

Econ206 Statistics

Midterm Exam Solutions

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1. Let $X \sim \text{Exp}(\lambda)$ with probability density function:

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0, \lambda > 0$$

Find the mean and variance using the moment generating function technique.

solution:

Moment Generating Function (MGF) is defined as $M_X(t) = E[e^{tX}]$.

For $X \sim \text{Exp}(\lambda)$ with PDF $f(x) = \lambda e^{-\lambda x}$, $x \geq 0$:

$$\begin{aligned} M_X(t) &= \int_0^{\infty} e^{tx} \lambda e^{-\lambda x} dx \\ &= \lambda \int_0^{\infty} e^{-(\lambda-t)x} dx \end{aligned}$$

This integral converges only if $\lambda - t > 0$, i.e. $t < \lambda$.

$$\begin{aligned} M_X(t) &= \lambda \left[\frac{e^{-(\lambda-t)x}}{-(\lambda-t)} \right]_0^{\infty} \\ &= \lambda \left[0 - \frac{1}{-(\lambda-t)} \right] \\ &= \frac{\lambda}{\lambda-t} \end{aligned}$$

So:

$$M_X(t) = \frac{\lambda}{\lambda-t}, \quad t < \lambda$$

We know $E[X] = M'_X(0)$.

First derivative:

$$\begin{aligned} M'_X(t) &= \frac{d}{dt} [\lambda(\lambda-t)^{-1}] \\ &= \lambda \cdot (-1)(\lambda-t)^{-2} \cdot (-1) \\ &= \lambda(\lambda-t)^{-2} \end{aligned}$$

So:

$$M'_X(t) = \frac{\lambda}{(\lambda-t)^2}$$

Evaluate at $t = 0$:

$$E[X] = \frac{\lambda}{(\lambda-0)^2} = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$$

Second Moment from MGF:

$$\begin{aligned}
M_X''(t) &= \frac{d}{dt} [\lambda(\lambda - t)^{-2}] \\
&= \lambda \cdot (-2)(\lambda - t)^{-3} \cdot (-1) \\
&= \frac{2\lambda}{(\lambda - t)^3}
\end{aligned}$$

Evaluate at $t = 0$:

$$E[X^2] = \frac{2\lambda}{(\lambda - 0)^3} = \frac{2\lambda}{\lambda^3} = \frac{2}{\lambda^2}$$

Variance:

$$\begin{aligned}
\text{Var}(X) &= E[X^2] - (E[X])^2 \\
&= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 \\
&= \frac{2-1}{\lambda^2} = \frac{1}{\lambda^2}
\end{aligned}$$

2. Let $X_1, X_2, \dots, X_n$ be a random sample from an exponential distribution with variance θ^2 . The pdf is given by:

$$f(x) = \frac{1}{\lambda} e^{-x/\lambda}, \quad x > 0$$

Find the variance of the maximum likelihood estimator of θ . (10 points)

Solution:

First, note that for an exponential distribution with pdf $f(x) = \frac{1}{\mu} e^{-x/\mu}$, we have:

$$\text{Var}(X) = \mu^2 = \theta^2 \quad \Rightarrow \quad \mu = \theta$$

The likelihood function is:

$$L(\theta) = \prod_{i=1}^n \frac{1}{\theta} e^{-x_i/\theta} = \theta^{-n} e^{-\frac{1}{\theta} \sum_{i=1}^n x_i}$$

The log-likelihood is:

$$\ell(\theta) = -n \ln \theta - \frac{1}{\theta} \sum_{i=1}^n x_i$$

Differentiating with respect to θ :

$$\frac{\partial \ell}{\partial \theta} = -\frac{n}{\theta} + \frac{1}{\theta^2} \sum_{i=1}^n x_i$$

Setting the derivative to zero:

$$-\frac{n}{\theta} + \frac{1}{\theta^2} \sum_{i=1}^n x_i = 0 \quad \Rightarrow \quad \hat{\theta} = \frac{1}{n} \sum_{i=1}^n x_i = \bar{X}$$

Since $\hat{\theta} = \bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$, and $X_i \sim \text{Exp}(\theta)$ with $E[X_i] = \theta$ and $\text{Var}(X_i) = \theta^2$, we have:

$$\begin{aligned}
\text{Var}(\hat{\theta}) &= \text{Var}\left(\frac{1}{n} \sum_{i=1}^n X_i\right) \\
&= \frac{1}{n^2} \text{Var}\left(\sum_{i=1}^n X_i\right) \\
&= \frac{1}{n^2} \sum_{i=1}^n \text{Var}(X_i) \quad (\text{since the } X_i \text{ are independent}) \\
&= \frac{1}{n^2} \cdot n \cdot \theta^2 \\
&= \frac{\theta^2}{n}
\end{aligned}$$

Therefore, the variance of the maximum likelihood estimator of θ is:

$$\text{Var}(\hat{\theta}) = \frac{\theta^2}{n}$$

3. Let $Y_1, Y_2, \dots, Y_n$ be a random sample of size n from a normal population with mean μ and variance σ^2 . Assuming that $n = 2k$ for some integer k , one possible estimator for σ^2 is given by

$$\hat{\sigma}^2 = \frac{1}{2k} \sum_{i=1}^k (Y_{2i} - Y_{2i-1})^2.$$

Show that $\hat{\sigma}^2$ is an unbiased estimator for σ^2 .

Solution We will prove unbiasedness by expanding the square and using moments.

For any fixed i , consider:

$$(Y_{2i} - Y_{2i-1})^2 = Y_{2i}^2 + Y_{2i-1}^2 - 2Y_{2i}Y_{2i-1}.$$

Taking expectations:

$$\begin{aligned}
E[Y_{2i}^2] &= \mu^2 + \sigma^2, \\
E[Y_{2i-1}^2] &= \mu^2 + \sigma^2, \\
E[Y_{2i}Y_{2i-1}] &= E[Y_{2i}]E[Y_{2i-1}] = \mu \cdot \mu = \mu^2 \quad (\text{by independence}).
\end{aligned}$$

Therefore:

$$\begin{aligned}
E[(Y_{2i} - Y_{2i-1})^2] &= (\mu^2 + \sigma^2) + (\mu^2 + \sigma^2) - 2\mu^2 \\
&= 2\mu^2 + 2\sigma^2 - 2\mu^2 \\
&= 2\sigma^2.
\end{aligned}$$

Now compute the expectation of the estimator:

$$\begin{aligned}
E[\hat{\sigma}^2] &= E\left[\frac{1}{2k} \sum_{i=1}^k (Y_{2i} - Y_{2i-1})^2\right] \\
&= \frac{1}{2k} \sum_{i=1}^k E[(Y_{2i} - Y_{2i-1})^2] \\
&= \frac{1}{2k} \sum_{i=1}^k 2\sigma^2 \\
&= \frac{1}{2k} \cdot k \cdot 2\sigma^2 \\
&= \sigma^2.
\end{aligned}$$

Thus, $\hat{\sigma}^2$ is an unbiased estimator for σ^2 .

4. Let $Y_1, Y_2, \dots, Y_n$ denote a random sample from the probability density function

$$f(y) = \begin{cases} \theta y^{\theta-1}, & 0 < y < 1, \\ 0, & \text{elsewhere,} \end{cases}$$

where $\theta > 0$.

(a) Find the method of moments estimator for θ .

(b) Show that it is consistent.

Solution

(a) Method of Moments Estimator

First, compute the first population moment:

$$E[Y] = \int_0^1 y \cdot \theta y^{\theta-1} dy = \theta \int_0^1 y^\theta dy = \theta \cdot \frac{1}{\theta+1} = \frac{\theta}{\theta+1}.$$

Let $\bar{Y} = \frac{1}{n} \sum_{i=1}^n Y_i$ be the first sample moment. Equating population and sample moments:

$$\frac{\theta}{\theta+1} = \bar{Y}.$$

Solving for θ :

$$\begin{aligned} \theta &= \bar{Y}(\theta+1) \\ \theta &= \bar{Y}\theta + \bar{Y} \\ \theta - \bar{Y}\theta &= \bar{Y} \\ \theta(1 - \bar{Y}) &= \bar{Y} \\ \theta &= \frac{\bar{Y}}{1 - \bar{Y}}. \end{aligned}$$

Thus, the method of moments estimator is:

$$\hat{\theta} = \frac{\bar{Y}}{1 - \bar{Y}}$$

(b) Consistency:

We know from the Weak Law of Large Numbers that:

$$\bar{Y} \xrightarrow{P} E[Y] = \frac{\theta}{\theta+1}.$$

Define the function $g(t) = \frac{t}{1-t}$, which is continuous for $t \in (0, 1)$. Since $0 < Y_i < 1$, we have $0 < \bar{Y} < 1$ (almost surely for large n , $\bar{Y}$ is bounded away from 1 because $\frac{\theta}{\theta+1} < 1$).

By the Continuous Mapping Theorem:

$$\hat{\theta} = g(\bar{Y}) \xrightarrow{P} g\left(\frac{\theta}{\theta+1}\right).$$

Compute:

$$g\left(\frac{\theta}{\theta+1}\right) = \frac{\frac{\theta}{\theta+1}}{1 - \frac{\theta}{\theta+1}} = \frac{\frac{\theta}{\theta+1}}{\frac{1}{\theta+1}} = \theta.$$

Therefore:

$$\hat{\theta} \xrightarrow{P} \theta,$$

which proves that $\hat{\theta}$ is a consistent estimator of θ .

5. Suppose, for a sample selected from a normally distributed population, $\bar{x} = 65.5$. The population standard deviation is known to be equal to 8.6.
- Construct a 95 percent confidence interval for μ assuming $n = 16$. (10 points)
 - Construct a 90 percent confidence interval for μ assuming $n = 16$. Is the width of the interval of the 90 percent confidence interval smaller or larger than the width of the 95 confidence interval in the previous part. Explain why. (10 points)
 - Construct a 95 percent confidence interval for μ assuming $n = 25$. Is it larger or smaller than the one you found in part b? Explain why. (10 points)

Solution:

(a) 95% Confidence Interval for μ with $n = 16$

Given:

$$\bar{x} = 65.5, \quad \sigma = 8.6, \quad n = 16$$

For a 95% confidence interval, $z_{0.025} = 1.96$.

$$\text{Margin of error} = z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} = 1.96 \cdot \frac{8.6}{\sqrt{16}} = 1.96 \cdot \frac{8.6}{4} = 1.96 \cdot 2.15 = 4.214$$

$$\text{CI} = \bar{x} \pm \text{Margin of error} = 65.5 \pm 4.214$$

$$\text{CI} = (61.286, 69.714)$$

(b) 90% Confidence Interval for μ with $n = 16$

For a 90% confidence interval, $z_{0.05} = 1.645$.

$$\text{Margin of error} = 1.645 \cdot \frac{8.6}{\sqrt{16}} = 1.645 \cdot 2.15 = 3.53675$$

$$\text{CI} = 65.5 \pm 3.537 = (61.963, 69.037)$$

Comparison of widths:

- 95% CI width: $69.714 - 61.286 = 8.428$
- 90% CI width: $69.037 - 61.963 = 7.074$

The 90% confidence interval is **narrower** than the 95% confidence interval.

Reason: A lower confidence level means we require less certainty, so we can have a smaller margin of error, resulting in a narrower interval.

(c) 95% Confidence Interval for μ with $n = 25$

$$n = 25, \quad z_{0.025} = 1.96$$

$$\text{Margin of error} = 1.96 \cdot \frac{8.6}{\sqrt{25}} = 1.96 \cdot \frac{8.6}{5} = 1.96 \cdot 1.72 = 3.3712$$

$$\text{CI} = 65.5 \pm 3.371 = (62.129, 68.871)$$

Comparison with part (b):

- Part (b) 90% CI width ($n = 16$): 7.074
- Part (c) 95% CI width ($n = 25$): $68.871 - 62.129 = 6.742$

The 95% CI with $n = 25$ is **narrower** than the 90% CI with $n = 16$.

Reason: The larger sample size ($n = 25$ vs $n = 16$) reduces the standard error $\sigma/\sqrt{n}$, which shrinks the margin of error more than the increase from 90% to 95% confidence level widens it.