2}}{2\sqrt{p_1 p_2 p_3 p_4}} \\ &\Rightarrow \max_{\{m\}} \frac{2m(\sqrt{p_3 p_4} - \sqrt{p_1 p_2}) + 2I\sqrt{p_1 p_2}}{2\sqrt{p_1 p_2 p_3 p_4}} \end{aligned}$$

Thus, it is obvious to see that:

If $\sqrt{p_3 p_4} < \sqrt{p_1 p_2} \Rightarrow m^* = 0$, so $x_1^* = 0$ and $x_2^* = 0$.

If $\sqrt{p_3 p_4} > \sqrt{p_1 p_2} \Rightarrow m^* = I$ (all of the income will be spent on x_1 & x_2)

If $\sqrt{p_3 p_4} = \sqrt{p_1 p_2} \Rightarrow m^* \in [0, I]$

Thus;

If $\sqrt{p_1 p_2} < \sqrt{p_3 p_4}$	$x_1^* = I/2p_1$	$x_2^* = I/2p_2$	$x_3^* = 0$	$x_4^* = 0$
$\sqrt{p_1 p_2} > \sqrt{p_3 p_4}$	$x_1^* = 0$	$x_2^* = 0$	$x_3^* = I/2p_3$	$x_4^* = I/2p_4$
$\sqrt{p_1 p_2} = \sqrt{p_3 p_4}$	$x_1^* = \alpha/2p_1$	$x_2^* = \alpha/2p_2$	$x_3^* = (I-\alpha)/2p_3$	$x_4^* = (I-\alpha)/2p_4$

(f) $u(x_1, x_2) = e^{(x_1 + \ln(x_2))^{1/3}} \Rightarrow v(x_1, x_2) = [\ln u(x_1, x_2)]^3$

Using Lagrangian method, will be easier if we use monotonic transformation of the utility function.

$$v(x_1, x_2) = f(u(x_1, x_2)) = x_1 + \ln(x_2)$$

$$\mathcal{L}(x_1, x_2, \lambda) = x_1 + \ln x_2 + \lambda(I - p_1 x_1 - p_2 x_2)$$

$$\left. \begin{aligned} \partial \mathcal{L} / \partial x_1 &= 1 - \lambda p_1 = 0 \\ \partial \mathcal{L} / \partial x_2 &= \frac{1}{x_2} - \lambda p_2 = 0 \end{aligned} \right\} \frac{1}{p_1} = \frac{1}{x_2 p_2} \Rightarrow x_2^* = \frac{p_1}{p_2}$$

$$\partial \mathcal{L} / \partial \lambda = I - p_1 x_1 - p_2 x_2 = 0 \quad \text{--- Insert } x_2^* \text{ to the 3rd FOC:}$$

$$I - p_1 x_1 - p_2 (p_1 / p_2) = 0 \Rightarrow x_1^* = \frac{I - p_1}{p_1}$$

Then:

$$x_1^*(p_1, p_2, I) = \begin{cases} \frac{I - p_1}{p_1} & \text{if } I \geq p_1 \\ 0 & \text{o.w.} \end{cases}$$

$$x_2^*(p_1, p_2, I) = \begin{cases} p_1 / p_2 & \text{if } I \geq p_1 \\ I / p_2 & \text{o.w.} \end{cases}$$

④ $u(x,y) = x^{1/2} + y^{1/2}$

(a) Use Lagrangian method

$$\mathcal{L} = x^{1/2} + y^{1/2} + \lambda (m - p_x \cdot x - p_y \cdot y)$$

$$\left. \begin{aligned} \partial \mathcal{L} / \partial x: \frac{1}{2} x^{-1/2} - \lambda p_x &= 0 \\ \partial \mathcal{L} / \partial y: \frac{1}{2} y^{-1/2} - \lambda p_y &= 0 \\ \partial \mathcal{L} / \partial \lambda: m - p_x \cdot x - p_y \cdot y &= 0 \end{aligned} \right\} \begin{aligned} \frac{y}{x} &= \frac{p_x^2}{p_y^2} \\ x \cdot p_x + x \cdot \frac{p_x^2}{p_y^2} \cdot p_y &= m \\ x &= \frac{p_y \cdot m}{p_x (p_x + p_y)} \quad \text{and} \quad y = \frac{p_x \cdot m}{p_y (p_x + p_y)} \end{aligned}$$

(b) $\partial \mathcal{L} / \partial m = \lambda \rightsquigarrow$ marginal change in utility when income changes by one unit.

(c) $\partial y / \partial m > 0$ since $p_x, p_y, m > 0$

(d) $p_x = 2, p_y = 1, m = 12 \Rightarrow y = 8, x = 2.$

⑤ (a) Budget constraint: $20G + 10M = 60$

The Lagrangian is:

$$\mathcal{L} = G^{1/2} + M^{1/2} + \lambda (60 - 20G - 10M)$$

The FOCs are:

$$\left. \begin{aligned} \partial \mathcal{L} / \partial G &= \frac{1}{2} G^{-1/2} - 20\lambda = 0 \\ \partial \mathcal{L} / \partial M &= \frac{1}{2} M^{-1/2} - 10\lambda = 0 \\ \partial \mathcal{L} / \partial \lambda &= 60 - 20G - 10M = 0 \end{aligned} \right\} \begin{aligned} \left(\frac{M}{G}\right)^{1/2} &= 2 \Rightarrow M = 4G \\ 60 - 20G - 10 \cdot (4G) &= 0 \\ 20G + 40G &= 60 \\ G^* &= 1 \text{ and } M^* = 4 \end{aligned}$$

(b) $v(G, M) = f(u(G, M)) = \ln(G^{1/2} + M^{1/2})$

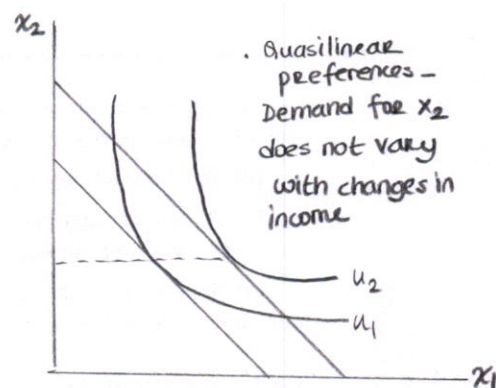
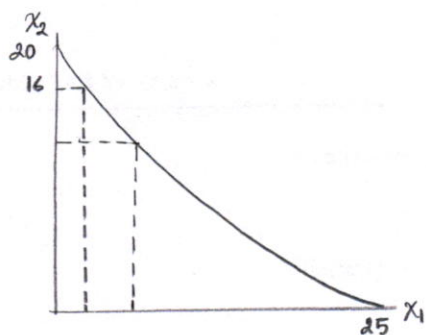
$$\mathcal{L} = \ln(G^{1/2} + M^{1/2}) + \lambda (60 - 20G - 10M)$$

$$\left. \begin{aligned} \partial \mathcal{L} / \partial G &= \frac{1}{2} \frac{G^{-1/2}}{G^{1/2} + M^{1/2}} - 20\lambda = 0 \\ \partial \mathcal{L} / \partial M &= \frac{1}{2} \frac{M^{-1/2}}{G^{1/2} + M^{1/2}} - 10\lambda = 0 \end{aligned} \right\} \left(\frac{M}{G}\right)^{1/2} = 2 \Rightarrow M = 4G$$

$$\partial \mathcal{L} / \partial \lambda = 60 - 20G - 10M \Rightarrow G^* = 1 \text{ and } M^* = 4$$

The solution is the same since we have just applied an increasing function to the utility function.

- ① (a) we simply solve the eqn. $x_2 = 20 - 4\sqrt{x_1}$ to obtain, for example, the points $(x_1, x_2) = (25, 0), (16, 4), (9, 8), (4, 12), (1, 16), (0, 20)$.



- (b) Leo's problem can be written as

$$\max_{x_1, x_2 \geq 0} 4\sqrt{x_1} + x_2 \text{ subject to } x_1 + 2x_2 \leq 24.$$

To solve this problem using the Lagrangian method, we write down the Lagrangian

$$\mathcal{L}(x_1, x_2, \lambda) = 4\sqrt{x_1} + x_2 + \lambda(24 - x_1 - 2x_2)$$

The first order conditions are:

$$\partial \mathcal{L} / \partial x_1 = 4 \cdot \left(\frac{1}{2}\right) x_1^{-1/2} - \lambda = 0$$

$$\partial \mathcal{L} / \partial x_2 = 1 - 2\lambda = 0$$

$$\partial \mathcal{L} / \partial \lambda = x_1 + 2x_2 - 24 = 0$$

The second eqn. gives us $\lambda^* = 1/2$ which we can plug into the first equation to give us $x_1^* = 16$, which then plugged into the third eqn. yields $x_2^* = 4$.

Note that this utility function is well-behaved: it is increasing in x_1 and x_2 , MRS is diminishing (check!), and it is continuous & differentiable. Thus, this problem can also be solved by setting the slope of the indifference curve equal to the slope of the budget line. To do so, we set:

$$MRS = \frac{P_1}{P_2} = \frac{1}{2}$$

$$\text{Note that } MRS = \frac{MU_1}{MU_2} \Rightarrow \frac{\partial u / \partial x_1}{\partial u / \partial x_2} = \frac{4 \cdot \left(\frac{1}{2}\right) x_1^{-1/2}}{1} = \frac{2}{\sqrt{x_1}}$$

and so we set $\frac{2}{\sqrt{x_1}} = \frac{1}{2}$ to yield $x_1^* = 16$, which we substitute into the budget line eqn.

$$x_1 + 2x_2 = 24 \text{ to yield } x_2^* = 4.$$

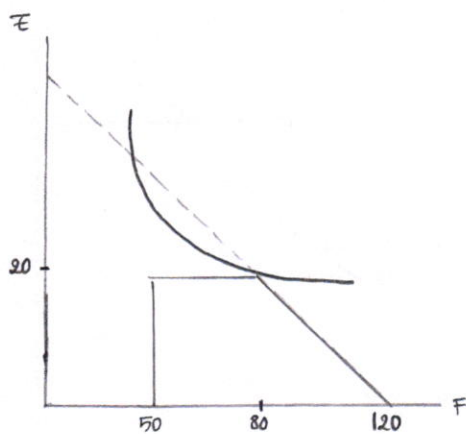
- (c) The new budget constraint is $x_1 + 2x_2 \leq 34$. Leo now buys 16 units of nuts and 9 units of berries. Notice the previous calculation shows that as long as the prices for nuts and berries remain 1 and 2 respectively, Leo always purchases 16 units of nuts. (This is an example of quasilinear) utility function. A consumer who has a quasilinear utility function consumes a fixed amount of one good, if he can afford it.) Thus, we only need to substitute $x_1 = 16$ in the new budget line to get $x_2 = 9$.

- (d) The new budget constraint is $x_1 + 2x_2 \leq 9$, and so Leo can no longer afford 16 units of nuts. The (absolute value of the) slope of the indifference curve at $(9, 0)$ is

$$\frac{\partial u / \partial x_1 |_{x_1=9}}{\partial u / \partial x_2 |_{x_2=0}} = \frac{4 \cdot \left(\frac{1}{2}\right) x_1^{-1/2} |_{x_1=9}}{1 |_{x_2=0}} = \frac{2}{3},$$

which is steeper than the slope of the budget line, which is (in absolute value) $P_1/P_2 = 1/2$. Therefore, Leo cannot afford anything that gives him higher utility.

7 (a)

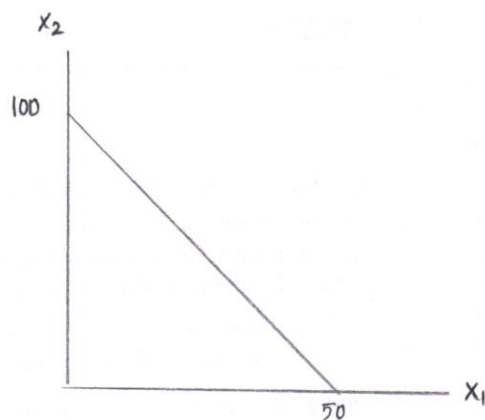
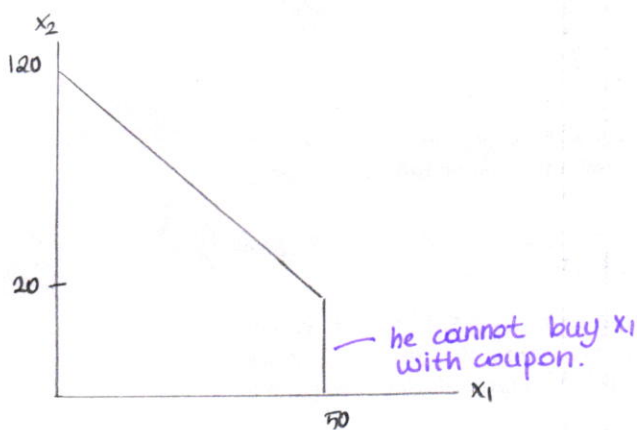


(b) This is a monotonic transformation of $u(f, e) = f^{1/2} e^{1/2}$.

by Cobb-Douglas solution, $e = \frac{m}{p_e} \cdot \frac{1}{2} = 30$ and $f = \frac{m}{p_f} \cdot \frac{1}{2} = 60$. But, $e = 30$ is not allowed. Thus, the consumer will consume as much e as he can, the maximum allowed, that is, $e^* = 20$. Then, from the budget eqn. we get $f^* = (600 - (10 \cdot 20)) / 5 = 80$. You can see the optimal bundle in the graph.

8 (a) This is a perfect substitutes case. $mu_1 = 3$ and $p_1 = 2$, and $mu_2 = 1$ and $p_2 = 1$. Thus, $MRS_{12} = \frac{mu_1}{mu_2} = \frac{3}{1} = 3 > \frac{p_1}{p_2} = \frac{2}{1} = 2$, that is, $\frac{mu_1}{p_1} > \frac{mu_2}{p_2}$. Thus, this consumer will spend all income on good 1, which means $x_1^* = 50$, $x_2^* = 0$.

(b) The budget set is drawn below. Since \$20 coupon can only be spent on good 2, he will use the coupon to buy good 2, and use all income to buy good 1, as the prices did not change, that is, $\frac{mu_1}{p_1} > \frac{mu_2}{p_2}$ is still the case. Thus we get, $x_1^* = 50$, $x_2^* = 20$ as the optimal consumption bundle.



← The regular budget set