

2025-2026
ECON 205

Problem Set 2 - Answers

1) Suppose that $P(A_1 \cap A_2 \cap A_3) > 0$ and show that

$$P(A_1 \cap A_2 \cap A_3 \cap A_4) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2)P(A_4|A_1 \cap A_2 \cap A_3)$$

SOLUTION

Let's say $B = A_1 \cap A_2 \cap A_3$

$$P(A_1 \cap A_2 \cap A_3 \cap A_4) = P(B \cap A_4) \quad (1)$$

$$P(A_1 \cap A_2 \cap A_3 \cap A_4) = P(B)P(A_4|B) \quad (2)$$

$$P(A_1 \cap A_2 \cap A_3 \cap A_4) = P(A_1 \cap A_2 \cap A_3)P(A_4|A_1 \cap A_2 \cap A_3) \quad (3)$$

Let's say $C = A_1 \cap A_2$

$$P(A_1 \cap A_2 \cap A_3) = P(C \cap A_3) \quad (4)$$

$$P(A_1 \cap A_2 \cap A_3) = P(C)P(A_3|C) \quad (5)$$

$$P(A_1 \cap A_2 \cap A_3) = P(A_1 \cap A_2)P(A_3|A_1 \cap A_2) \quad (6)$$

And we know that $P(A_1 \cap A_2) = P(A_1)P(A_2|A_1)$

$$P(A_1 \cap A_2 \cap A_3 \cap A_4) = P(A_1 \cap A_2 \cap A_3)P(A_4|A_1 \cap A_2 \cap A_3) \quad (7)$$

$$P(A_1 \cap A_2 \cap A_3 \cap A_4) = P(A_1 \cap A_2)P(A_3|A_1 \cap A_2)P(A_4|A_1 \cap A_2 \cap A_3) \quad (8)$$

$$\begin{aligned} P(A_1 \cap A_2 \cap A_3 \cap A_4) \\ = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2)P(A_4|A_1 \cap A_2 \cap A_3) \end{aligned} \quad (9)$$

2) A, B, and C are mutually independent events such that

$$P(A) = 0.4, P(B) = 0.25, P(C) = 0.35$$

$$P[(A' \cap B') \cup C] = ?$$

SOLUTION

$$P[(A' \cap B') \cup C] = P((A \cup B)' \cup C) = P((A \cup B \cup C)' \cup C) \quad (1)$$

$$P[(A' \cap B') \cup C] = P((A \cup B \cup C)') + P(C) = 1 - P(A \cup B \cup C) + P(C) \quad (2)$$

By the mutual independence of the events A, B, C,

$$P[(A' \cap B') \cup C] = 1 - P(A) - P(B) + P(A)P(B) + P(B)P(C) \quad (3)$$

$$+ P(A)P(C) - P(A)P(B)P(C)$$

$$P[(A' \cap B') \cup C] = 1 - 0.4 - 0.25 + (0.4)(0.25) + (0.25)(0.35) + (0.4)(0.35) - (0.4)(0.25)(0.35) \quad (4)$$

$$P[(A' \cap B') \cup C] = 0.6425 \quad (5)$$

3) Many public schools are implementing a “no-pass, no-play” rule for athletes. Under this system, a student who fails a course is disqualified from participating in extracurricular activities during the next grading period. Suppose that the probability is .15 that an athlete who has not previously been disqualified will be disqualified next term. For athletes who have been previously disqualified, the probability of disqualification next term is .5. If 30% of the athletes have been disqualified in previous terms, what is the probability that a randomly selected athlete will be disqualified during the next grading period?

SOLUTION

Let A = athlete disqualified previously

B = athlete disqualified next term

Then, we know $P(B | \bar{A}) = .15, P(B | A) = .5, P(A) = .3$. To find $P(B)$, use the law of total probability: $P(B) = .3(.5) + .7(.15) = 0.255$.

- 4) Consider independent trials consisting of rolling a pair of fair dice, over and over. What is the probability that a sum of 6 appears before a sum of 9?

SOLUTION

Let A be the event that we see a sum of 6 before a sum of 9. We want to compute $P(A)$. The easiest way to solve this is by conditioning on the outcome of the first roll: Let B be the event that the first roll is a 6; let C be the event that the first roll is a 9; let D be the event that the first roll is a sum other than 6 or 9. Then B, C, D partition the sample space (a set of possible cases). Then, by "the law of total probability":

$$P(A) = P(A | B)P(B) + P(A | C)P(C) + P(A | D)P(D)$$

Now, we know that $P(B) = 5/36$, $P(C) = 4/36$, and $P(D) = 27/36$. Also, given that the first roll is a 6, the probability we get a 6 before a 9 is 1: $P(A | B) = 1$. Similarly, given that the first roll is a 9, the probability we get a 6 before a 9 is 0: $P(A | C) = 0$. Now, if the first roll is neither a 6 nor a 9, we can think of the process starting all over again: the chance we get a 6 before 9 is just like it was ($P(A)$) before we started rolling: $P(A | D) = P(A)$.

Thus,

$$\begin{aligned} P(A) &= P(A | B)P(B) + P(A | C)P(C) + P(A | D)P(D) \\ P(A) &= 1 * \frac{5}{36} + 0 * \frac{4}{36} + P(A) * \frac{27}{36} \end{aligned}$$

which gives us an equation in one unknown ($P(A)$)

$$\begin{aligned} \frac{9}{36} * P(A) &= \frac{5}{36} \\ P(A) &= \frac{5}{9} \end{aligned}$$

Then, we find $P(A)$ as $\frac{5}{9}$.

2nd solution (as an alternative);

A: Observing a sum of 6

B: Observing a sum other than 6 and other than 9

$$P(A) = \frac{5}{36} \text{ and } P(B) = \frac{27}{36}$$

X: Observing a sum of 6 before a 9

A sum of 6 is observed before a 9 if the following outcomes are observed:

A, BA, BBA, BBBA,

$$\begin{aligned}
P(X) &= P(A) + P(BA) + P(BBA) + P(BBBA) + \dots \\
P(X) &= \frac{5}{36} + \frac{27}{36} \frac{5}{36} + \frac{27}{36} \frac{27}{36} \frac{5}{36} + \frac{27}{36} \frac{27}{36} \frac{27}{36} \frac{5}{36} + \dots \\
P(X) &= \frac{5}{36} \left[1 + \frac{27}{36} + \left(\frac{27}{36}\right)^2 + \left(\frac{27}{36}\right)^3 + \dots \right] \\
P(X) &= \frac{5}{36} \left(\frac{1}{1-r} \right) \text{ where } r = \frac{27}{36} \\
P(X) &= \frac{5}{9}
\end{aligned}$$

- 5) Let Y be a discrete random variable with mean μ and variance σ^2 . If a and b are constants, prove that $V(aY+b)=a^2V(Y)=a^2\sigma^2$

SOLUTION

$$V(aY+b)=E(aY+b-a\mu-b)^2=E(aY-a\mu)^2=a^2E(Y-\mu)^2=a^2\sigma^2$$

- 6) Let Y be a random variable with p(y) and p(0)=0.2, p(1)=0.3, p(2)=0.4, p(4)=0.1. Find E(Y), E(Y²) and V(Y).

SOLUTION

$$E(Y)=0*0.2+1*0.3+2*0.4+4*0.1=1.5$$

$$E(Y^2)=0*0.2+1*0.3+4*0.4+16*0.1=3.5$$

$$V(Y)=E(Y^2)-(E(Y))^2=3.5-(1.5)^2=1.25$$

- 7) A contractor is interested in the total cost of a Project on which she intends to bid. She estimates that materials will cost \$25,000 and that her labor will be \$900 per day. If the project takes X days to complete, the total labor cost will be 900X, and the total cost of project (in dollars) will be as follows:

$$C = 25,000 + 900X$$

Using her experience, the contractor forms probabilities of likely completion times for the project.

- Find the mean and variance for completion time X.
- Find $E(1/X)$ and $E(\frac{1}{X} - 1)$.
- Find the mean and variance for total cost C.

Probability Distribution for Completion Times

COMPLETION TIMES X (DAYS)	10	11	12	13	14
Probability	0.1	0.3	0.3	0.2	0.1

SOLUTION

a)

$$\begin{aligned}\mu = E(X) &= \sum_x x \cdot p(x) = 10 * 0.1 + 11 * 0.3 + 12 * 0.3 + 13 * 0.2 + 14 * 0.1 = 11.9 \\ \sigma^2 &= [E(x - \mu)^2] \\ &= \sum_x (x - \mu)^2 * p(x) = (10 - 11.9)^2 * 0.1 + (11 - 11.9)^2 * 0.3 + (12 - 11.9)^2 * 0.3 + (13 - 11.9)^2 * 0.2 + (14 - 11.9)^2 * 0.1 = 1.29\end{aligned}$$

b)

$$\begin{aligned}E\left(\frac{1}{X}\right) &= \sum_x \frac{1}{x} p(x) = \frac{1}{10} * 0.1 + \frac{1}{11} * 0.3 + \frac{1}{12} * 0.3 + \frac{1}{13} * 0.2 + \frac{1}{14} * 0.1 = 0.084 \\ E(1/X - 1) &= 0.084 - 1 = -0.916\end{aligned}$$

c)

$$\begin{aligned}\mu_c &= E(25000 + 900X) = 25000 + 900E(X) = 25000 + 900 * 11.9 = 35710 \\ \sigma_c^2 &= V(25000 + 900X) = 900^2 V(X) = 810000 * 1.29 = 1044900\end{aligned}$$

8) Two gamblers bet \$1 each on the successive tosses of a coin. Each has a bank of \$6. What is the probability that they

- a) break even after six tosses of the coin?
- b) one player -say, Jones- wins all the money on the tenth toss of the coin?

SOLUTION

a. The gamblers break even if each win three times and lose three times. Considering the possible sequences of "wins" and "losses", there are $\binom{6}{3} = 20$ possible orderings. Since each has probability $\left(\frac{1}{2}\right)^6$, the probability of breaking even is $20 \left(\frac{1}{2}\right)^6 = 0.3125$.

b. In order for this event to occur, the gambler Jones must have \$11 at trial 9 and must win on trial 10. So, in the nine remaining trials, seven "wins" and two "losses" must be placed. So, there are $\binom{9}{2} = 36$ ways to do this. However, this includes cases where Jones would win before the 10th trial. Now, Jones can only win the game on an even trial (since he must gain \$6). Included in the 36 possibilities, there are three ways Jones could win on trial 6: *WWWWWWL*, *WWWWWWLL*, *WWWWWWLWL*, and there are six ways Jones could win on trial 8: *LWWWWWWL*, *WLWWWWWWL*, *WWLWWWWWWL*,

WWWLWWWWL, WWWLWWWWL, WWWWWLWWL. So, these nine cases must be removed from the 36. So, the probability is $27 \left(\frac{1}{2}\right)^{10}$.

- 9) Airlines sometimes overbook flights. Suppose that for a plane with 50 seats, 55 passengers have tickets. Define the random variables Y as the number of ticketed passengers who show up for the flight. The probability mass function of Y appears in the accompanying table.

y	45	46	47	48	49	50	51	52	53	54	55
p(y)	.05	.10	.12	.14	.25	.17	.06	.05	.03	.02	.01

- What is the probability that the flight will accommodate all ticketed passengers who show up?
- What is the probability that not all ticketed passengers who show up can be accommodated?
- If you are the first person on the standby list consisting of people without tickets (which means you will be the first one to get on the plane if there are any seats available after all ticketed passengers have been accommodated), what is the probability that you will be able to take the flight? What is this probability if you are the third person on the standby list?

SOLUTION

- In order for the flight to accommodate all the ticketed passengers who show up, no more than 50 can show up. We need $Y \leq 50$.
 $P(Y \leq 50) = .05 + .10 + .12 + .14 + .25 + .17 = .83$
- Using the information in a. above, $P(Y > 50) = 1 - P(Y \leq 50) = 1 - .83 = .17$
- For you to get on the flight, at most 49 of the ticketed passengers must show up. $P(Y \leq 49) = .05 + .10 + .12 + .14 + .25 = .66$. For the 3rd person on the standby list, at most 47 of the ticketed passengers must show up. $P(Y \leq 47) = .05 + .10 + .12 = .27$

- 10) Suppose that Y possesses the probability mass function:

$$f(y) = \begin{cases} cy, & y = 2, 4, 6, 8, 10 \\ 0, & \text{otherwise} \end{cases}$$

- a) Find the value of c that makes $f(y)$ a probability function.
 b) Find $P(Y \geq 3)$

SOLUTION

a)

$$\sum P(Y = y) = 1$$

$$\sum P(Y = y) = \sum_y P(Y = y) = \sum_y cy = 2c + 4c + 6c + 8c + 10c = 30c = 1$$

$$c = 1/30$$

b)

$$P(Y \geq 3) = P(Y = 4) + P(Y = 6) + P(Y = 8) + P(Y = 10) = \frac{1}{30}(4 + 6 + 8 + 10) = \frac{14}{15}$$

11) Let $m(t) = \left(\frac{1}{6}\right)e^t + \left(\frac{2}{6}\right)e^{2t} + \left(\frac{3}{6}\right)e^{3t}$ be the moment-generating function of a random variable Y . Find the following:

- a) $E(Y)$
 b) $V(Y)$

a) First derivative of the moment generating function:

$$\left(\frac{1}{6}\right)e^t + \left(\frac{2 * 2}{6}\right)e^{2t} + \left(\frac{3 * 3}{6}\right)e^{3t}$$

Let $t=0$.

$$(1/6) + (4/6) + (9/6) = (14/6) = (7/3) = E(Y)$$

b) Second derivative of the moment generating function:

$$\left(\frac{1}{6}\right)e^t + \left(\frac{2 * 2 * 2}{6}\right)e^{2t} + \left(\frac{3 * 3 * 3}{6}\right)e^{3t}$$

$$\text{Let } t=0. E(Y^2) = (1/6) + (8/6) + (27/6) = 6$$

$$V(Y) = E(Y^2) - (E(Y))^2 = 6 - (7/3)^2 = 5/9$$