

Test for Variances

$$H_0: \sigma^2 = \sigma_0^2$$

$$H_1: \sigma^2 \neq \sigma_0^2 \text{ or } \sigma^2 > \sigma_0^2 \text{ or } \sigma^2 < \sigma_0^2$$

$$\chi^2 = \frac{(n-1)S^2}{\sigma_0^2} \sim \chi^2_{n-1}$$

Reject H_0 :

$$- H_1: \sigma^2 \neq \sigma_0^2 \text{ if } \chi^2 \geq \chi^2_{\alpha/2, n-1} \text{ or } \chi^2 \leq \chi^2_{1-\alpha/2, n-1}$$

$$- H_1: \sigma^2 > \sigma_0^2 \text{ if } \chi^2 > \chi^2_{\alpha, n-1}$$

$$- H_1: \sigma^2 < \sigma_0^2 \text{ if } \chi^2 < \chi^2_{1-\alpha, n-1}$$

Ex:

In order to assess variability in production of an assembly line threshold level of efficiency should equal to 2 units. For a random sample of 20 controls, Sample variance is found as 2.36 units. Test at 1% significance level that population variance is 2 units against it is bigger than 2 units. Assume population is distributed normally

$$H_0: \sigma^2 = 2 \quad \chi^2 = \frac{(20-1) \cdot 2.36}{2} = 22.42$$

$$H_1: \sigma^2 > 2$$

$$\chi^2_{0.05, 19} = 30.14 \quad \chi^2 < \chi^2_{0.05, 19}$$

$$n=20, S^2=2.36$$

$$\alpha=0.05 \quad \sigma_0^2=2$$

↳ not reject H_0

for testing equality of two different populations variances

$$H_0: \sigma_1^2 = \sigma_2^2$$

$$H_1: \sigma_1^2 \neq \sigma_2^2 \text{ or } \sigma_1^2 > \sigma_2^2 \text{ or } \sigma_1^2 < \sigma_2^2$$

$$F = \frac{S_1^2}{S_2^2}$$

Reject H_0 if

$$-\sigma_1^2 \neq \sigma_2^2 \quad S_1^2/S_2^2 \geq f_{\alpha/2, n_1-1, n_2-1} \text{ if } S_1^2 \geq S_2^2$$

$$S_2^2/S_1^2 \geq f_{\alpha/2, n_2-1, n_1-1} \text{ if } S_1^2 < S_2^2$$

$$-\sigma_1^2 > \sigma_2^2, \quad S_1^2/S_2^2 \geq f_{\alpha, n_1-1, n_2-1}$$

$$-\sigma_1^2 < \sigma_2^2, \quad S_2^2/S_1^2 \geq f_{\alpha, n_2-1, n_1-1}$$

Ex: For a random sample of 17 firms variance of productivity is found as 123.35 units. For another sample of 11 firms in another industry, variance of productivity is 108.02 at 10% significance level test the hypothesis that two population variances are equal

$$n_1 = 17 \quad S_1^2 = 123.35 \quad n_2 = 11 \quad S_2^2 = 108.02 \quad \alpha = 0.10$$

$$S_1^2 > S_2^2$$

$$\sigma_1^2 \neq \sigma_2^2 \quad S_1^2/S_2^2 \geq f_{\alpha/2, n_1-1, n_2-1} \text{ if } S_1^2 \geq S_2^2$$

$$F = \frac{123.35}{108.02} = 1.142$$

$$F_{0.05, 16, 10} = 2.49 \quad F < F_{0.05, 16, 10} \Rightarrow \text{Not reject } H_0$$

Measuring Power of a Test

$1 - \beta \Rightarrow$ power of Test $\rightarrow$ prob of not committing type II error

$\beta \rightarrow$ Prob of committing type II error

power function of a test of a statistical hypc. of H_0 against H_1 is $\pi(\theta) = \begin{cases} \alpha(\theta) & \text{for values assumed under } H_0 \\ 1 - \beta(\theta) & \text{" " " " " " } H_1 \end{cases}$

Values of power functions are probabilities of rejecting H_0 for various values of parameter θ

For values of θ assumed under $H_0 \Rightarrow$ power function gives prob of committing type I error

for values of θ assumed under $H_1 \Rightarrow$ power func. gives prob. of not committing Type II error

Therefore these probs can be used as a procedure to assess the power of the test

Ex: Given a random sample of 16 obs
test

$$(\alpha = 0.05)$$

$$H_0: \mu = 5$$

$$H_1: \mu > 5$$

population distr is assumed to be normal with $\sigma = 0.1$

$$\text{Reject } H_0 \text{ if } \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} > z_{\alpha} \quad z_{0.05} = 1.645$$

$$H_0 \text{ is not rejected if } \frac{\bar{X} - 5}{0.1/\sqrt{16}} > 1.645$$

$$\bar{X} > 1.645(0.025) + 5 = 5.041 \quad \left. \begin{array}{l} H_0 \text{ is not rejected if } \\ \bar{X} \leq 5.041 \end{array} \right\} \begin{array}{l} \text{test procedure will lead to} \\ \text{non-rejection of } H_0 \text{ if sample} \\ \text{mean weight does not exceed} \\ 5.041 \end{array}$$

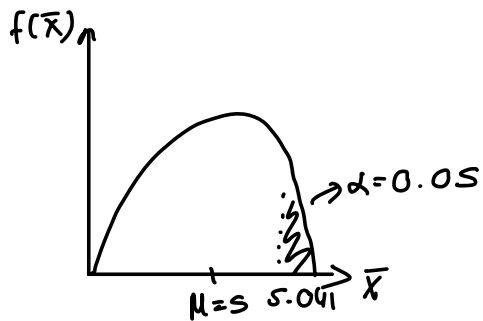
Suppose that we want to determine prob. that H_0 will not be rejected if true mean is 5.05 (μ_1)

$$P(X \leq 5.041) = P\left(Z \leq \frac{5.041 - \mu_1}{\sigma/\sqrt{n}}\right) = P\left(Z \leq \frac{5.041 - 5.05}{0.025}\right)$$

$$= P(Z < -0.36) = 1 - 0.6406 = 0.3594$$

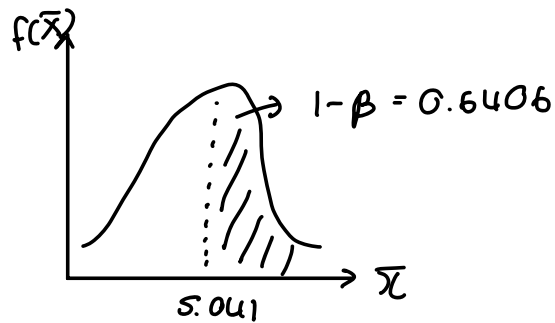
Prob. β of Type II error is involved in not rejecting H_0 when true mean is 5.05 $\Rightarrow 0.3594$

$$\text{Power of test } 1 - \beta = 1 - 0.3594 = \underline{\underline{0.6406}}$$

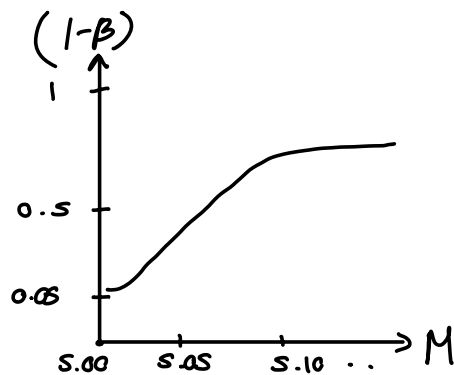


Prob. that sample mean exceeds 5.041 is 0.05

by using similar of population mean



Shaded area show prob. that sample mean exceeds 5.041 when true pop. mean is 5.05
calculation, we find powers of test for



- ① Further true mean (μ_1) hypothesized mean (μ_0), greater the power of test. It means that we are more likely to detect larger than small discrepancies from hypothesized mean.
- ② Smaller the level of significance of test, smaller the power reducing prob of type I error will $\uparrow$ prob of type II error
- ③ Larger population variance, lower the power of test. We are less likely to detect small departures from a hypothesized mean when there is greater variability in population
- ④ Larger the sample size, greater the power of test
More information is obtained from population

Ex: $H_0: \mu = 25.0$ $\sigma = 2.4$ ($\alpha = 0.05$)
 $n = 30$

$H_1: \mu > 25.0$ $\bar{x} = 25.718$

Computing prob of committing type I error ↗ level of significance

$$\begin{aligned} P(\text{Type I error}) &= P(\text{Reject } H_0 | H_0 \text{ is true}) \\ &= P(\bar{x} > 25.718 | \mu = 25.0) \\ &= P\left(\frac{\bar{x} - 25.0}{2.4/\sqrt{30}} \geq \frac{25.718 - 25.0}{2.4/\sqrt{30}}\right) \end{aligned}$$

$$p = P(z > 1.64) \cong 0.05$$

Computing Prob of committing Type II error
 let's true mean is $\mu = 25.750$

$$\begin{aligned} P(\text{Type II error} | \mu = 25.750) &= P(\text{we do not reject } H_0 | \mu = 25.750) \\ &= P(\bar{x} < 25.718 | \mu = 25.750) \\ p &= P\left(\frac{\bar{x} - 25.75}{2.4/\sqrt{30}} < \frac{25.718 - 25.750}{2.4/\sqrt{30}}\right) = \\ &P(z < -0.07) = 0.4721 \\ 1 - \beta &= 0.5279 \end{aligned}$$

let's true mean $\mu = 26.8$

$$P(\bar{x} < 25.718 | \mu = 26.8)$$

$$P\left(\frac{\bar{x} - 26.8}{2.4/\sqrt{30}} < \frac{25.718 - 26.8}{2.4/\sqrt{30}}\right) = P(z < -2.47) = 0.0068$$

↓

$$1 - \beta = 0.9932$$

Ex: *estki sinav sorusu*

Suppose an experimenter wishes to test

$$H_0: \mu = 100$$

$$H_1: \mu > 100$$

at $\alpha = 0.05$ wants $1 - \beta$ to equal 0.66 when $\mu = 133$

what is the smallest sample size that will achieve this objective?

Assume that variable being measured is normally distributed with $\sigma = 14$

Consider consequence of level of significance, $\alpha = 0.05$
 $\alpha = P(\text{Reject } H_0 | H_0 \text{ is true}) = P(\bar{X} \geq \bar{X}^* | \mu = 100)$

$$= P\left(\frac{\bar{X} - 100}{14/\sqrt{n}} \geq \frac{\bar{X}^* - 100}{14/\sqrt{n}}\right) = 0.05 \quad P(Z \geq 1.64) = 0.05$$

$$\frac{\bar{X}^* - 100}{14/\sqrt{n}} = 1.64$$

$$\bar{X}^* = 100 + 1.64 \left(\frac{14}{\sqrt{n}}\right)$$

$$1 - \beta = P(\text{we rejected } H_0 | H_1 \text{ is true})$$

$$= P(\bar{X} \geq \bar{X}^* | \mu = 103)$$

$$= P\left(\frac{\bar{X} - 103}{14/\sqrt{n}} \geq \frac{\bar{X}^* - 103}{14/\sqrt{n}}\right) = 0.60$$

$$P(Z \geq 0.25)$$

$$\frac{\bar{X}^* - 103}{14/\sqrt{n}} = 0.25 \Rightarrow \bar{X}^* = 103 - 0.25 \left(\frac{14}{\sqrt{n}}\right)$$

$$100 + 1.64 \left(\frac{14}{\sqrt{n}}\right) = 103 - 0.25 \left(\frac{14}{\sqrt{n}}\right) \Rightarrow n \cong 78$$

Ex: eski sınav sorusu

Suppose $n=36$ obs. are taken from normal distr. pop. where $\sigma = 8.0$ for testing

$$H_0: \mu = 60$$

$$H_1: \mu \neq 60 \quad \text{at } \alpha = 0.07$$

the lead investigator skipped the statistics class on the day decision rules were being discussed. and s/he intends to reject H_0 if $\bar{X}$ falls in region $(60 - \bar{X}^*, 60 + \bar{X}^*)$

a) Find $\bar{X}^*$

b) what is power of test when true mean is $\mu = 62$?

a) in order $\alpha = 0.07$

$$P(60 - \bar{X}^* < \bar{X} < 60 + \bar{X}^* | \mu = 60) = 0.07$$

$$P\left(\frac{60 - \bar{X}^* - 60}{8/\sqrt{36}} < \frac{\bar{X} - 60}{8/\sqrt{36}} < \frac{60 + \bar{X}^* - 60}{8/\sqrt{36}}\right) =$$

$$P(-0.75 \bar{x}^* < z < 0.75) = 0.07$$

$$= P(-0.09 \leq z < 0.09) = 0.07$$

$$0.75 \bar{x}^* = 0.09 \rightarrow \bar{x}^* = 0.12$$

$$b) 1 - \beta = P(\text{Reject } H_0 \mid H_1 \text{ is true})$$

$$= P(59.88 < \bar{x} < 60.12 \mid \mu = 62) =$$

$$P\left(\frac{59.88 - 62}{8 / \sqrt{36}} < z < \frac{60.12 - 62}{8 / \sqrt{36}}\right)$$

$$P(-1.59 < z < 1.41) = 0.0793 - 0.0559 \\ = 0.0234$$