

Spring 2024-  
2025  
ECON 205

### Problem Set 4- Answers

- 1) If  $Y$  is a continuous random variable with mean  $\mu$  and variance  $\sigma^2$ ,  $a$  and  $b$  are constants, prove the following:

a)  $E(aY+b) = aE(Y)+b = a\mu+b$

$$E(aY + b) = \int_{-\infty}^{\infty} (ay + b)f(y)dy \quad (1)$$

$$E(aY + b) = a \int_{-\infty}^{\infty} y f(y)dy + b \int_{-\infty}^{\infty} f(y)dy \quad (2)$$

$$E(aY + b) = a E(Y) + b,$$

$$\text{where } \int_{-\infty}^{\infty} y f(y)dy = E(Y) \text{ and } \int_{-\infty}^{\infty} f(y)dy = 1 \quad (3)$$

$$E(aY + b) = a\mu + b \quad (4)$$

b)  $V(aY+b) = a^2V(Y) = a^2\sigma^2$

$$V(aY + b) = E[(aY + b)^2] - [E(aY + b)]^2 \quad (5)$$

$$V(aY + b) = E[(a^2Y^2 + 2abY + b^2)] - [(aE(Y) + b)]^2 \quad (6)$$

$$V(aY + b) = a^2E(Y^2) + 2abE(Y) + b^2 - (a^2E(Y)^2 + 2abE(Y) + b^2) \quad (7)$$

$$V(aY + b) = a^2E(Y^2) - a^2E(Y)^2 \quad (8)$$

Rearrange the equation (8), then

$$V(aY + b) = a^2[E(Y^2) - E(Y)^2] \quad (9)$$

$$V(aY + b) = a^2V(Y) = a^2\sigma^2 \quad (10)$$

- 2) Mrs. Brown is pregnant and due in 100 days. Suppose that the probability density distribution function for having a child is approximately normal with mean 100 and standard deviation 8. Mrs. Brown has a business trip and will return in 85 days and have to go on another business trip in 107 days.
- What is the probability that the birth will occur before her second trip?
  - What is the probability that the birth will occur after she returns from her first business trip?
  - What is the probability that she will be here for the birth?

**SOLUTION**

- a)  $P(X < 107)$  is asked.

Compute the z-score

$$z = \frac{107 - 100}{8} = 0.88$$

$$P(Z < 0.88) = 0.8106 \text{ from the z-table}$$

Hence, there is 81% probability that the baby will be born before the second trip

- b) Now, we want  $P(X > 85)$

z-score:

$$z = \frac{85 - 100}{8} = -1.88$$

$$P(Z < -1.88) = 0.0301$$

$$P(Z > -1.88) = 1 - 0.0301 = 0.9699$$

Hence, there is 97% probability that the baby will be born after the first trip

- c)  $P(85 < X < 107)$

$$P(85 < X < 107) = P(X < 107) - P(X < 85) = 81\% - 3\% = 78\%$$

Hence, there is 78% probability that Mrs Brown will make it to the birth

- 3) Scores on an examination are assumed to be normally distributed with mean 78 and variance 36.

- a) What is the probability that a person taking the examination scores higher than 72?

$$\mu = 78 \text{ and } \sigma^2 = 36, \sigma = 6$$

$$P(Y > 72) = P\left(Z > \frac{72 - 78}{6}\right) \quad (1)$$

$$P(Z > -1) = 0.8413 \quad (2)$$

- b) Suppose that students scoring in the top 10% of this distribution are to receive an A grade. What is the minimum score a student must achieve to earn an A grade?

We seek  $c$  such that  $P(Y > c) = 0.1$ . For standard normal,  $P(Z > z_0) = 0.1$  when  $z_0 = 1.28$ . Therefore,  $c = 78 + (1.28)6 = 85.68$ .

- c) What must be the cut-off point for passing the examination if the examiner wants only the top 28.1% of all scores to be passing?

We seek  $c$  such that  $P(Y > c) = 0.281$ . For standard normal,  $P(Z > z_0) = 0.281$  when  $z_0 = 0.58$ . Therefore,  $c = 78 + (0.58)6 = 81.48$ .

- d) Approximately what proportion of students have scores 5 or more points above the score that cuts off the lowest 25%?

For the standard normal,  $P(Z < -0.67) = 0.25$ . Therefore, the score that cuts off the lowest 25% is given by  $(-0.67)6 + 78 = 73.98$ .

We need to find the proportion of students who have scores 78.98 or more.

$$P(X \geq 78.98) = P\left(Z \geq \frac{78.98 - 78}{6}\right) = P(Z \geq 0.163) \quad (3)$$

$$P(Z \geq 0.163) = 1 - P(Z < 0.163) \quad (4)$$

$$P(X \geq 78.98) = 1 - 0.5636 = 0.4364 \quad (5)$$

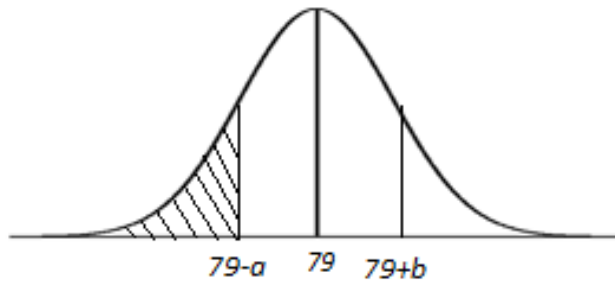
- e) If it is known that a student's score exceeds 72, what is the probability that his or her score exceeds 84?

$$P(Y > 84 | Y > 72) = \frac{P(Y > 84)}{P(Y > 72)} \quad (6)$$

$$P(Y > 84 | Y > 72) = \frac{P(Z > 1)}{P(Z > -1)} \quad (7)$$

$$P(Y > 84 | Y > 72) = \frac{0.1587}{0.8413} = .1886 \quad (8)$$

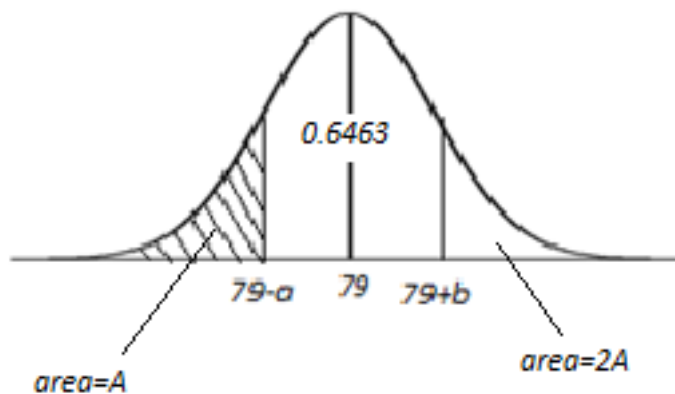
- 4) Let  $X$  be a normally distributed random variable with mean 79. Suppose it is known that  $P(79-a \leq X \leq 79+b) = 0.6463$ .



It is given that  $P(X \geq 79+b) = 2P(X \leq 79-a)$

- Calculate the shaded area.
- Find the value of  $b$ .

### SOLUTION



$$3A + 0.6463 = 1$$

- $A = 0.1179$
- $2A = 0.2358$   
From the z-table,  
 $z = 0.72$  Since  $z = \frac{(79+b)-79}{\sigma}$ , we obtain  $b = 0.72\sigma$ .

5) The density function of X is given by

$$f(x) = \begin{cases} ax + bx^2, & 0 \leq x \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

$E(X)=1/9$ . Find a and b.

**SOLUTION**

$$\int_0^1 (ax + bx^2) dx = 1$$

$$\frac{a}{2} + \frac{b}{3} = 1$$

$$\int_0^1 (ax + bx^2) x dx = \frac{1}{9}$$

$$\frac{a}{3} + \frac{b}{4} = \frac{1}{9}$$

$$a \cong 15.33, b = -20$$

6) The weekly repair cost, X, for a certain machine has a probability density function given by:

$$f(x) = \begin{cases} cx(1-x), & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

with measurements in \$100.

- a. Find the value of c that makes this function a valid density function.
- b. Find the cumulative distribution function of X.
- c. What is the probability that repair costs will exceed \$75 during a week?
- d. What is the probability that the repair costs will exceed \$75 during a week given that they will exceed \$50?

**SOLUTION**

a) Find the value of c that makes this function a valid density function.

$$\text{For any pdf, } \int_{-\infty}^{\infty} f(x) dx = 1$$

$$\int_0^1 cx(1-x)dx = c \int_0^1 (x-x^2)dx = c \left( \frac{x^2}{2} - \frac{x^3}{3} \right) \Big|_{x=0}^{x=1} = 1$$

$$c \left( \frac{1^2}{2} - \frac{1^3}{3} \right) - c \left( \frac{0^2}{2} - \frac{0^3}{3} \right) = \frac{c}{6} = 1$$

$$c=6$$

- b)** Find the cumulative distribution function of X.

For  $0 \leq x \leq 1$

$$F(x) = \int_0^x 6z(1-z)dz = 3x^2 - 2x^3$$

$$F(x) = \begin{cases} 0, & x < 0 \\ 3x^2 - 2x^3, & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases}$$

- c)** What is the probability that repair costs will exceed \$75 during a week?

$$\text{Find } P(X > 0.75) = 1 - P(X \leq 0.75) = 1 - F(0.75) = 1 - [3(0.75)^2 - 2(0.75)^3] \cong 0.16$$

- d)** What is the probability that the repair costs will exceed \$75 during a week given that they will exceed \$50?

$$P(X > 0.75 | X > 0.5) = \frac{P(X > 0.75 \cap X > 0.5)}{P(X > 0.5)} = \frac{P(X > 0.75)}{P(X > 0.5)} = \frac{1 - [3(0.75)^2 - 2(0.75)^3]}{1 - [3(0.5)^2 - 2(0.5)^3]} \cong 0.3$$

- 7)** Suppose X is a continuous random variable with density function:

$$f(x) = \begin{cases} \frac{1}{2} x^2 e^{-x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Find the moment generation function.

The moment generation function can be written as:

$$M_x(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{xt} f(x) \cdot dx$$

$$M_x(t) = E(e^{tX}) = \int_0^{\infty} e^{xt} * \frac{1}{2} * x^2 * e^{-x} * dx = \frac{1}{2} \int_0^{\infty} x^2 e^{(t-1)x} dx \quad (1)$$

Integrate by parts:  $\int f g' = f g - \int f' g$ , where  $f = x^2$ ,  $g' = e^{(t-1)x}$  and

$$f' = 2x, \quad g = \frac{e^{(t-1)x}}{t-1}$$

$$M_x(t) = \frac{1}{2} \left[ \frac{x^2 e^{(t-1)x}}{(t-1)} \Big|_0^{\infty} - \int_0^{\infty} \frac{2x e^{(t-1)x}}{t-1} dx \right] \quad (2)$$

$$M_x(t) = \frac{1}{2} \left[ \frac{x^2 e^{(t-1)x}}{(t-1)} \Big|_0^{\infty} - \frac{2}{t-1} \int_0^{\infty} x e^{(t-1)x} dx \right] \quad (3)$$

Now, integrate by parts:  $\int f g' = f g - \int f' g$ , where  $f = x$ ,  $g' = e^{(t-1)x}$  and

$$f' = 1, \quad g = \frac{e^{(t-1)x}}{t-1}$$

$$\int_0^{\infty} x e^{(t-1)x} dx = \frac{x e^{(t-1)x}}{t-1} \Big|_0^{\infty} - \int_0^{\infty} \frac{e^{(t-1)x}}{t-1} dx \quad (4)$$

$$\int_0^{\infty} x e^{(t-1)x} dx = \frac{x e^{(t-1)x}}{t-1} \Big|_0^{\infty} - \frac{e^{(t-1)x}}{(t-1)^2} \Big|_0^{\infty} \quad (5)$$

Plug (5) in (3)

$$M_x(t) = \frac{1}{2} \left[ \frac{x^2 e^{(t-1)x}}{(t-1)} \Big|_0^{\infty} - \frac{2}{t-1} \left[ \frac{x e^{(t-1)x}}{t-1} \Big|_0^{\infty} - \frac{e^{(t-1)x}}{(t-1)^2} \Big|_0^{\infty} \right] \right] \quad (6)$$

$$M_x(t) = \frac{1}{2} \left[ \left( \frac{x^2}{t-1} - \frac{2x}{(t-1)^2} + \frac{2}{(t-1)^3} \right) e^{(t-1)x} \right]_0^{\infty} = \frac{1}{(1-t)^3} \quad (7)$$

Where the last equality holds for  $t < 1$ . So the mgf is defined for  $t < 1$ .

- 8) The length of time required by students to complete a one-hour exam is a random variable with a density function given by

$$f(y) = \begin{cases} cy^2 + y, & 0 \leq y \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

- a) Find c.
- b) Find F(y).
- c) Find the probability that a randomly selected student will finish in less than half an hour.

**SOLUTION**

$$\text{a) } \int_0^1 (cy^2 + y) dy = \frac{cy^3}{3} + \frac{y^2}{2} \Big|_0^1 = \frac{c}{3} + \frac{1}{2} = 1$$

$$\frac{c}{3} = \frac{1}{2}$$

$$c = \frac{3}{2}$$

**b)**

For  $0 \leq y \leq 1$

$$F(y) = \int_0^y \left(\frac{3}{2}t^2 + t\right) dt = \frac{t^3}{2} + \frac{t^2}{2} \Big|_0^y = \frac{y^3}{2} + \frac{y^2}{2}$$

$$F(x) = \begin{cases} 0, & y < 0 \\ \frac{y^3}{2} + \frac{y^2}{2}, & 0 \leq y \leq 1 \\ 1, & y > 1 \end{cases}$$

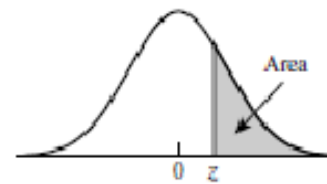
$$\text{c) } P(Y < 0.5) = F(0.5) = \frac{0.5^3}{2} + \frac{0.5^2}{2} = \frac{3}{16}$$

**Table A.3** Standard Normal Curve Areas (cont.)

$\Phi(z) = P(Z \leq z)$

| $z$ | .00   | .01   | .02   | .03   | .04   | .05   | .06   | .07   | .08   | .09   |
|-----|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 0.0 | .5000 | .5040 | .5080 | .5120 | .5160 | .5199 | .5239 | .5279 | .5319 | .5359 |
| 0.1 | .5398 | .5438 | .5478 | .5517 | .5557 | .5596 | .5636 | .5675 | .5714 | .5753 |
| 0.2 | .5793 | .5832 | .5871 | .5910 | .5948 | .5987 | .6026 | .6064 | .6103 | .6141 |
| 0.3 | .6179 | .6217 | .6255 | .6293 | .6331 | .6368 | .6406 | .6443 | .6480 | .6517 |
| 0.4 | .6554 | .6591 | .6628 | .6664 | .6700 | .6736 | .6772 | .6808 | .6844 | .6879 |
| 0.5 | .6915 | .6950 | .6985 | .7019 | .7054 | .7088 | .7123 | .7157 | .7190 | .7224 |
| 0.6 | .7257 | .7291 | .7324 | .7357 | .7389 | .7422 | .7454 | .7486 | .7517 | .7549 |
| 0.7 | .7580 | .7611 | .7642 | .7673 | .7704 | .7734 | .7764 | .7794 | .7823 | .7852 |
| 0.8 | .7881 | .7910 | .7939 | .7967 | .7995 | .8023 | .8051 | .8078 | .8106 | .8133 |
| 0.9 | .8159 | .8186 | .8212 | .8238 | .8264 | .8289 | .8315 | .8340 | .8365 | .8389 |
| 1.0 | .8413 | .8438 | .8461 | .8485 | .8508 | .8531 | .8554 | .8577 | .8599 | .8621 |
| 1.1 | .8643 | .8665 | .8686 | .8708 | .8729 | .8749 | .8770 | .8790 | .8810 | .8830 |
| 1.2 | .8849 | .8869 | .8888 | .8907 | .8925 | .8944 | .8962 | .8980 | .8997 | .9015 |
| 1.3 | .9032 | .9049 | .9066 | .9082 | .9099 | .9115 | .9131 | .9147 | .9162 | .9177 |
| 1.4 | .9192 | .9207 | .9222 | .9236 | .9251 | .9265 | .9278 | .9292 | .9306 | .9319 |
| 1.5 | .9332 | .9345 | .9357 | .9370 | .9382 | .9394 | .9406 | .9418 | .9429 | .9441 |
| 1.6 | .9452 | .9463 | .9474 | .9484 | .9495 | .9505 | .9515 | .9525 | .9535 | .9545 |
| 1.7 | .9554 | .9564 | .9573 | .9582 | .9591 | .9599 | .9608 | .9616 | .9625 | .9633 |
| 1.8 | .9641 | .9649 | .9656 | .9664 | .9671 | .9678 | .9686 | .9693 | .9699 | .9706 |
| 1.9 | .9713 | .9719 | .9726 | .9732 | .9738 | .9744 | .9750 | .9756 | .9761 | .9767 |
| 2.0 | .9772 | .9778 | .9783 | .9788 | .9793 | .9798 | .9803 | .9808 | .9812 | .9817 |
| 2.1 | .9821 | .9826 | .9830 | .9834 | .9838 | .9842 | .9846 | .9850 | .9854 | .9857 |
| 2.2 | .9861 | .9864 | .9868 | .9871 | .9875 | .9878 | .9881 | .9884 | .9887 | .9890 |
| 2.3 | .9893 | .9896 | .9898 | .9901 | .9904 | .9906 | .9909 | .9911 | .9913 | .9916 |
| 2.4 | .9918 | .9920 | .9922 | .9925 | .9927 | .9929 | .9931 | .9932 | .9934 | .9936 |
| 2.5 | .9938 | .9940 | .9941 | .9943 | .9945 | .9946 | .9948 | .9949 | .9951 | .9952 |
| 2.6 | .9953 | .9955 | .9956 | .9957 | .9959 | .9960 | .9961 | .9962 | .9963 | .9964 |
| 2.7 | .9965 | .9966 | .9967 | .9968 | .9969 | .9970 | .9971 | .9972 | .9973 | .9974 |
| 2.8 | .9974 | .9975 | .9976 | .9977 | .9977 | .9978 | .9979 | .9979 | .9980 | .9981 |
| 2.9 | .9981 | .9982 | .9982 | .9983 | .9984 | .9984 | .9985 | .9985 | .9986 | .9986 |
| 3.0 | .9987 | .9987 | .9987 | .9988 | .9988 | .9989 | .9989 | .9989 | .9990 | .9990 |
| 3.1 | .9990 | .9991 | .9991 | .9991 | .9992 | .9992 | .9992 | .9992 | .9993 | .9993 |
| 3.2 | .9993 | .9993 | .9994 | .9994 | .9994 | .9994 | .9994 | .9995 | .9995 | .9995 |
| 3.3 | .9995 | .9995 | .9995 | .9996 | .9996 | .9996 | .9996 | .9996 | .9996 | .9997 |
| 3.4 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9998 |

Normal Curve Areas  
 Standard normal probability in right-hand  
 tail (for negative values of  $z$ , areas are found by symmetry)



| Second decimal place of $z$ |       |       |       |       |       |       |       |       |       |       |
|-----------------------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $z$                         | .00   | .01   | .02   | .03   | .04   | .05   | .06   | .07   | .08   | .09   |
| 0.0                         | .5000 | .4960 | .4920 | .4880 | .4840 | .4801 | .4761 | .4721 | .4681 | .4641 |
| 0.1                         | .4602 | .4562 | .4522 | .4483 | .4443 | .4404 | .4364 | .4325 | .4286 | .4247 |
| 0.2                         | .4207 | .4168 | .4129 | .4090 | .4052 | .4013 | .3974 | .3936 | .3897 | .3859 |
| 0.3                         | .3821 | .3783 | .3745 | .3707 | .3669 | .3632 | .3594 | .3557 | .3520 | .3483 |
| 0.4                         | .3446 | .3409 | .3372 | .3336 | .3300 | .3264 | .3228 | .3192 | .3156 | .3121 |
| 0.5                         | .3085 | .3050 | .3015 | .2981 | .2946 | .2912 | .2877 | .2843 | .2810 | .2776 |
| 0.6                         | .2743 | .2709 | .2676 | .2643 | .2611 | .2578 | .2546 | .2514 | .2483 | .2451 |
| 0.7                         | .2420 | .2389 | .2358 | .2327 | .2296 | .2266 | .2236 | .2206 | .2177 | .2148 |
| 0.8                         | .2119 | .2090 | .2061 | .2033 | .2005 | .1977 | .1949 | .1922 | .1894 | .1867 |
| 0.9                         | .1841 | .1814 | .1788 | .1762 | .1736 | .1711 | .1685 | .1660 | .1635 | .1611 |
| 1.0                         | .1587 | .1562 | .1539 | .1515 | .1492 | .1469 | .1446 | .1423 | .1401 | .1379 |
| 1.1                         | .1357 | .1335 | .1314 | .1292 | .1271 | .1251 | .1230 | .1210 | .1190 | .1170 |
| 1.2                         | .1151 | .1131 | .1112 | .1093 | .1075 | .1056 | .1038 | .1020 | .1003 | .0985 |
| 1.3                         | .0968 | .0951 | .0934 | .0918 | .0901 | .0885 | .0869 | .0853 | .0838 | .0823 |
| 1.4                         | .0808 | .0793 | .0778 | .0764 | .0749 | .0735 | .0722 | .0708 | .0694 | .0681 |
| 1.5                         | .0668 | .0655 | .0643 | .0630 | .0618 | .0606 | .0594 | .0582 | .0571 | .0559 |
| 1.6                         | .0548 | .0537 | .0526 | .0516 | .0505 | .0495 | .0485 | .0475 | .0465 | .0455 |
| 1.7                         | .0446 | .0436 | .0427 | .0418 | .0409 | .0401 | .0392 | .0384 | .0375 | .0367 |
| 1.8                         | .0359 | .0352 | .0344 | .0336 | .0329 | .0322 | .0314 | .0307 | .0301 | .0294 |
| 1.9                         | .0287 | .0281 | .0274 | .0268 | .0262 | .0256 | .0250 | .0244 | .0239 | .0233 |
| 2.0                         | .0228 | .0222 | .0217 | .0212 | .0207 | .0202 | .0197 | .0192 | .0188 | .0183 |
| 2.1                         | .0179 | .0174 | .0170 | .0166 | .0162 | .0158 | .0154 | .0150 | .0146 | .0143 |
| 2.2                         | .0139 | .0136 | .0132 | .0129 | .0125 | .0122 | .0119 | .0116 | .0113 | .0110 |
| 2.3                         | .0107 | .0104 | .0102 | .0099 | .0096 | .0094 | .0091 | .0089 | .0087 | .0084 |
| 2.4                         | .0082 | .0080 | .0078 | .0075 | .0073 | .0071 | .0069 | .0068 | .0066 | .0064 |
| 2.5                         | .0062 | .0060 | .0059 | .0057 | .0055 | .0054 | .0052 | .0051 | .0049 | .0048 |
| 2.6                         | .0047 | .0045 | .0044 | .0043 | .0041 | .0040 | .0039 | .0038 | .0037 | .0036 |
| 2.7                         | .0035 | .0034 | .0033 | .0032 | .0031 | .0030 | .0029 | .0028 | .0027 | .0026 |
| 2.8                         | .0026 | .0025 | .0024 | .0023 | .0023 | .0022 | .0021 | .0021 | .0020 | .0019 |
| 2.9                         | .0019 | .0018 | .0017 | .0017 | .0016 | .0016 | .0015 | .0015 | .0014 | .0014 |