

Spring 2024-
2025
ECON 205

Problem Set 6 Answers

- 1) Suppose you roll two fair, six-sided dice, one of which is red and the other of which is green. Define the following random variables:

X = The number shown on the red die

$$Y = \begin{cases} 0 & \text{if the two dice show the same number} \\ 1 & \text{if the number on the green die is bigger than the number on the red die} \\ 2 & \text{if the number on the red die is bigger than the number on the green die} \end{cases}$$

- a) Write down a table showing the joint probability mass function for X and Y .
b) Find the marginal probability mass function for Y , and compute its expected value.

SOLUTION

- a. Write down a table showing the joint probability mass function for X and Y .

		$X =$					
		1	2	3	4	5	6
$Y =$	0	1/36	1/36	1/36	1/36	1/36	1/36
	1	5/36	4/36	3/36	2/36	1/36	0
	2	0	1/36	2/36	3/36	4/36	5/36

- b. Find the marginal probability mass function for Y , and compute its expected value.

y	$P(Y = y)$
0	6/36
1	15/36
2	15/36

$$E(Y) = 0 \times (6/36) + 1 \times (15/36) + 2 \times (15/36) = 45/36$$

2)

The random variable X has a range of $\{0,1,2\}$ and the random variable Y has a range of $\{1,2\}$. The joint distribution of X and Y is given by the following table.

		y	
		1	2
x	$p(x,y)$		
	0	0.2	0.1
	1	0.0	0.2
	2	0.3	0.2

- Write down tables for marginal distributions of X and Y .
- Compute $E(X)$ and $E(Y)$.
- Compute $E(XY)$

SOLUTION

- Write down tables for marginal distributions of X and Y .

To find marginal distributions of X and Y , we use the definitions below.

$$p_X(x) = \sum_y p(x,y)$$

$$p_Y(y) = \sum_x p(x,y)$$

Then we can write down the tables for marginal distributions of X and Y.

x	$p_X(x)$
0	0.3
1	0.2
2	0.5

y	$p_Y(y)$
1	0.5
2	0.5

ii) Compute $E(X)$ and $E(Y)$.

$$E(X) = \sum_x x \cdot p_X(x) = 0 * (0.3) + 1 * (0.2) + 2 * (0.5) = 1.2$$

$$E(Y) = \sum_y y \cdot p_Y(y) = 1 * (0.5) + 2 * (0.5) = 1.5$$

iii) Compute $E(XY)$

$$E(XY) = \sum_x \sum_y xy \cdot p(x, y)$$

$$\begin{aligned} E(XY) &= (0 * 1)(0.2) + (0 * 2)(0.1) + (1 * 1)(0.0) + (1 * 2)(0.2) \\ &\quad + (2 * 1)(0.3) + (2 * 2)(0.2) = 1.8 \end{aligned}$$

- 3) Of nine executives in a business firm, four are married, three have never been married, and two are divorced. Three of executives are to be selected for promotion. Let Y_1 denote the number of married executives and Y_2 denote the number of never-married executives among three selected for promotion.

i) Assuming that the three are randomly selected from nine available, find the joint probability of Y_1 and Y_2 .

ii) Find the $\text{Cov}(Y_1, Y_2)$.

SOLUTION

i)

The joint probability function is

$$p(y_1, y_2) = \frac{\binom{4}{y_1} \binom{3}{y_2} \binom{2}{3-y_1-y_2}}{\binom{9}{3}} \quad (1)$$

where $0 \leq y_1, 0 \leq y_2$ and $1 \leq y_1 + y_2 \leq 3$

In the table format, this is

$p(y_1, y_2)$		y_2			
		0	1	2	3
y_1	0	0	3/84	6/84	1/84
	1	4/84	24/84	12/84	0
	2	12/84	18/84	0	0
	3	4/84	0	0	0

ii)

To find covariance between Y_1 and Y_2 , we first compute $E(Y_1)$, $E(Y_2)$ and $E(Y_1Y_2)$.

$$E(Y_1) = \sum_y y_1 \cdot p_{Y_1}(y_1) = \frac{10}{84} * 0 + \frac{40}{84} * 1 + \frac{30}{84} * 2 + \frac{4}{84} * 3 = \frac{4}{3} \quad (2)$$

$$E(Y_2) = \sum_y y_2 \cdot p_{Y_2}(y_2) = \frac{20}{84} * 0 + \frac{45}{84} * 1 + \frac{18}{84} * 2 + \frac{1}{84} * 3 = 1 \quad (3)$$

$$E(Y_1Y_2) = 1 * 1 * \frac{24}{84} + 2 * 1 * \frac{12}{84} + 1 * 2 * \frac{18}{84} = 1 \quad (4)$$

Therefore, $Cov(Y_1, Y_2)$ can be calculated as

$$Cov(Y_1, Y_2) = 1 - \left(\frac{4}{3}\right) 1 = -\frac{1}{3}$$

4) Suppose that Y_1 and Y_2 are random variables and show that

$$V(Y_1+Y_2)=V(Y_1)+V(Y_2)+2Cov(Y_1,Y_2).$$

SOLUTION

Proof. Let $Z = Y_1 + Y_2$. Using the definition of variance, we have

$$\begin{aligned} V(Z) &= E[(Z - \mu_Z)^2] \\ &= E\{[(Y_1 + Y_2) - E(Y_1 + Y_2)]^2\} \\ &= E[(Y_1 + Y_2 - \mu_{Y_1} - \mu_{Y_2})^2] \\ &= E\{[(Y_1 - \mu_{Y_1}) + (Y_2 - \mu_{Y_2})]^2\} \\ &= E[(Y_1 - \mu_{Y_1})^2 + (Y_2 - \mu_{Y_2})^2 + 2 \underbrace{(Y_1 - \mu_{Y_1})(Y_2 - \mu_{Y_2})}_{\text{cross product}}] \\ &= E[(Y_1 - \mu_{Y_1})^2] + E[(Y_2 - \mu_{Y_2})^2] + 2E[(Y_1 - \mu_{Y_1})(Y_2 - \mu_{Y_2})] \\ &= V(Y_1) + V(Y_2) + 2Cov(Y_1, Y_2). \end{aligned}$$

5) Let Y_1 and Y_2 have joint density function

$$f(y_1, y_2) = \begin{cases} e^{-(y_1+y_2)}, & y_1 > 0, y_2 > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

- a. What is $P(Y_1 < 1, Y_2 > 5)$?
- b. What is $P(Y_1 + Y_2 < 3)$?

SOLUTION

a. $P(Y_1 < 1, Y_2 > 5) = \int_0^1 \int_5^\infty e^{-(y_1+y_2)} dy_1 dy_2 = \left[\int_0^1 e^{-y_1} dy_1 \right] \left[\int_5^\infty e^{-y_2} dy_2 \right] = [1 - e^{-1}] e^{-5} = .00426.$

b. $P(Y_1 + Y_2 < 3) = P(Y_1 < 3 - Y_2) = \int_0^3 \int_0^{3-y_2} e^{-(y_1+y_2)} dy_1 dy_2 = 1 - 4e^{-3} = .8009.$

6) The joint density function of Y_1 and Y_2 is given by

$$f(y_1, y_2) = \begin{cases} 30y_1y_2^2, & y_1 - 1 \leq y_2 \leq 1 - y_1, 0 \leq y_1 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

- a. Find $F(1/2, 1/2)$.
- b. Find $F(1/2, 2)$.
- c. Find $P(Y_1 > Y_2)$.

SOLUTION

a. $F(1/2, 1/2) = \int_0^{1/2} \int_{y_1-1}^{1/2} 30y_1y_2^2 dy_2 dy_1 = \frac{9}{16}.$

b. Note that:

$$F(1/2, 2) = F(1/2, 1) = P(Y_1 \leq 1/2, Y_2 \leq 1) = P(Y_1 \leq 1/2, Y_2 \leq 1/2) + P(Y_1 \leq 1/2, Y_2 > 1/2)$$

So, the first probability statement is simply $F(1/2, 1/2)$ from part a. The second probability statement is found by

$$P(Y_1 \leq 1/2, Y_2 > 1/2) = \int_{1/2}^1 \int_0^{1-y_2} 30y_1y_2^2 dy_1 dy_2 = \frac{4}{16}.$$

Thus, $F(1/2, 2) = \frac{9}{16} + \frac{4}{16} = \frac{13}{16}.$

c. $P(Y_1 > Y_2) = 1 - P(Y_1 \leq Y_2) = 1 - \int_0^{1/2} \int_{y_1}^{1-y_1} 30y_1y_2^2 dy_2 dy_1 = 1 - \frac{11}{32} = \frac{21}{32} = .65625.$

7)

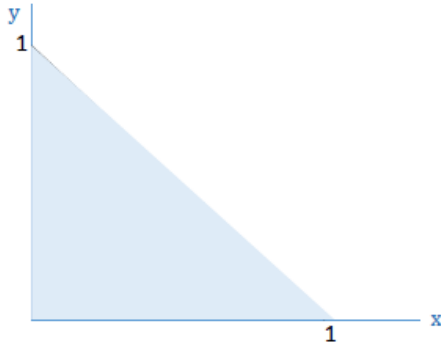
Let X and Y be jointly continuous random variables with joint pdf

$$f(x, y) = \begin{cases} cx + 1, & x, y \geq 0, x + y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

- i) Show the range of (X, Y) , in the x-y plane.
- ii) Find the constant c .
- iii) Find the marginal pdf's $f_X(x)$ and $f_Y(y)$.
- iv) Find $P(Y < 2X^2)$.

SOLUTION

- i) Show the range of (X, Y) , in the x - y plane.



- ii) Find the constant c .

To find the constant c , we write

$$\begin{aligned}\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy &= 1 \\ \int_0^1 \int_0^{1-x} (cx + 1) dy dx &= 1 \\ \int_0^1 (cx + 1)(1 - x) dx &= 1 \\ \int_0^1 (-cx^2 + (c - 1)x + 1) dx &= 1 \\ \frac{1}{2} + \frac{1}{6}c &= 1 \rightarrow c = 3\end{aligned}$$

- iii) Find the marginal pdf's $f_X(x)$ and $f_Y(y)$.

We first note that $R_X = R_Y = [0, 1]$.

$$\begin{aligned}f_X(x) &= \int_{-\infty}^{\infty} f(x, y) dy \\ f_X(x) &= \int_0^{1-x} (3x + 1) dy \\ f_X(x) &= (3x + 1)(1 - x), \text{ for } x \in [0, 1]\end{aligned}$$

Thus, we have

$$f_X(x) = \begin{cases} (3x+1)(1-x), & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Similarly, we obtain

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx$$
$$f_Y(y) = \int_0^{1-y} (3x+1)dx$$

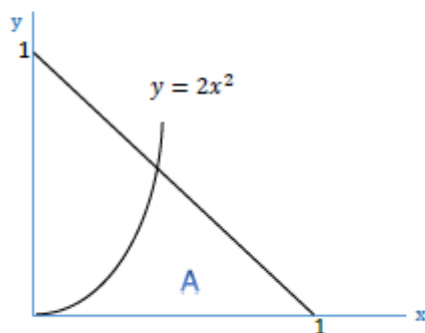
$$f_Y(y) = \frac{1}{2}(1-y)(5-3y), \text{ for } y \in [0, 1]$$

Thus, we have

$$f_Y(y) = \begin{cases} \frac{1}{2}(1-y)(5-3y), & 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

iv) Find $P(Y < 2X^2)$.

To find $P(Y < 2X^2)$, we need to integrate $f(x,y)$ over the region A in figure below.



We have

$$\begin{aligned}
P(Y < 2X^2) &= \int_{-\infty}^{\infty} \int_{-\infty}^{2x^2} f(x, y) dy dx \\
P(Y < 2X^2) &= \int_0^1 \int_0^{\min(2x^2, 1-x)} (3x+1) dy dx \\
P(Y < 2X^2) &= \int_0^1 (3x+1) \min(2x^2, 1-x) dx \\
P(Y < 2X^2) &= \int_0^{\frac{1}{2}} 2x^2(3x+1) dx + \int_{\frac{1}{2}}^1 (3x+1)(1-x) dx \\
P(Y < 2X^2) &= \left(\frac{3x^4}{2} + \frac{2x^3}{3} \right) \Big|_{x=0}^{x=1/2} + (-x^3 + x^2 + x) \Big|_{x=1/2}^{x=1} \\
P(Y < 2X^2) &= \frac{53}{96}
\end{aligned}$$

8)

Assume we have the following form;

$$f(x, y) = \begin{cases} 6yx^2, & 0 < y < 1 \text{ and } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Is $f(x, y)$ a joint probability density function? If so calculate the following:

a) The marginal densities of X and Y .

b) $P(X + Y > 1)$

c) $P\left(\frac{1}{3} < Y < \frac{1}{2}\right)$.

SOLUTION

$$f(x, y) = \begin{cases} 6yx^2, & 0 < y < 1 \text{ and } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1 \quad \forall x, y \quad 0 \leq f(x, y)$$

$$\int_0^1 \int_0^1 6yx^2 dx dy = 1 \text{ since}$$

$$\int_0^1 6yx^2 dx = \frac{6yx^3}{3} \Big|_{x=0}^{x=1} = 2y$$

$$\int_0^1 2y dy = y^2 \Big|_{y=0}^{y=1} = 1$$

a) $f_x(x) = \int_{-\infty}^{\infty} f(x, y) dy$

If $0 < x < 1$, $f_x(x) = \int_0^1 6yx^2 dy = 3y^2 x^2 \Big|_{y=0}^{y=1} = 3x^2$

$$f_x(x) = \begin{cases} 3x^2, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_y(y) = \int_{-\infty}^{\infty} f(x, y) dx$$

If $0 < y < 1$, $f_y(y) = \int_0^1 6yx^2 dx = 2yx^3 \Big|_{x=0}^{x=1} = 2y$

$$f_y(y) = \begin{cases} 2y, & 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

b) $P(x+y > 1)$

$$\begin{aligned} P(y > 1-x) &= \int_0^1 \int_{1-x}^1 6yx^2 dy dx = \int_0^1 \left(3y^2 x^2 \right) \Big|_{y=1-x}^{y=1} dx = \int_0^1 (3x^2 - 3(1-x)^2 x^2) dx \\ &= \int_0^1 (6x^3 - 3x^4) dx = \frac{6}{4} x^4 \Big|_0^1 - \frac{3}{5} x^5 \Big|_0^1 = \frac{6}{4} - 0 - \left(\frac{3}{5} - 0 \right) = \frac{9}{10} \end{aligned}$$

c) $P\left(\frac{1}{3} < Y < \frac{1}{2}\right) = \int_{1/3}^{1/2} f_y(y) dy = \int_{1/3}^{1/2} 2y dy = y^2 \Big|_{1/3}^{1/2} = \frac{1}{4} - \frac{1}{9} = \frac{5}{36}$