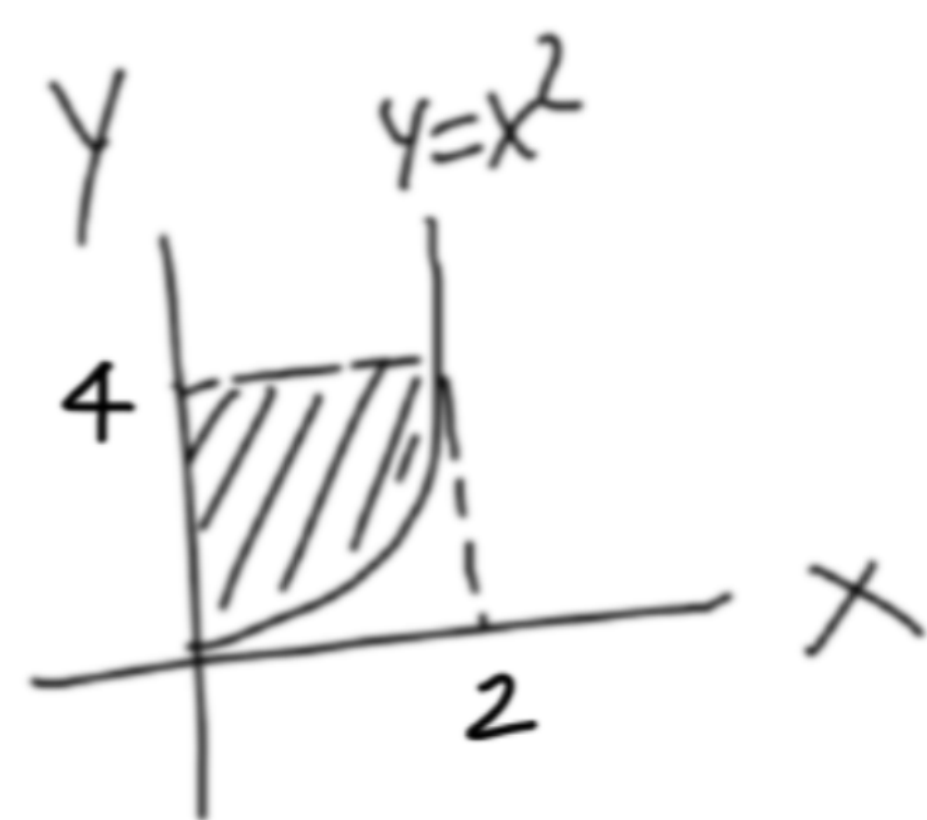


ECON 205 Quiz 2 Solutions

$$1) f(x,y) = \begin{cases} \frac{3}{16} & \text{if } x^2 \leq y \leq 4 \text{ and } 0 < x < 2 \\ 0 & \text{o.w.} \end{cases}$$

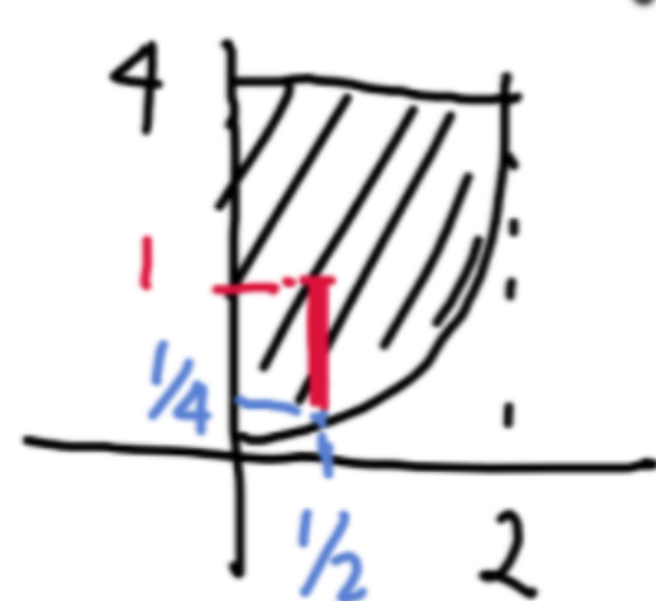
$$i) f(y|x) = \frac{f(x,y)}{f_X(x)}$$



$$f_X(x) = \begin{cases} \int_{x^2}^4 \frac{3}{16} dy = \frac{3}{16} (4 - x^2) & \text{if } 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{For } 0 < x < 2: f(y|x) = \begin{cases} \frac{\frac{3}{16}}{\frac{3}{16}(4-x^2)} = \frac{1}{4-x^2} & \text{if } x^2 \leq y < 4 \\ 0 & \text{otherwise} \end{cases}$$

$$ii) P(Y < 1 | X = \frac{1}{2}) = \int_{\frac{1}{4}}^1 \frac{1}{4 - \frac{1}{4}} dy = \frac{4}{15} \left[y \right]_{\frac{1}{4}}^1 = \frac{4}{15} \left(1 - \frac{1}{4} \right) = \frac{1}{5} //$$



$$iii) \text{For } 0 < x < 2: E(Y|X=x) = \int_{-\infty}^{\infty} y f(y|x) dy = \int_{x^2}^4 y \cdot \frac{1}{(4-x^2)} dy = \frac{1}{(4-x^2)} \left[\frac{y^2}{2} \right]_{x^2}^4 = \frac{16 - x^4}{2(4-x^2)} = \frac{4+x^2}{2} //$$

$$E(Y|X=\frac{1}{2}) = \frac{4 + \frac{1}{4}}{2} = \frac{17}{8} //$$

$$iv) \text{For } 0 < x < 2: E(Y^2|X) = \int_{-\infty}^{\infty} y^2 f(y|x) dy = \int_{x^2}^4 y^2 \frac{1}{(4-x^2)} dy = \frac{1}{(4-x^2)} \left[\frac{y^3}{3} \right]_{x^2}^4 = \frac{16 + 4x^2 + x^4}{3}$$

$$\text{For } 0 < x < 2: V(Y|X) = E(Y^2|X) - [E(Y|X)]^2 = \frac{x^4 + 4x^2 + 16}{3} - \frac{(x^2 + 4)^2}{4} = \frac{(x^2 - 4)^2}{12}$$

$$V(Y|X=\frac{1}{2}) = \frac{(\frac{1}{4} - 4)^2}{12} = \frac{15^2}{4^2 \cdot 12} = \frac{75}{64} //$$

2) We can use the transformation method since V is an increasing function of X .

$$V=10X+5 \Rightarrow X = \frac{V-5}{10} \Rightarrow \frac{dx}{dv} = \frac{1}{10}$$

$$0 < X < 1 \Leftrightarrow 0 < \frac{V-5}{10} < 1 \Leftrightarrow 0 < V-5 < 10 \Leftrightarrow 5 < V < 15$$

$$f_V(v) = \begin{cases} 2 \cdot \left(\frac{V-5}{10}\right) \left|\frac{1}{10}\right|, & 5 < V < 15 \\ 0, & \text{o.w.} \end{cases} = \begin{cases} \frac{V-5}{50}, & 5 < V < 15 \\ 0, & \text{o.w.} \end{cases}$$

or using the distribution function method:

$$F_V(v) = P(V \leq v) = P(10X+5 \leq v) = P\left(X \leq \frac{V-5}{10}\right)$$

$$\frac{V-5}{10} = 0 \Leftrightarrow V=5$$

$$\frac{V-5}{10} = 1 \Leftrightarrow V=15$$

$$F_V(v) = \begin{cases} 0 & \text{if } v \leq 5 \\ \int_0^{(v-5)/10} 2x dx & \text{if } 5 < v < 15 \\ 1 & \text{if } v \geq 15 \end{cases} = \begin{cases} 0 & , v \leq 5 \\ \frac{(V-5)^2}{100} & , 5 < v < 15 \\ 1 & , v \geq 15 \end{cases}$$

$$\Rightarrow \frac{dF_V(v)}{dv} = f_V(v) = \begin{cases} \frac{2(V-5)}{100}, & 5 < v < 15 \\ 0 & \text{o.w.} \end{cases} = \begin{cases} \frac{V-5}{50}, & 5 < v < 15 \\ 0 & , \text{o.w.} \end{cases}$$

$$\begin{aligned}
3) \operatorname{Cov}(aX+b, cY+d) &= E[(aX+b)(cY+d)] - E(aX+b) \cdot E(cY+d) \\
&= E(acXY + adX + bcY + bd) - [aE(X) + b][cE(Y) + d] \\
&= acE(XY) + \cancel{adE(X)} + \cancel{bcE(Y)} + \cancel{bd} - acE(X)E(Y) \\
&\quad - \cancel{adE(X)} - \cancel{bcE(Y)} - \cancel{bd} \\
&= ac(E(XY) - E(X)E(Y)) = ac \operatorname{Cov}(X, Y) \quad \square
\end{aligned}$$