

Calculating Probability of an Event

- ↳ Sample point method
- ↳ Event composition method

Sample point method

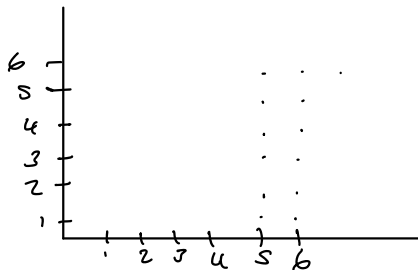
- 1- Define experiment with clearly determining & how to describe one simple event
- 2- List sample events associated with experiment & make certain that it cannot be further decomposed
- 3- Assign reasonable probabilities to sample points in the sample space (S), making certain that $P(E_i) \geq 0$
$$\sum_{i=1}^k P(E_i) = 1$$
- 4- Define event of interest, A, as a specific collection of sample points
- 5- Find $P(A)$, by summing probs. of sample points in A

Ex. Consider tossing a pair of dice. Calculate the probability that numbers on dice is greater than 8

36 sample points

$$P(E_i) = 1/36$$

$$P(A) = 10/36$$



If sample space contain N equiprobable sample points & an event A contains exactly n_A sample points then $P(A) = n_A / N$

With m elements $a_1, a_2, \dots, a_m$ & n elements $b_1, b_2, \dots, b_n$, it is possible to form $mn = m \times n$ points containing one element from each group

Consider tossing a pair of dice $m=6$ $n=6$
 $m \times n = 36$

An ordered arrangement of distinct objects is called permutation

Number of ways of ordering n distinct objects taken r at a time will be denoted by P_r^n

$$P_r^n = n(n-1)(n-2) \dots (n-r+1) = \frac{n!}{(n-r)!}$$

Ex: what is the number of different sequences will be included in an experiment choosing two letters at a time from the letters A, B, C, D

$$P_2^4 = \frac{4!}{2!} = \underline{12}$$

Number of permutations of n distinct objects arranged in a circle is $(n-1)!$ this is called circular permutation

Ex: How many circular permutations are there of four person playing bridge

$$(4-1)! = 6$$

one thousand exercise
in Probability

number of ways of partitioning n distinct objects into k distinct groups containing $n_1, n_2, \dots, n_k$ objects respectively, where each object appears exactly in one group & $\sum_{i=1}^k n_i = n$

$$N = \binom{n}{n_1, n_2, \dots, n_k} = \frac{n!}{n_1! n_2! \dots n_k!}$$

ordered

multinomial coefficients $\rightarrow$ occurs in expansion of multinomial term

$y + y_1 + y_2 + \dots + y_k$ raised to the n^{th} power

$$(y_1 + y_2 + \dots + y_k)^n = \sum \binom{n}{n_1, n_2, \dots, n_k} y_1^{n_1} y_2^{n_2} \dots y_k^{n_k}$$

$$() \quad n_1 + n_2 + \dots + n_k = n$$

Ex: How many different ways can 3 copies of one novel & one copy of each of four other novels be arranged on a shelf

$$N = \binom{7}{3 \ 1 \ 1 \ 1 \ 1} = \frac{7!}{3! \cdot 1! \cdot 1! \cdot 1! \cdot 1!} = \underline{840}$$

Combinations

Number of combinations of n objects taken r at a time is number of subsets, each of size r that can be formed from n objects. Denoted by C_r^n or $\binom{n}{r}$

of unordered subset of size r chosen without replacement from n possible objects

$$\underbrace{\binom{n}{r}}_{\text{binomial coefficients}} = C_r^n = \frac{n!}{r! (n-r)!}$$

binomial expansion

$$(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \binom{n}{2} x^{n-2} y^2 + \dots + \binom{n}{n} x^0 y^n$$

Ex: In how many different ways can tosses of a coin yield two heads & four tails

$$\binom{6}{2} = \frac{6!}{2! 4!} = 15$$

Ex: How many different committees of two economists & one politician can be formed from four economists & three politicians

Four economists $\binom{4}{2} = \frac{4!}{2!2!} = 6$
 for politeness $\binom{3}{1} = \frac{3!}{2!1!} = 3$ } number of committees
 $6 \times 3 = \underline{\underline{18}}$

Conditional Prob. & Independence of Events

Prob. of an event sometimes depends on occurrence/non-occurrence of another event

Ex:

If we ask prob. that a person working in a bank earns TL 15000 per month

we may get several different answers since, they all may be correct

Working in appropriate or public bank, with 10 years of experience etc.

Since choice of S is by self-evident? that is set of all possibilities under consideration

$P(A|S)$ to denote conditional probability of any event A in S or .prob of A given S

Conditional prob. of an event $\rightarrow$ is prob. of event given fact that one or more events have already occurred

Ex! A consumer research organization has studied service under warranty period by 20 new car dealers in a city

	<u>Good Service</u>	<u>Bad Service</u>
in business 10 or more years	16 0.8	4 0.2
in business less than 10 years	10 0.5	10 0.5

a) If a person randomly selects one of these dealers, what is the prob that he/she gets good service under warranty period

By randomly selects $\rightarrow$ each case, all possible selections are equally likely

$G \rightarrow$ selection of dealer providing good service

$n(G) \rightarrow$ number of element in G

$n(S) \rightarrow$ number of elements in sample space

$$P(G) = \frac{n(G)}{n(S)} = \frac{16+10}{20} = 0.8$$

b) If a person randomly selects one of the dealers who is in business for 10 or more years what is the

probability that he/she gets one who provides good service under warranty

limit our selves to a reduced sample space $\rightarrow T$

$$P(G|T) = \frac{16}{20} = 0.80 \quad P(G|T) > P(G)$$

numerator 16 of $P(G|T)$ is $n(T \cap G) = 16$, denominator is $n(T)$

$$P(G|T) = \frac{n(T \cap G)}{n(T)} \quad \text{if we divide both numerator \& denominator by } n(S) \quad \frac{n(T \cap G)/n(S)}{n(T)/n(S)}$$
$$= \frac{P(T \cap G)}{P(T)}$$

The conditional prob of an event A , given an event B has occurred is $P(A|B) = \frac{P(A \cap B)}{P(B)}$, where $P(B) > 0$

$$\text{or } P(B|A) = \frac{P(A \cap B)}{P(A)}, \text{ where } P(A) > 0$$

Ex: for our previous example, what is the prob. that one of the dealers who has been in business less than 10 years will provide good service

$T \rightarrow$ dealers in business 10 years or more

$G \rightarrow$ good service

$\hookrightarrow \bar{T}$
 $\hookrightarrow$ complement of T

$$P(\bar{T} \cap G) = 10/50 = 0.2$$

$$P(\bar{T}) = 10+20 / 50 = 0.6$$

$$P(G|\bar{T}) = \frac{P(\bar{T} \cap G)}{P(\bar{T})} = \frac{0.2}{0.6} = 1/3$$

If we multiply both sides of our definition by $P(B)$ we obtain multiplication rule:

$$P(A|B) \cdot P(B) = \frac{P(A \cap B)}{P(B)} \cdot P(B) \Rightarrow \begin{matrix} P(A \cap B) = P(A|B) \cdot P(B) \\ \text{or} \\ P(A \cap B) = P(B|A) \cdot P(A) \end{matrix}$$

Probability of occurrence of an event $A \rightarrow$ is unaffected by occurrence or non-occurrence of event B

$\hookrightarrow$ If this is the case we say that A & B are independent

Two events are independent if any of following holds

- $P(A|B) = P(A)$
- $P(B|A) = P(B)$
- $P(A \cap B) = P(A) \cdot P(B)$

Ex! A coin is tossed three times & 8 possible

outcomes are

HHH	H TT	T T H	T H T
H H T	T T T	T H H	H T H

and they are
equally likely

$A \rightarrow$ Event that H occurs on each first coin tosses

$B \rightarrow$ Event that T occurs on third toss

$C \rightarrow$ Event that exactly 2 head occurs in 3 tosses

Show that events A & B are independent

Show that events B & C are dependent

$$A = \{HHH, HHT\} \quad B = \{HHT, HTT, THT, TTT\}$$

$$C = \{HTT, THT, TTH\}$$

$$A \cap B = \{HHT\} \quad B \cap C = \{HHT, THT\}$$

$$P(A) = 0.25 \quad P(B) = 0.5 \quad P(C) = 3/8$$

$$P(A \cap B) = 1/8 \quad P(B \cap C) = 1/4$$

$$a) P(A) \cdot P(B) = 1/4 \cdot 1/2 = 1/8 = P(A \cap B)$$

$$b) P(B) \cdot P(C) = 1/2 \cdot 3/8 = 3/16 \neq P(B \cap C)$$

Quiz 1, October 31, 10:40

Laws of Probability

Multiplicative Rule of Prob

Prob. of intersection of two events is

$$\begin{aligned}P(A \cap B) &= P(A) \cdot P(B|A) \\ &= P(B) \cdot P(A|B)\end{aligned}$$

if A & B are independent, $P(A \cap B) = P(A) \cdot P(B)$

$$\begin{aligned}\text{for probability } P(A \cap B \cap C) &= P[(A \cap B) \cap C] \\ &= P(A \cap B) \cdot P(C|A \cap B) \\ &= P(A) \cdot P(B|A) \cdot P(C|A \cap B)\end{aligned}$$

if they are independent

$$P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$$

$$P(A_1 \cap A_2 \cap A_3 \dots \cap A_k) = P(A_1) \cdot P(A_2|A_1) \cdot P(A_3|A_1 \cap A_2) \dots P(A_k|A_1 \cap A_2 \cap \dots \cap A_{k-1})$$

Additive Law of Prob.

Prob of union of two events A & B is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

If events A & B are mutually exclusive $P(A \cap B) = 0$

$P(A \cup B) = P(A) + P(B)$ in this case

$$\begin{aligned}P(A \cup B \cup C) &= P[A \cup (B \cup C)] = P(A) + P(B \cup C) - P[A \cap (B \cup C)] \\ &= P(A) + P(B) + P(C) - P(B \cap C) - P[(A \cap B) \cup (A \cap C)] \\ &= P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap B) - P(A \cap C)\end{aligned}$$

since $(A \cap B) \cap (A \cap C) = A \cap B \cap C$

If A is an event then

$$P(A) = 1 - P(\bar{A})$$

Ex1 A box contains 20 fuses of which 5 are defective. If 3 of them are selected randomly without replacement. What is probability that all 3 fuses are defective

$A \rightarrow 1^{\text{st}}$ defective $P(A) = 5/20$

$B \rightarrow 2^{\text{nd}}$ defective $P(B|A) = 4/19$

$C \rightarrow 3^{\text{rd}}$ defective $P(C|A \cap B) = 3/18$

$$\begin{aligned} P(A \cap B \cap C) &= P(A) \cdot P(B|A) \cdot P(C|A \cap B) \\ &= 5/20 \times 4/19 \times 3/18 = 1/114 \end{aligned}$$

Event Composition method

- 1- Define experiment
- 2- Visualise nature of sample. Identify your thinking
- 3- Write the equation expressing event of interest, A , as to composition of two or more events using union, intersections & compliments
- 4- Apply additive & multiplicative laws of prob. to composition obtained in step 3
- 5- Calculate required prob

Ex:

Three balls are drawn from a box containing
6 red 4 white, 5 blue balls

Find prob. that they are drawn in order RWB
with replacement & without replacement

a) with replacement

$$\begin{array}{l} R_1 \\ W_2 \\ B_3 \end{array} \quad P(R_1 \cap W_2 \cap B_3) = P(R_1) \cdot P(W_2 | R_1) \cdot P(B_3 | R_1 \cap W_2) \\ = P(R_1) \cdot P(W_2) \cdot P(B_3) = \frac{6}{15} \cdot \frac{4}{15} \cdot \frac{5}{15}$$

b) without replacement

$$\begin{array}{l} R_1 \\ W_2 \\ B_3 \end{array} \quad P(R_1 \cap W_2 \cap B_3) = P(R_1) \cdot P(W_2 | R_1) \cdot P(B_3 | R_1 \cap W_2) \\ = \frac{6}{6+4+5} = \frac{4}{5+4+5} + \frac{5}{5+3+1} = \boxed{\frac{6}{15} \cdot \frac{4}{14} \cdot \frac{5}{13}}$$

Ex: Of the items produced daily by a factory, 40%
comes from line I, & 60% from line II

line I has a defect rate of 1/8, line II has 1/10
if an item is chosen randomly, find prob. that it
will not be defective

I $\Rightarrow$ item from line 1, II $\Rightarrow$ item from line 2

D $\Rightarrow$ item is defective

$$P(\bar{D}) = P[\bar{D} \cap (I \cup II)] = P(\bar{D} \cap I) + P(\bar{D} \cap II)$$

$$\begin{aligned} &= P(\bar{D}|I) \cdot P(I) + P(\bar{D}|II) \cdot P(II) \\ &= (0.92)(0.4) + (0.90)(0.6) = 0.908 \end{aligned}$$