

Statistics for Economists II

Week 7: Confidence Intervals for the Mean

Güneş A. Aşık

Department of Economics
Middle East Technical University

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Roadmap for Today

- 1 From Point Estimates to Intervals
- 2 Confidence Intervals: Definition and Interpretation
- 3 Case 1: CI for the Mean — Known Variance
- 4 Case 2: CI for the Mean — Unknown Variance
- 5 Economics Applications
- 6 CI for a Population Proportion
- 7 Sample Size Determination
- 8 Practice Problems
- 9 Key Takeaways

Why a Single Number Is Not Enough

- In previous weeks we found **point estimators**:
 $\hat{\theta}$ is our best single guess for θ

A point estimate is like saying “the temperature tomorrow will be 20°C.” An interval estimate is like saying “between 17°C and 23°C”—it **quantifies our ignorance**.

The Goal of This Week

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 - Researcher A: $\hat{\mu} = 48,000$ TL ($n = 10$)
 - Researcher B: $\hat{\mu} = 48,000$ TL ($n = 10,000$)
- Same estimate, but vastly different **reliability**!

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The Estimation Error Is Random

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Since $\hat{\theta}$ is computed from random data, ε is itself a **random variable**.

We cannot know the exact error for any particular estimate, but we *can* make **probability statements** about its size:

$$\mathbb{P}(|\hat{\theta} - \theta| < b) = \mathbb{P}(\theta - b < \hat{\theta} < \theta + b)$$

Think of it this way: every time we draw a new sample, $\hat{\theta}$ changes and so does the error. We want to find a margin b such that the error is “small” with high probability. This leads directly to the idea of a confidence interval.

Using Standard Error and Chebyshev to Bound Errors

Even **without knowing the distribution** of $\hat{\theta}$, if it is unbiased we can use Chebyshev's inequality. Setting $b = k \sigma_{\hat{\theta}}$:

$$\mathbb{P}\left(|\hat{\theta} - \theta| < k \sigma_{\hat{\theta}}\right) \geq 1 - \frac{1}{k^2}$$

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k	Chebyshev bound	Normal probability
1	$\geq 0\%$	68.3%
2	$\geq 75\%$	95.4%
3	$\geq 88.9\%$	99.7%

Chebyshev gives a **distribution-free guarantee**—it works for any distribution. But it is very conservative. When we know (or can assume) normality, we get **much tighter** intervals. This is exactly what the CLT enables!

Formal Definition of a Confidence Interval

Definition

Let θ be an unknown parameter. If, based on sample data, we find random variables A and B such that

$$\mathbb{P}(A < \theta < B) = 1 - \alpha$$

where $0 < \alpha < 1$, then the interval (a, b) computed from a specific sample is a $100(1 - \alpha)\%$ **confidence interval** for θ .

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Key terminology:

- a and b are the **lower and upper confidence limits**
- $1 - \alpha$ is the **confidence level** (or confidence coefficient)
- α is the **significance level**

A confidence interval satisfies two competing goals: (1) it should **contain** the true θ , and (2) it should be as **narrow** as possible. Higher confidence $\Rightarrow$ wider interval. There is always a trade-off between confidence and precision.

The Correct Interpretation—Read Carefully!

Common Misconception (WRONG!)

“There is a 95% probability that μ lies in the interval (2.5, 3.8).”

Why is this wrong? Once you compute a specific interval from your data, μ is **either in it or not**—there is nothing random anymore. The number μ is a fixed (unknown) constant, not a random variable.

Correct Interpretation

If we repeatedly draw random samples and compute a 95% confidence interval each time, then **95% of those intervals will contain the true μ** . Any single interval either captures μ or it does not—we just do not know which case we are in.

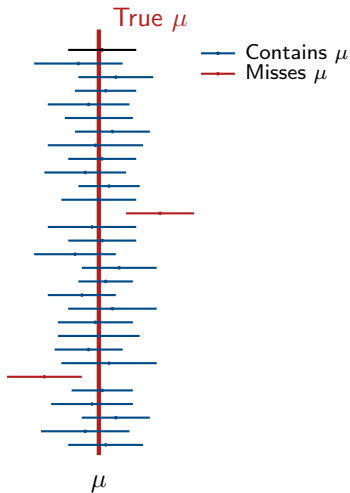
The Correct Interpretation: An Analogy

Analogy: Imagine a factory that produces nets. 95% of the nets successfully catch the fish. Before you cast a particular net, you are 95% “confident” it will work. But once cast, the fish is either caught or not. The 95% refers to the **manufacturing process** (repeated sampling), not to any single net.

Key principle: Confidence is a property of the **procedure**, not of any single computed interval. Once we have a specific interval (a, b) , μ either lies inside it or it does not—there is no probability involved anymore.

Statement	Correct?
“ μ is in this interval with probability 95%”	No
“95% of intervals from this procedure contain μ ”	Yes

The “100 Intervals” Visualization



Each horizontal line is one 95% CI from a different random sample. About 95% of them (blue) contain the true μ ; about 5% (red) miss it entirely. **The randomness is in the interval, not in μ .**

Setting Up the z-Interval

Setup: $X_1, \dots, X_n \stackrel{\text{i.i.d.}}{\sim} N(\mu, \sigma^2)$ with σ^2 **known**, μ unknown.

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We know that

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$$

Starting from $\mathbb{P}(-z_{\alpha/2} < Z < z_{\alpha/2}) = 1 - \alpha$, we rearrange:

$$\begin{aligned} 1 - \alpha &= \mathbb{P}\left(-z_{\alpha/2} < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < z_{\alpha/2}\right) \\ &= \mathbb{P}\left(\bar{X} - z_{\alpha/2} \frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + z_{\alpha/2} \frac{\sigma}{\sqrt{n}}\right) \end{aligned}$$

100(1 - α)% Confidence Interval for μ (Known σ)

$$\bar{X} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

Margin of Error and Width

Margin of Error (ME):

$$ME = z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

Width of the interval:

$$w = 2 \times ME = 2 z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

Three factors control width:

- 1 σ — population variability (fixed)
- 2 n — sample size (we control this)
- 3 $1 - \alpha$ — confidence level (we choose this)

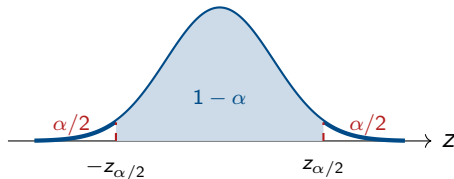
- Larger $\sigma \Rightarrow$ wider CI (more noise)
- Larger $n \Rightarrow$ narrower CI (more data)
- Higher confidence $\Rightarrow$ wider CI (more cautious)

Key insight: Width shrinks with $1/\sqrt{n}$.
To halve the width, you must **quadruple** the sample size!

Table of Critical Values $z_{\alpha/2}$

Confidence Level $1 - \alpha$	α	$z_{\alpha/2}$
90%	0.10	1.645
95%	0.05	1.960
98%	0.02	2.326
99%	0.01	2.576
99.9%	0.001	3.291

How to read: For a 95% CI, we use $z_{0.025} = 1.96$ because $\alpha/2 = 0.025$ in each tail.



Example 4.1: Circuit Resistance (Known σ)

Problem: A sample of 11 circuits from a large normal population has a mean resistance of 2.20 ohms. From past testing, $\sigma = 0.35$ ohms. Find a 95% CI for the true mean resistance.

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Solution:

- $\bar{x} = 2.20$, $\sigma = 0.35$, $n = 11$, $z_{0.025} = 1.96$

Standard error: $\frac{\sigma}{\sqrt{n}} = \frac{0.35}{\sqrt{11}} = \frac{0.35}{3.317} \approx 0.1055$

Margin of error: $ME = 1.96 \times 0.1055 \approx 0.2068$

95% CI: 2.20 ± 0.2068

$(1.993, 2.407) \text{ ohms}$

We are 95% confident that the true mean resistance lies between 1.993 and 2.407 ohms. Although this particular interval may or may not contain μ , 95% of intervals formed this way will.

Example 4.2: Shopping Times and Changing Confidence

Problem: $n = 64$ customers at a supermarket; $\bar{x} = 33$ min, $\sigma^2 = 256 \Rightarrow \sigma = 16$ min. Find a 90% CI for the true average shopping time.

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Solution:

$$ME = z_{0.05} \times \frac{\sigma}{\sqrt{n}} = 1.645 \times \frac{16}{\sqrt{64}} = 1.645 \times 2 = 3.29 \text{ min}$$

$$33 \pm 3.29 = (29.71, 36.29) \text{ minutes}$$

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$$33 \pm 3.29 = (29.71, 36.29) \text{ minutes}$$

What if we wanted 95% instead?

$$ME_{95} = 1.96 \times 2 = 3.92 \Rightarrow (29.08, 36.92)$$

Notice: **higher confidence = wider interval**. Going from 90% to 95% widened the interval from 6.58 to 7.84 minutes. We pay for extra confidence with less precision. In economics, the 95% level is the most common convention, but 90% is also widely used.

Why We Need the t -Distribution

In practice, σ is **almost never known**. When we replace σ with the sample standard deviation s :

$$T = \frac{\bar{X} - \mu}{s/\sqrt{n}}$$

this quantity is **no longer** standard normal.

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this quantity is **no longer** standard normal.

Why? There is extra randomness from estimating σ with s . The resulting distribution has **heavier tails**—extreme values are more likely than under the normal.

Key Point

- Known σ : use $Z \sim N(0, 1)$
- Unknown σ : use $T \sim t_{n-1}$
- As $n \rightarrow \infty$: $t_{n-1} \rightarrow N(0, 1)$

Using z when σ is unknown would produce intervals that are **too narrow**—they would miss μ more often than advertised. The t -distribution automatically **widens** the interval to compensate for the extra uncertainty in s .

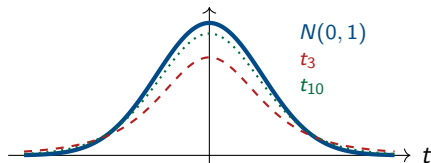
Student's t -Distribution: A Quick Recap

If $X_1, \dots, X_n \stackrel{\text{i.i.d.}}{\sim} N(\mu, \sigma^2)$, then

$$T = \frac{\bar{X} - \mu}{s/\sqrt{n}} \sim t_{n-1}$$

Properties of t_ν :

- Symmetric around 0
- Bell-shaped but **heavier tails** than $N(0, 1)$
- Shape depends on $\nu = n - 1$ (degrees of freedom)
- As $\nu \rightarrow \infty$, $t_\nu \rightarrow N(0, 1)$
- For $\nu > 30$, practically indistinguishable from $N(0, 1)$



The t -Interval for μ (Unknown σ)

100(1 - α)% Confidence Interval for μ (Unknown σ)

$$\bar{x} \pm t_{n-1, \alpha/2} \frac{s}{\sqrt{n}}$$

where $s = \sqrt{\frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2}$ is the sample standard deviation, and $t_{n-1, \alpha/2}$ is the critical value from the t -distribution with $n - 1$ degrees of freedom.

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Comparison with the z -interval:

	Known σ	Unknown σ
Formula	$\bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$	$\bar{x} \pm t_{n-1, \alpha/2} \frac{s}{\sqrt{n}}$
Distribution used	$N(0, 1)$	t_{n-1}
Critical value	$z_{\alpha/2}$	$t_{n-1, \alpha/2} > z_{\alpha/2}$
Width	Narrower	Wider (penalty for estimating σ)

Example 4.3: t -Interval in Action

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Solution:

- $df = n - 1 = 24$
- $t_{24, 0.025} = 2.064$ (from t -table)

$$ME = t_{24, 0.025} \times \frac{s}{\sqrt{n}} = 2.064 \times \frac{8}{\sqrt{25}} = 2.064 \times 1.6 = 3.302$$

$$50 \pm 3.302 = (46.70, 53.30)$$

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Compare: if $\sigma = 8$ were **known**, we would use $z_{0.025} = 1.96$ and get $ME = 1.96 \times 1.6 = 3.136$, yielding a **narrower** interval (46.86, 53.14). The extra 0.166 in margin of error is the “price” of not knowing σ .

Example 4.4: Gasoline Consumption (Small Sample)

Problem: Gasoline consumption (gallons/km) of 6 randomly selected cars from a normal population: 18.6, 18.4, 19.2, 20.8, 19.4, 20.5. Find a 90% CI for the mean.

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Solution:

$$\bar{x} = \frac{116.9}{6} = 19.483, \quad s^2 = \frac{\sum (x_i - \bar{x})^2}{5} = 0.961, \quad s = 0.980$$

- $df = 5, \quad t_{5,0.05} = 2.015$

$$ME = 2.015 \times \frac{0.980}{\sqrt{6}} = 2.015 \times 0.400 = 0.806$$

$19.483 \pm 0.806 = (18.68, 20.29) \text{ gallons/km}$

With only $n = 6$, the t -critical value (2.015) is noticeably larger than the corresponding z -value (1.645). The small sample forces us to use a wider interval to maintain our stated confidence level.

Economics Example: Estimating Average Income

Problem: An economist surveys $n = 100$ randomly selected households in Ankara and finds $\bar{x} = 42,500$ TL monthly income with $s = 12,000$ TL. Construct a 95% CI for the true average household income.

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Problem: An economist surveys $n = 100$ randomly selected households in Ankara and finds $\bar{x} = 42,500$ TL monthly income with $s = 12,000$ TL. Construct a 95% CI for the true average household income.

Solution: Since $n = 100$ is large, $t_{99, 0.025} \approx 1.984 \approx 1.96$.

$$ME = 1.984 \times \frac{12,000}{\sqrt{100}} = 1.984 \times 1,200 = 2,381 \text{ TL}$$

$$42,500 \pm 2,381 = (40,119; 44,881) \text{ TL}$$

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Policy implication: If the government claims average income is 45,000 TL, our data does not contradict this (45,000 is close to the upper limit). But if they claimed 50,000 TL, this falls **outside** our interval—casting serious doubt on their claim. This is the link between confidence intervals and hypothesis testing (coming in Weeks 9–11).

Economics Example: Inflation and Unemployment

Inflation Rate

A basket of $n = 400$ goods shows average monthly price increase $\bar{x} = 2.1\%$ with $s = 3.6\%$. 95% CI:

$$2.1 \pm 1.96 \times \frac{3.6}{\sqrt{400}}$$

$$= 2.1 \pm 0.353$$

$$(1.75\%, 2.45\%)$$

Unemployment Rate

Survey of $n = 2,000$ workers finds 160 unemployed: $\hat{p} = 0.08$.

95% CI for p :

$$0.08 \pm 1.96 \sqrt{\frac{0.08 \times 0.92}{2000}}$$

$$= 0.08 \pm 0.012$$

$$(6.8\%, 9.2\%)$$

Central banks and statistical offices report economic indicators with confidence intervals (or margins of error) precisely because these numbers come from **samples**, not censuses. Policy decisions must account for this **statistical uncertainty**.

Confidence Interval for a Proportion p

Setup: $X_1, \dots, X_n \stackrel{\text{i.i.d.}}{\sim} \text{Bernoulli}(p)$. The sample proportion is $\hat{p} = \bar{X}$.

By the CLT (for large n):

$$\frac{\hat{p} - p}{\sqrt{p(1-p)/n}} \stackrel{\text{approx}}{\sim} N(0, 1)$$

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By the CLT (for large n):

$$\frac{\hat{p} - p}{\sqrt{p(1-p)/n}} \stackrel{\text{approx}}{\sim} N(0, 1)$$

Replacing the unknown p in the standard error with $\hat{p}$:

100(1 - α)% CI for Population Proportion

$$\hat{p} \pm z_{\alpha/2} \sqrt{\frac{\hat{p}(1 - \hat{p})}{n}}$$

Condition: $n\hat{p}(1 - \hat{p}) \geq 5$ (for the normal approximation to be adequate)

This is the formula behind every election poll margin of error you see in the news. When a poll says “Candidate A leads with 52% $\pm$ 3%,” they are reporting a confidence interval for p .

Example: Election Poll

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Solution:

- $\hat{p} = 0.78$, $n = 1,000$, $z_{0.025} = 1.96$
- Check: $n\hat{p}(1 - \hat{p}) = 1000 \times 0.78 \times 0.22 = 171.6 \geq 5 \checkmark$

$$SE = \sqrt{\frac{0.78 \times 0.22}{1000}} = \sqrt{0.0001716} = 0.0131$$

$$ME = 1.96 \times 0.0131 = 0.0257$$

$$0.78 \pm 0.026 = (75.4\%, 80.6\%)$$

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Why Do Polls Report $\pm 3.1\%$?

Polls often use the **conservative** estimate $p = 0.5$ (which maximizes $p(1 - p)$), giving $ME = 1.96\sqrt{0.25/1000} = 3.1\%$. This guarantees the stated margin regardless of the true p .

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For the mean (known σ):

$$ME_0 = z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \quad \Rightarrow \quad n = \left\lceil \left(\frac{z_{\alpha/2} \sigma}{ME_0} \right)^2 \right\rceil$$

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For a proportion:

$$ME_0 = z_{\alpha/2} \sqrt{\frac{p(1-p)}{n}} \implies n = \left\lceil \frac{z_{\alpha/2}^2 p(1-p)}{ME_0^2} \right\rceil$$

If p is unknown, use the conservative $p = 0.5$.

Sample size determination is crucial in **research design**. Collecting too few observations wastes the effort (results will be imprecise). Collecting too many wastes resources. This formula tells you **exactly** how many observations you need to achieve a target precision at a given confidence level.

Example: Sample Size for a Survey

Problem: An economist wants to estimate the average monthly expenditure of Turkish households with a margin of error of 500 TL at 95% confidence. Past studies suggest $\sigma \approx 4,000$ TL. How many households should be surveyed?

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Solution:

$$n = \left(\frac{z_{\alpha/2} \sigma}{ME_0} \right)^2 = \left(\frac{1.96 \times 4000}{500} \right)^2 = \left(\frac{7840}{500} \right)^2 = (15.68)^2 = 245.9$$

$$n = 246 \text{ households}$$

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$$n = 246 \text{ households}$$

What if we wanted $ME = 250$ TL instead?

$$n = \left(\frac{1.96 \times 4000}{250} \right)^2 = (31.36)^2 = 983.4 \Rightarrow n = 984$$

Halving the margin of error **quadrupled** the required sample size (from 246 to 984). Precision is expensive! This is why national statistical surveys require thousands of respondents but still report margins of error.

Practice Problem 1: Average Wage Estimation

Problem: A labor economist surveys $n = 49$ randomly selected workers in the manufacturing sector. She finds $\bar{x} = 8,200$ TL monthly and the population standard deviation is known to be $\sigma = 1,400$ TL.

- (a) Find a 95% confidence interval for the true mean wage.
- (b) Find a 99% confidence interval.
- (c) Compare the widths. What happens to precision as confidence increases?

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- (c) Compare the widths. What happens to precision as confidence increases?

Solution (a): $SE = \frac{1400}{\sqrt{49}} = 200$

$$ME_{95} = 1.96 \times 200 = 392$$

$95\% \text{ CI : } 8,200 \pm 392 = (7,808; 8,592) \text{ TL}$

Practice Problem 1 (continued)

Solution (b):

$$ME_{99} = 2.576 \times 200 = 515.2$$

$$99\% \text{ CI : } 8,200 \pm 515 = (7,685; 8,715) \text{ TL}$$

Practice Problem 1 (continued)

Solution (b):

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$$99\% \text{ CI : } 8,200 \pm 515 = (7,685; 8,715) \text{ TL}$$

Solution (c):

- Width at 95%: $2 \times 392 = 784$ TL
- Width at 99%: $2 \times 515 = 1,030$ TL

The 99% CI is 31% wider than the 95% CI. This illustrates the fundamental trade-off: **greater confidence demands less precision** (wider intervals), and vice versa. In applied economics, 95% is the standard because it balances confidence and precision.

Practice Problem 2: Unknown Variance

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- $df = 15$, $t_{15, 0.025} = 2.131$

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Note that $t_{15, 0.025} = 2.131 > z_{0.025} = 1.96$. The t -interval is wider, penalizing us for not knowing σ . With $n = 16$, this penalty is noticeable. Had $n = 100$, we would have $t_{99, 0.025} \approx 1.984$ —barely different from 1.96.

Practice Problem 3: Sample Size for a Proportion

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Solution: Since p is unknown, use the conservative $p = 0.5$:

$$\begin{aligned} n &= \frac{z_{\alpha/2}^2 \times p(1-p)}{ME_0^2} = \frac{(1.96)^2 \times 0.25}{(0.02)^2} = \frac{3.8416 \times 0.25}{0.0004} \\ &= \frac{0.9604}{0.0004} = 2,401 \end{aligned}$$

$$n = 2,401 \text{ voters}$$

This explains why major opinion polls survey 1,000–2,500 people. With $n = 2,401$, the margin of error is $\pm 2\%$ —the standard for high-quality election polls. Reducing the margin to $\pm 1\%$ would require about 9,604 respondents!

Practice Problem 4: Interpreting a CI

Problem: A researcher computes a 95% CI for the average annual inflation rate and obtains (3.2%, 5.8%). Which of the following statements are correct?

- (a) There is a 95% probability that μ is between 3.2% and 5.8%.
- (b) 95% of all inflation rates in the population fall between 3.2% and 5.8%.
- (c) If we repeated the sampling process many times, about 95% of the resulting CIs would contain μ .
- (d) We are 95% confident that the sample mean $\bar{x}$ is in this interval.

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Answers

- (a) **WRONG.** μ is fixed; the interval either contains it or not.
- (b) **WRONG.** The CI is about the *mean*, not individual observations.
- (c) **CORRECT.** This is the frequentist interpretation.
- (d) **WRONG.** $\bar{x}$ is always in the CI by construction (it is the center!).

Practice Problem 5: From CI Width to Sample Size

Problem: A development economist is studying per-capita income in a rural region. A pilot study gives $s = 3,000$ TL. She wants a 95% CI with **total width** no more than 1,000 TL. What is the minimum sample size?

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Solution:

Width $= 2 \times ME$, so $ME = 500$ TL.

Using s as an approximation for σ :

$$n = \left(\frac{z_{\alpha/2} \times \sigma}{ME} \right)^2 = \left(\frac{1.96 \times 3000}{500} \right)^2 = (11.76)^2 = 138.3$$

$$n = 139 \text{ households}$$

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In practice, the pilot-study estimate of σ introduces some uncertainty into the required n . Researchers often add 10–20% to the computed n as a safety buffer, or they use sequential sampling designs that adjust n as data arrives.

Key Takeaways from Week 7

- 1 A **point estimate** alone is incomplete—we need to quantify uncertainty through **confidence intervals**
- 2 A $100(1 - \alpha)\%$ CI means: if we repeat the procedure many times, $100(1 - \alpha)\%$ of the intervals will contain θ
- 3 **Known σ** : use $\bar{x} \pm z_{\alpha/2} \sigma / \sqrt{n}$ (z-interval)
- 4 **Unknown σ** : use $\bar{x} \pm t_{n-1, \alpha/2} s / \sqrt{n}$ (t-interval, wider)
- 5 **Proportion**: use $\hat{p} \pm z_{\alpha/2} \sqrt{\hat{p}(1 - \hat{p})/n}$
- 6 **Width** depends on σ (or s), n , and the confidence level—there is always a **trade-off between confidence and precision**
- 7 To halve the margin of error, **quadruple** the sample size

Coming Up Next Week

Confidence intervals for the difference of two means, paired samples, and confidence intervals for population variance (χ^2 -based intervals).

Further Reading

- Wackerly, Mendenhall & Scheaffer, Ch. 8: Estimation
- Newbold, Carlson & Thorne, Ch. 8: Confidence Interval Estimation
- Hogg, Tanis & Zimmerman, Ch. 6: Point Estimation and Confidence Intervals
- Devore, Ch. 7: Statistical Intervals Based on a Single Sample