

ECON 205

The Answers of the Review Problem Set I

1)

A survey classified a large number of adults according to whether they were diagnosed as needing eyeglasses to correct their reading vision and whether they use eyeglasses when reading. The proportions falling into the four resulting categories are given in the following table:

Needs glasses	Uses Eyeglasses for Reading	
	Yes	No
Yes	.44	.14
No	.02	.40

If a single adult is selected from the large group, find the probabilities of the events defined below. The adult

- a needs glasses.
- b needs glasses but does not use them.
- c uses glasses whether the glasses are needed or not.

SOLUTION

a. $P(\text{needs glasses}) = .44 + .14 = 0.58$

b. $P(\text{needs glasses but doesn't use them}) = .14$

c. $P(\text{uses glasses}) = .44 + .02 = .46$

2)

A local fraternity is conducting a raffle where 50 tickets are to be sold—one per customer. There are three prizes to be awarded. If the four organizers of the raffle each buy one ticket, what is the probability that the four organizers win

- a all of the prizes?
- b exactly two of the prizes?
- c exactly one of the prizes?
- d none of the prizes?

SOLUTION

There are $\binom{50}{3} = 19,600$ ways to choose the 3 winners. Each of these is equally likely.

a. There are $\binom{4}{3} = 4$ ways for the organizers to win all of the prizes. The probability is $4/19600$.

b. There are $\binom{4}{2}\binom{46}{1} = 276$ ways the organizers can win two prizes and one of the other 46 people to win the third prize. So, the probability is $276/19600$.

c. $\binom{4}{1}\binom{46}{2} = 4140$. The probability is $4140/19600$.

d. $\binom{46}{3} = 15,180$. The probability is $15180/19600$.

3) A sample of 10 nurses are selected from a total of 90 nurses employed by the hospital.

- a) How many different samples of 10 nurses can be selected?
- b) 20 nurses of 90 nurses are male. If 10 nurses are randomly selected from those employed by the hospital, what is the probability that the sample will include exactly 4 male nurses?

SOLUTION

$$\text{a. } \binom{90}{10} \qquad \text{b. } \frac{\binom{20}{4}\binom{70}{6}}{\binom{90}{10}} = 0.111$$

4) A survey of consumers in a particular community showed that 10% were dissatisfied with plumbing jobs done in their homes. Assume that all consumers who are dissatisfied with plumbing jobs file complaints. Half the complaints dealt with plumber A, who does 40% of the plumbing jobs in the town. Find the probability that a consumer will obtain

- a) an unsatisfactory plumbing job, given that the plumber was A.
- b) a satisfactory plumbing job, given that the plumber was A.

SOLUTION

Define the events: U: job is unsatisfactory, A: plumber A does the job

$$\text{a) } P(U|A) = \frac{P(A \cap U)}{P(A)} = \frac{P(A|U)P(U)}{P(A)} = \frac{0.5 \cdot 0.1}{0.4} = 0.125$$

$$\text{b) } 1 - P(U|A) = 1 - 0.125 = 0.875$$

5)

A population of voters contains 40% Republicans and 60% Democrats. It is reported that 30% of the Republicans and 70% of the Democrats favor an election issue. A person chosen at random from this population is found to favor the issue in question. Find the conditional probability that this person is a Democrat.

SOLUTION

Apply Bayes' rule:

$$P(D|F) = \frac{P(F|D)P(D)}{P(F|D)P(D) + P(F|R)P(R)} = \frac{0.7 * 0.6}{0.7 * 0.6 + 0.3 * 0.4} = \frac{0.42}{0.54} = \frac{7}{9}$$

6)

Medical case histories indicate that different illnesses may produce identical symptoms. Suppose that a particular set of symptoms, denoted H , occurs only when any one of three illnesses, I_1 , I_2 , or I_3 , occurs. Assume that the simultaneous occurrence of more than one of these illnesses is impossible and that

$$P(I_1) = .01, \quad P(I_2) = .005, \quad P(I_3) = .02.$$

The probabilities of developing the set of symptoms H , given each of these illnesses, are known to be

$$P(H|I_1) = .90, \quad P(H|I_2) = .95, \quad P(H|I_3) = .75.$$

Assuming that an ill person exhibits the symptoms, H , what is the probability that the person has illness I_1 ?

SOLUTION

Apply Bayes' rule:

$$\begin{aligned} P(I_1|H) &= \frac{P(H|I_1)P(I_1)}{P(H|I_1)P(I_1) + P(H|I_2)P(I_2) + P(H|I_3)P(I_3)} \\ &= \frac{0.9 * 0.01}{0.9 * 0.01 + 0.95 * 0.005 + 0.75 * 0.02} = 0.313 \end{aligned}$$

7) Suppose that 30% of the applicants for a certain industrial job possess advanced training in computer programming. Applicants are interviewed sequentially and are selected at random from the pool. Assuming the pool of applicants is large, find the probability that the first applicant with advanced training in programming is found on the fifth interview.

SOLUTION

$$(0.7)^4 * 0.3 = 0.07203$$

8)

A corporation is sampling without replacement for $n = 3$ firms to determine the one from which to purchase certain supplies. The sample is to be selected from a pool of six firms, of which four are local and two are not local. Let Y denote the number of nonlocal firms among the three selected.

a $P(Y = 1)$.

b $P(Y \geq 1)$.

c $P(Y \leq 1)$.

SOLUTION

a. $P(Y = 1) = \frac{\binom{4}{2}\binom{2}{1}}{\binom{6}{3}} = .6$.

b. $P(Y \geq 1) = p(1) + p(2) = \frac{\binom{4}{2}\binom{2}{1}}{\binom{6}{3}} + \frac{\binom{4}{1}\binom{2}{2}}{\binom{6}{3}} = .8$

c. $P(Y \leq 1) = p(0) + p(1) = .8$.

9)

Ten percent of the engines manufactured on an assembly line are defective. If engines are randomly selected one at a time and tested, what is the probability that the first nondefective engine will be found on the second trial?

SOLUTION

Let $Y = \#$ of trials until this first non-defective engine is found. Then, Y is geometric with $p = .9$. $P(Y = 2) = .9(.1) = .09$.

10)

An oil exploration firm is formed with enough capital to finance ten explorations. The probability of a particular exploration being successful is .1. Assume the explorations are independent. Find the mean and variance of the number of successful explorations.

SOLUTION

Using expression for the mean and variance of $Y = \#$ of successful explorations, a binomial random variable with $n = 10$ and $p = .1$, $E(Y) = 10(.1) = 1$, and $V(Y) = 10(.1)(.9) = 0.9$.

11)

In a gambling game a person draws a single card from an ordinary 52-card playing deck. A person is paid \$15 for drawing a jack or a queen and \$5 for drawing a king or an ace. A person who draws any other card pays \$4. If a person plays this game, what is the expected gain?

SOLUTION

Define G to be the gain to a person in drawing one card. The possible values for G are \$15, \$5 or \$-4 with probabilities $2/13$, $2/13$ and $9/13$ respectively. So, $E(G) = 15 \cdot (2/13) + 5 \cdot (2/13) - 4 \cdot (9/13) = 4/13$.

12)

A balanced die is tossed six times, and the number on the uppermost face is recorded each time. What is the probability that the numbers recorded are 1, 2, 3, 4, 5, and 6 in any order?

SOLUTION

$$6! \left(\frac{1}{6} \right)^6 = 5/324.$$

13) Suppose that a die has been altered so that the faces are 1, 2, 3, 4, 5 and 5. If the die is tossed five times, what is the probability that the numbers recorded are 1, 2, 3, 4 and 5 in any order?

SOLUTION

$$5! \cdot \left(\frac{1}{6} \right)^4 \cdot \frac{2}{6} = 5/162$$

14)

Goranson and Hall (1980) explain that the probability of detecting a crack in an airplane wing is the product of p_1 , the probability of inspecting a plane with a wing crack; p_2 , the probability of inspecting the detail in which the crack is located; and p_3 , the probability of detecting the damage.

- a What assumptions justify the multiplication of these probabilities?
- b Suppose $p_1 = .9$, $p_2 = .8$, and $p_3 = .5$ for a certain fleet of planes. If three planes are inspected from this fleet, find the probability that a wing crack will be detected on at least one of them.

SOLUTION

- a. Assume independence of the three inspection events.
- b. Let $Y = \#$ of plane with wing cracks that are detected. Then, Y is binomial with $n = 3$ and $p = .9(.8)(.5) = .36$. Then, $P(Y \geq 1) = 1 - P(Y = 0) = 0.737856$.

15)

Show that, for any integer $n \geq 1$,

- a $\binom{n}{n} = 1$. Interpret this result.
- b $\binom{n}{0} = 1$. Interpret this result.
- c $\binom{n}{r} = \binom{n}{n-r}$. Interpret this result.

SOLUTION

- a. $\binom{n}{n} = \frac{n!}{n!(n-n)!} = 1$. There is only one way to choose all of the items.
- b. $\binom{n}{0} = \frac{n!}{0!(n-0)!} = 1$. There is only one way to choose none of the items.
- c. $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!(n-(n-r))!} = \binom{n}{n-r}$. There are the same number of ways to choose r out of n objects as there are to choose $n-r$ out of n objects.

16)

If two events, A and B , are such that $P(A) = .5$, $P(B) = .3$, and $P(A \cap B) = .1$, find the following:

- a $P(A|B)$
- b $P(B|A)$
- c $P(A|A \cup B)$
- d $P(A|A \cap B)$
- e $P(A \cap B|A \cup B)$

SOLUTION

- a. $P(A|B) = .1/.3 = 1/3$.
- b. $P(B|A) = .1/.5 = 1/5$.
- c. $P(A|A \cup B) = .5/(.5+.3-.1) = 5/7$
- d. $P(A|A \cap B) = 1$
- e. $\frac{P((A \cap B) \cap (A \cup B))}{P(A \cup B)} = \frac{P(A \cap B)}{P(A \cup B)} = \frac{0.1}{0.5+0.3-0.1} = \frac{0.1}{0.7} = \frac{1}{7}$

17)

Diseases I and II are prevalent among people in a certain population. It is assumed that 10% of the population will contract disease I sometime during their lifetime, 15% will contract disease II eventually, and 3% will contract both diseases.

- a Find the probability that a randomly chosen person from this population will contract at least one disease.
- b Find the conditional probability that a randomly chosen person from this population will contract both diseases, given that he or she has contracted at least one disease.

SOLUTION

Define the events: I: disease I is contracted II: disease II is contracted. Then, $P(I) = 0.1$, $P(II) = 0.15$, and $P(I \cap II) = 0.03$.

- a. $P(I \cup II) = .1 + .15 - .03 = 0.22$
- b. $P(I \cap II | I \cup II) = .03/.22 = 3/22$.

18) The number of imperfections in the weave of a certain textile has a Poisson distribution with a mean 4 per square yard. The cost of repairing the imperfections in the weave is \$10 per imperfection. Find the mean and standard deviation of the repair cost for an 8-square-yard bolt of the textile.

SOLUTION

For an 8-square yard bolt, let $X = \#$ of imperfections so that X is Poisson with $\lambda = 32$. Thus, $C = 10X$ is the cost to repair the weave and

$$E(C) = 10E(X) = \$320 \text{ and } V(C) = 100V(X) = 3200.$$

19)

The number of bacteria colonies of a certain type in samples of polluted water has a Poisson distribution with a mean of 2 per cubic centimeter (cm^3).

- a If four 1-cm^3 samples are independently selected from this water, find the probability that at least one sample will contain one or more bacteria colonies.
- b How many 1-cm^3 samples should be selected in order to have a probability of approximately .95 of seeing at least one bacteria colony?

SOLUTION

a. Let $Y = \#$ of samples with at least one bacteria colony. Then, Y is binomial with $n = 4$ and $p = P(\text{at least one bacteria colony}) = 1 - P(\text{no bacteria colonies}) = 1 - e^{-2} = .865$ (by the Poisson). Thus, $P(Y \geq 1) = 1 - P(Y = 0) = 1 - (.135)^4 = .9997$.

b. Following the above, we require $1 - (.135)^n = .95$ or $(.135)^n = .05$. Solving for n , we have $n = \frac{\ln(.05)}{\ln(.135)} = 1.496$, so take $n = 2$.

20) Suppose that X has pmf

x	13.5	15.9	19.1
$p(x)$	.2	.5	.3

Compute $E(X)$, $E(X^2)$, and $V(X)$.

SOLUTION

$$E(X) = (13.5)(.2) + (15.9)(.5) + (19.1)(.3) = 16.38,$$

$$E(X^2) = (13.5)^2(.2) + (15.9)^2(.5) + (19.1)^2(.3) = 272.298,$$

$$V(X) = 272.298 - (16.38)^2 = 3.9936$$

21)

Random variable Y has the following probability density function

$$f(y) = \begin{cases} \frac{1}{\ln 3} \frac{1}{y} & \text{for } 1 < y < 3 \\ 0 & \text{elsewhere} \end{cases}$$

Find $E(X)$ and $V(X)$.

SOLUTION

$$E(X) = \int_1^3 \frac{1}{\ln 3} \frac{1}{y} y dy = \left[\frac{y}{\ln 3} \right]_1^3 = \frac{2}{\ln 3}$$

$$E(X^2) = \int_1^3 \frac{1}{\ln 3} \frac{1}{y} y^2 dy = \left[\frac{y^2}{2 \ln 3} \right]_1^3 = \frac{8}{2 \ln 3}$$

$$V(X) = E(X^2) - [E(X)]^2 = \frac{4}{\ln 3} - \left(\frac{2}{\ln 3} \right)^2 = \frac{4(\ln 3 - 1)}{(\ln 3)^2}$$

22)

For any events A and B with $P(B) > 0$, show that $P(A | B) + P(A' | B) = 1$.

SOLUTION

$$P(A | B) + P(A' | B) = \frac{P(A \cap B)}{P(B)} + \frac{P(A' \cap B)}{P(B)} = \frac{P(A \cap B) + P(A' \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

23) The cdf of checkout duration X is described as:

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{x^2}{4}, & 0 \leq x < 2 \\ 1, & x \geq 2 \end{cases}$$

Find the median checkout duration $\tilde{\mu}$.

SOLUTION

$$.5 = F(\tilde{\mu}) = \frac{\tilde{\mu}^2}{4} \Rightarrow \tilde{\mu}^2 = 2 \Rightarrow \tilde{\mu} = \sqrt{2} \approx 1.414$$

24) Let X be a Bernoulli random variable with pmf as

$$p(x) = \begin{cases} 1 - p & x = 0 \\ p & x = 1 \\ 0 & x \neq 0, 1 \end{cases}$$

- a) Compute $E(X^2)$.
- b) Show that $V(X) = p(1-p)$.

SOLUTION

$$\text{a. } E(X^2) = \sum_{x=0}^1 x^2 \cdot p(x) = (0^2)(1-p) + (1^2)(p) = (1)(p) = p$$

$$\text{b) } E(X) = \sum_{x=0}^1 x \cdot p(x) = 0 \cdot (1-p) + 1 \cdot p = p$$

$$V(X) = E(X^2) - (E(X))^2 = p - p^2 = p(1-p)$$

25)

A supplier of kerosene has a 150-gallon tank that is filled at the beginning of each week. His weekly demand shows a relative frequency behavior that increases steadily up to 100 gallons and then levels off between 100 and 150 gallons. If Y denotes weekly demand in hundreds of gallons, the relative frequency of demand can be modeled by

$$f(y) = \begin{cases} y, & 0 \leq y \leq 1, \\ 1, & 1 < y \leq 1.5, \\ 0, & \text{elsewhere.} \end{cases}$$

- a Find $F(y)$.
- b Find $P(0 \leq Y \leq .5)$.
- c Find $P(.5 \leq Y \leq 1.2)$.

SOLUTION

a. For $0 \leq y \leq 1$, $F(y) = \int_0^y t dt = y^2 / 2$. For $1 < y \leq 1.5$, $F(y) = \int_0^1 t dt + \int_1^y dt = 1/2 + y - 1 = y - 1/2$. Hence,

$$F(y) = \begin{cases} 0 & y < 0 \\ y^2 / 2 & 0 \leq y \leq 1 \\ y - 1/2 & 1 < y \leq 1.5 \\ 1 & y > 1.5 \end{cases}$$

b. $P(0 \leq Y \leq .5) = F(.5) = 1/8$.

c. $P(.5 \leq Y \leq 1.2) = F(1.2) - F(.5) = 1.2 - 1/2 - 1/8 = .575$.

26)

Y has density function

$$f(y) = \begin{cases} (3/2)y^2 + y, & 0 \leq y \leq 1, \\ 0, & \text{elsewhere,} \end{cases}$$

find mean and variance of Y.

SOLUTION

$$E(Y) = \int_0^1 (1.5y^3 + y^2) dy = \left[\frac{3y^4}{8} + \frac{y^3}{3} \right]_0^1 = 17/24 = .708.$$

$$E(Y^2) = \int_0^1 (1.5y^4 + y^3) dy = \left[\frac{3y^5}{10} + \frac{y^4}{4} \right]_0^1 = 3/10 + 1/4 = .55.$$

$$V(Y) = 0.55 - (0.708)^2 = 0.0487$$

27)

Y has density function

$$f(y) = \begin{cases} .2, & -1 < y \leq 0, \\ .2 + (1.2)y, & 0 < y \leq 1, \\ 0, & \text{elsewhere,} \end{cases}$$

find mean and variance of Y.

SOLUTION

$$E(Y) = \int_{-1}^0 .2y dy + \int_0^1 (.2y + 1.2y^2) dy = .4, \quad E(Y^2) = \int_{-1}^0 .2y^2 dy + \int_0^1 (.2y^2 + 1.2y^3) dy = 1.3/3.$$

$$V(Y) = \frac{1.3}{3} - 0.4^2 = 0.2733$$

28)

The proportion of time Y that an industrial robot is in operation during a 40-hour week is a random variable with probability density function

$$f(y) = \begin{cases} 2y, & 0 \leq y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

- a Find $E(Y)$ and $V(Y)$.
- b For the robot under study, the profit X for a week is given by $X = 200Y - 60$. Find $E(X)$ and $V(X)$.

SOLUTION

a. $E(Y) = \int_0^1 2y^2 dy = 2/3$, $E(Y^2) = \int_0^1 2y^3 dy = 1/2$. Thus, $V(Y) = 1/2 - (2/3)^2 = 1/18$.

b. With $X = 200Y - 60$, $E(X) = 200(2/3) - 60 = 220/3$

$$V(X) = 40000 \cdot V(Y) = 2222.222$$

29)

As a measure of intelligence, mice are timed when going through a maze to reach a reward of food. The time (in seconds) required for any mouse is a random variable Y with a density function given by

$$f(y) = \begin{cases} \frac{b}{y^2}, & y \geq b, \\ 0, & \text{elsewhere,} \end{cases}$$

where b is the minimum possible time needed to traverse the maze.

- a Show that $f(y)$ has the properties of a density function.
- b Find $F(y)$.
- c Find $P(Y > b + c)$ for a positive constant c .
- d If c and d are both positive constants such that $d > c$, find $P(Y > b + d | Y > b + c)$.

SOLUTION

a. For $b \geq 0$, $f(y) \geq 0$. Also, $\int_{-\infty}^{\infty} f(y) dy = \int_b^{\infty} \frac{b}{y^2} dy = \frac{-b}{y} \Big|_b^{\infty} = 1$

b. $F(y) = 1 - b/y$, for $y \geq b$, 0 elsewhere.

c. $P(Y > b + c) = 1 - F(b + c) = b/(b + c)$.

d. Applying part c., $P(Y > b + d \mid Y > b + c) = (b + c)/(b + d)$.

30) Let $m(t) = 2e^{-t} + 1/3e^{4t} + 1/4e^{5t}$. Find $E(Y)$, $V(Y)$.

SOLUTION

$$E(Y) = \left. \frac{dm(t)}{dt} \right|_{t=0} = (-2e^{-t} + \frac{4}{3}e^{4t} + \frac{5}{4}e^{5t})|_{t=0} = -2 + \frac{4}{3} + \frac{5}{4} = \frac{7}{12}$$

$$E(Y^2) = \left. \frac{d^2m(t)}{dt^2} \right|_{t=0} = (2e^{-t} + \frac{16}{3}e^{4t} + \frac{25}{4}e^{5t})|_{t=0} = 2 + \frac{16}{3} + \frac{25}{4} = 13.583$$

$$V(Y) = E(Y^2) - (E(Y))^2 = 13.583 - \frac{49}{144} = 13.24$$

31)

If $M_X(t) = e^{5(e^t-1)}$ then find $E(X)$ and $V(X)$ by differentiating

$$R_X(t)$$

where $R_X(t) = \ln(M_X(t))$.

SOLUTION

We define $R_X(t)$ as

$$R_X(t) = \ln[M_X(t)]. \text{ Then } R'_X(0) = E(X), R''_X(0) = V(X)$$

$$R_X(t) = \ln[e^{5(e^t-1)}] = 5(e^t - 1)$$

The first and second derivative of $R_X(t)$ with respect to t gives the expected value of X and the variance of X , respectively.

$$R'_X(t) = 5e^t \rightarrow R'_X(0) = 5e^0 = 5 \rightarrow E(X) = 5$$

$$R''_X(t) = 5e^t \rightarrow R''_X(0) = 5e^0 = 5 \rightarrow V(X) = 5$$