

Spring 2024-
2025
ECON 205

Problem Set 5 - Answers

- 1) The proportion of time Y that an industrial robot is in operation during a 40- hour week is a random variable with probability density function

$$f(y) = \begin{cases} 2y, & 0 \leq y \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

- a) Find $E(Y)$ and $V(Y)$.
- b) For the robot under study, the profit X for a week is given by $X = 200Y - 60$. Find $E(X)$ and $V(X)$.
- c) Find an interval in which the profit should lie for at least 75% of the weeks that the robot is in use.

SOLUTION

a. $E(Y) = \int_0^1 2y^2 dy = 2/3$, $E(Y^2) = \int_0^1 2y^3 dy = 1/2$. Thus, $V(Y) = 1/2 - (2/3)^2 = 1/18$.

b. With $X = 200Y - 60$, $E(X) = 200(2/3) - 60 = 220/3$, $V(X) = 20000/9$.

c. Using Tchebysheff's theorem, a two standard deviation interval about the mean is given by $220/3 \pm 2\sqrt{20000/9}$ or $(-20.948, 167.614)$.

- 2) A 12-in. bar clamped at both ends is subjected to an increasing amount of stress until it snaps. Let Y = the distance from the left end at which the break occurs. Suppose Y has pdf

$$f(y) = \frac{y}{24} \left(1 - \frac{y}{12}\right) \quad 0 \leq y \leq 12$$

Compute the following:

- The cumulative distribution function (CDF) of Y .
- $P(Y \leq 4)$, $P(Y > 6)$ and $P(4 \leq Y \leq 6)$.
- $E(Y)$, $E(Y^2)$, and $V(Y)$.
- The probability that the break point occurs more than 2 inches from the expected break point.
- The expected length of the shorter segment when the break occurs.

SOLUTION

- a. For

$$0 \leq y \leq 12$$

$$F(y) = \frac{1}{24} \int_0^y \left(u - \frac{u^2}{12} \right) du = \frac{1}{24} \left(\frac{u^2}{2} - \frac{u^3}{36} \right) \bigg|_0^y = \frac{y^2}{48} - \frac{y^3}{864}. \text{ Thus}$$

$$F(y) = \begin{cases} 0 & y < 0 \\ \frac{y^2}{48} - \frac{y^3}{864} & 0 \leq y \leq 12 \\ 1 & y > 12 \end{cases}$$

- b. $P(Y \leq 4) = F(4) = .259$. $P(Y > 6) = 1 - F(6) = .5$.
 $P(4 \leq Y \leq 6) = F(6) - F(4) = .5 - .259 = .241$.

c. $E(Y) = \int_0^{12} y \cdot \frac{1}{24} y \left(1 - \frac{y}{12} \right) dy = \frac{1}{24} \int_0^{12} \left(y^2 - \frac{y^3}{12} \right) dy = \frac{1}{24} \left[\frac{y^3}{3} - \frac{y^4}{48} \right]_0^{12} = 6 \text{ inches.}$

$$E(Y^2) = \frac{1}{24} \int_0^{12} \left(y^3 - \frac{y^4}{12} \right) dy = 43.2, \text{ so } V(Y) = 43.2 - 36 = 7.2.$$

- d. $P(Y < 4 \text{ or } Y > 8) = 1 - P(4 \leq Y \leq 8) = 1 - [F(8) - F(4)] = .518$.

- e. The shorter segment has length equal to $\min(Y, 12 - Y)$, and

$$\begin{aligned} E[\min(Y, 12 - Y)] &= \int_0^{12} \min(y, 12 - y) \cdot f(y) dy = \int_0^6 \min(y, 12 - y) \cdot f(y) dy \\ &+ \int_6^{12} \min(y, 12 - y) \cdot f(y) dy = \int_0^6 y \cdot f(y) dy + \int_6^{12} (12 - y) \cdot f(y) dy = \frac{90}{24} = 3.75 \text{ inches.} \end{aligned}$$

- 3) A package delivery service divides their packages into weight classes. Suppose that packages in the 14 to 20 pound class are uniformly distributed, meaning that all weights within that class are equally likely to occur.
- Find the probability that a package weighs between 15 and 16.5 pounds.
 - Find the probability that a package weighs at least 18 pounds.

SOLUTION

- a) X = the weight of a package in the 14 to 20 pound weight class $X \sim U(14, 20)$ The graph of the uniform distribution is a rectangle. The rectangle starts at 14 and ends at 20. The height of the rectangle is $1/6$ so that the area of the entire rectangle is 1.

$$\text{Area} = 1 = (\text{base})(\text{height}) \quad \text{base} = 20 - 14 = 6$$

The equation for area becomes $1 = 6h$. Solve for height h : $h = 1/6$

The probability that a package weighs between 15 and 16.5 pounds.

$$P(15 \leq X \leq 16.5) = (16.5 - 15)(1/6)$$

$$P(15 \leq X \leq 16.5) = 1.5 * 1/6 = 1/4 = 0.25$$

- b) The probability that a package weighs at least 18 pounds.

$$P(X \geq 18) = (20 - 18)/6 = 1/3 \approx 0.33$$

- 4) The operator of a pumping station has observed that demand for water during early afternoon hours has an approximately exponential distribution with mean 100 cfs (cubic feet per second).
- Find the probability that the demand will exceed 200 cfs during the early afternoon on a randomly selected day.
 - What water-pumping capacity should the station maintain during early afternoons so that the probability that demand will exceed capacity on a randomly selected day is only .01?

SOLUTION

Let Y = water demand in the early afternoon. Then, Y is exponential with $\beta = 100$ cfs.

$$\text{a. } P(Y > 200) = \int_{200}^{\infty} \frac{1}{100} e^{-y/100} dy = e^{-2} = .1353.$$

- b. We require the 99th percentile of the distribution of Y :

$$P(Y > \phi_{.99}) = \int_{\phi_{.99}}^{\infty} \frac{1}{100} e^{-y/100} dy = e^{-\phi_{.99}/100} = .01. \text{ So, } \phi_{.99} = -100 \ln(.01) = 460.52 \text{ cfs.}$$

- 5) The time till failure of an electronic component has an exponential distribution and it is known that 10% of components have failed by 1000 hours. What is the probability that a component is still working after 5000 hours?

SOLUTION

Let Y be a random variable represents the time until failure occurs in hours

$$f(y) = \begin{cases} ve^{-vy}, & 0 \leq y, \\ 0, & elsewhere \end{cases}$$

First, we need to find v . To find it, we can use the knowledge in the question (e.g. it is known that...).

$$P(Y \leq 1000) = \int_0^{1000} ve^{-vy} \cdot dy = -e^{-vy} \Big|_{x=0}^{x=1000} = 1 - e^{-1000v}$$

$$P(Y \leq 1000) = 0.1 = 1 - e^{-1000v} \rightarrow v = 1.05 * 10^{-4}$$

We want to find $P(Y > 5000)$

$$P(Y > 5000) = \int_{5000}^{\infty} ve^{-vy} \cdot dy = -e^{-vy} \Big|_{y=5000}^{y=\infty} = e^{-5000v} \approx 0.59$$

- 6) Suppose that when a type of transistor is subjected to an accelerated life test, the lifetime X (in weeks) has a gamma distribution with mean 24 weeks and standard deviation 12 weeks.
- a) What is the probability that a transistor will last between 12 and 24 weeks?
 - b) What is the probability that a transistor will last at most 24 weeks? Is the median of the lifetime distribution less than 24? Why or why not?
 - c) What is the 99th percentile of the lifetime distribution?

SOLUTION

Let X be the lifetime of transistor. And, we know X has a gamma distribution, and the expected value and standard deviation.

$$\mu = \alpha * \beta \text{ \& } \sigma^2 = \alpha * \beta^2$$

$$\alpha\beta = 24 \text{ \& } \alpha\beta^2 = 144$$

Now we have two unknowns and two equations, we can solve it.

$$\frac{\alpha\beta}{\alpha\beta^2} = \frac{24}{144} \rightarrow \beta = 6 \text{ \& } \alpha = 4$$

a)

Now, we can calculate the probability by using the Table A.4 in the textbook which gives us cdf of X .

$$P(12 < X < 24) = F\left(\frac{24}{6}; 4\right) - F\left(\frac{12}{6}; 4\right) = F(4; 4) - F(2; 4)$$

Table A.4 The Incomplete Gamma Function

$x \backslash \alpha$	1	2	3	4	5
1	.632	.264	.080	.019	.004
2	.865	.594	.323	.143	.053
3	.950	.801	.577	.353	.185
4	.982	.908	.762	.567	.371
5	.993	.960	.875	.735	.560

$$P(12 < X < 24) = 0.567 - 0.143 = 0.424$$

b)

$$P(X \leq 24) = F\left(\frac{24}{6}; 4\right) = F(4; 4) = 0.567$$

Table A.4 The Incomplete Gamma Function

$x \backslash \alpha$	1	2	3	4	5
1	.632	.264	.080	.019	.004
2	.865	.594	.323	.143	.053
3	.950	.801	.577	.353	.185
4	.982	.908	.762	.567	.371
5	.993	.960	.875	.735	.560

The median of the lifetime distribution is less than 24, since $P(X \leq 24)$ is equal to 0.567. A number that smaller than 24 can give the probability of 0.5, and this number can be the median of lifetime distribution.

c)

We are trying to find $P(X \leq c) = 0.99$

$$P(X \leq c) = F\left(\frac{c}{6}; 4\right) = 0.99$$

$x \backslash \alpha$	1	2	3	4
1	.632	.264	.080	.019
2	.865	.594	.323	.143
3	.950	.801	.577	.353
4	.982	.908	.762	.567
5	.993	.960	.875	.735
6	.998	.983	.938	.849
7	.999	.993	.970	.918
8	1.000	.997	.986	.958
9		.999	.994	.979
10		1.000	.997	.990
11			.999	.995

$$F(10; 4) = 0.99 \rightarrow \frac{c}{6} = 10. \text{ So, } c = 6 * 10 = 60$$

- 7) The weekly amount of downtime Y (in hours) for an industrial machine has approximately a gamma distribution with $\alpha = 3$ and $\beta = 2$. The loss L (in dollars) to the industrial operation as a result of this downtime is given by $L = 30Y + 2Y^2$. Find the expected value and variance of L .

SOLUTION

Let Y have a gamma distribution with $\alpha = 3$ and $\beta = 2$. Then, the loss $L = 30Y + 2Y^2$.

Then,

$$E(L) = E(30Y + 2Y^2) = 30E(Y) + 2E(Y^2) = 30(6) + 2(12 + 6^2) = 276,$$

$$V(L) = E(L^2) - [E(L)]^2 = E(900Y^2 + 120Y^3 + 4Y^4) - 276^2.$$

$$E(Y^3) = \int_0^{\infty} \frac{y^3}{16} e^{-y/2} dy = 480$$

$$E(Y^4) = \int_0^{\infty} \frac{y^4}{16} e^{-y/2} dy = 5760$$

$$\text{Thus, } V(L) = 900(48) + 120(480) + 4(5760) - 276^2 = 47,664.$$

8) The lifetime of X (in hundreds of hours) of a light bulb has Weibull distribution with parameters $\alpha=2$ and $\beta=3$.

- a) Find $E(X)$ and $V(X)$
- b) Compute $P(X \leq 6)$.
- c) Compute $P(2.8 \leq X \leq 4)$.
- d) Compute $P(X \geq 3)$.

SOLUTION

Let X denote the lifetime (in hundreds of hours) of light bulb.

Using the pdf definition of the Weibull Distribution with parameters $\alpha > 0$ and $\beta > 0$,

$$f(x; \alpha, \beta) = \begin{cases} \frac{\alpha}{\beta^\alpha} x^{\alpha-1} e^{-(x/\beta)^\alpha}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

The cumulative distribution function of X is

$$F(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-(x/\beta)^\alpha}, & x \geq 0 \end{cases}$$

a)

The expected value of X

$$E(X) = \beta \Gamma\left(1 + \frac{1}{\alpha}\right) \quad (3)$$

$$E(X) = 3 \Gamma\left(1 + \frac{1}{2}\right) = 3 \Gamma\left(\frac{3}{2}\right) \quad (4)$$

For this part of the question, we use two important properties of Gamma function.

$$\diamond \text{ For any } \alpha > 1, \Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$$

$$\diamond \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$E(x) = 3 * \left(\frac{3}{2} - 1\right) \Gamma\left(\frac{3}{2} - 1\right) = 3 * \frac{1}{2} \Gamma\left(\frac{1}{2}\right) \quad (5)$$

$$E(X) = \frac{3}{2} * \sqrt{\pi} = \frac{3}{2} * 1.7725 = 2.6587 \quad (6)$$

The variance of X ,

$$V(X) = \sigma^2 = \beta^2 \left\{ \Gamma\left(1 + \frac{2}{\alpha}\right) - \left[\Gamma\left(1 + \frac{1}{\alpha}\right) \right]^2 \right\}$$

$$V(X) = 3^2 \left\{ \Gamma\left(1 + \frac{2}{2}\right) - \left[\Gamma\left(1 + \frac{1}{2}\right) \right]^2 \right\}$$

Recall another property of the Gamma function,

❖ For any positive integer, n , $\Gamma(n) = (n - 1)!$

$$V(X) = 9\{\Gamma(2) - [\Gamma(3/2)]^2\} = 9 \left[1 - \left(\frac{1}{2} * \Gamma\left(\frac{1}{2}\right) \right)^2 \right]$$

$$V(X) = 9 \left[1 - \left(\frac{\sqrt{\pi}}{2} \right)^2 \right] = 9 \left[1 - \left(\frac{\sqrt{3.1416}}{2} \right)^2 \right] = 1.931246$$

b)

$$P(X \leq 6) = F(6)$$

$$P(X \leq 6) = 1 - e^{-\left(\frac{6}{3}\right)^2} = 1 - e^{-(2)^2} = 1 - e^{-4}$$

$$P(X \leq 6) = 1 - 0.0183 = 0.9817$$

c)

$$P(2.8 \leq X \leq 4) = F(4) - F(2.8)$$

$$P(2.8 \leq X \leq 4) = \left[1 - e^{-\left(\frac{4}{3}\right)^2} \right] - \left[1 - e^{-\left(\frac{2.8}{3}\right)^2} \right] = -e^{-1.78} + e^{-0.87}$$

$$P(2.8 \leq X \leq 4) = -0.1686 + 0.4190 = 0.2504$$

d)

$$P(X \geq 3) = 1 - P(X < 3) = 1 - F(3)$$

$$P(X \geq 3) = 1 - \left[1 - e^{-\left(\frac{3}{3}\right)^2} \right] = e^{-(1)^2} = 0.3679$$

Table A.4 The incomplete gamma function
$$G(x; \alpha) = \int_0^x \frac{1}{\Gamma(\alpha)} y^{\alpha-1} e^{-y} dy$$

x	α									
	1	2	3	4	5	6	7	8	9	10
1	.632	.264	.080	.019	.004	.001	.000	.000	.000	.000
2	.865	.594	.323	.143	.053	.017	.005	.001	.000	.000
3	.950	.801	.577	.353	.185	.084	.034	.012	.004	.001
4	.982	.908	.762	.567	.371	.215	.111	.051	.021	.008
5	.993	.960	.875	.735	.560	.384	.238	.133	.068	.032
6	.998	.983	.938	.849	.715	.554	.394	.256	.153	.084
7	.999	.993	.970	.918	.827	.699	.550	.401	.271	.170
8	1.000	.997	.986	.958	.900	.809	.687	.547	.407	.283
9		.999	.994	.979	.945	.884	.793	.676	.544	.413
10		1.000	.997	.990	.971	.933	.870	.780	.667	.542
11			.999	.995	.985	.962	.921	.857	.768	.659
12			1.000	.998	.992	.980	.954	.911	.845	.758
13				.999	.996	.989	.974	.946	.900	.834
14				1.000	.998	.994	.986	.968	.938	.891
15					.999	.997	.992	.982	.963	.930

Table A.3 Standard Normal Curve Areas (cont.)

$$\Phi(z) = P(Z \leq z)$$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9278	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998