

Spring 2024-
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ECON 205

Problem Set 3 – Answers

- 1) A single cell can either die, with probability .1, or split into two cells, with probability .9, producing a new generation of cells. Each cell in the new generation dies or splits into two cells independently with the same probabilities as the initial cell. Find the probability distribution for the number of cells in the next generation (generation at $t=2$).

SOLUTION

Note that the number of cells cannot be odd.

$$p(0) = P(\text{no cells in the next generation}) = P(\text{the first cell dies or the first cell splits and both die}) = .1 + .9(.1)(.1) = 0.109$$

$$p(4) = P(\text{four cells in the next generation}) = P(\text{the first cell splits and both created cells split}) = .9(.9)(.9) = 0.729.$$

$$p(2) = 1 - .109 - .729 = 0.162.$$

- 2) Suppose that we survey 20 individuals working for a large company and ask each whether they favor implementation of a new policy regarding retirement funding. If, in our sample, 6 favored the new policy, find the estimate for p , the true but unknown proportion of employees that favor the new policy.

SOLUTION

If Y denotes the number among the 20 who favor the new policy, it is reasonable to conclude that Y has a binomial distribution with $n = 20$ for some value of p . Whatever the true value for p , we conclude that the probability of observing 6 out of 20 in favor of the policy is

$$P(Y = 6) = \binom{20}{6} p^6 (1 - p)^{14}$$

We will use as our estimate for p the value that maximizes the probability of observing the value that we actually observed (6 in favor in 20 trials). How do we find the value of p that maximizes $P(Y = 6)$?

Because $\binom{20}{6}$ is a constant (relative to p) and $\ln(w)$ is an increasing function of w , the value of that maximizes $P(Y=6) = \binom{20}{6} p^6 (1 - p)^{14}$ is the same as the value of p that maximizes $\ln[p^6(1 - p)^{14}] = [6 \ln(p) + 14 \ln(1 - p)]$.

If we take the derivative of $[6 \ln(p) + 14 \ln(1 - p)]$ with respect to p , we obtain

$$\frac{d[6 \ln(p) + 14 \ln(1 - p)]}{dp} = \frac{6}{p} - \frac{14}{1-p}$$

The value of p that maximizes (or minimizes) $[6 \ln(p) + 14 \ln(1 - p)]$ [and, more important, $P(Y = 6)$] is the solution to the equation

$\frac{6}{p} - \frac{14}{1-p} = 0$. Solving, we obtain $p = 6/20$. Because the second derivative of $[6 \ln(p) + 14 \ln(1 - p)]$ is negative when $p = 6/20$, it follows that $[6 \ln(p) + 14 \ln(1 - p)]$ [and $P(Y = 6)$] is maximized when $p = 6/20$.

- 3) A large stockpile of used pumps contains 20% that are in need of repair. A maintenance worker is sent to the stockpile with three repair kits. She selects pumps at random and tests them one at a time. If the pump works, she sets it aside for future use. However, if the pump does not work, she uses one of her repair kits on it. Suppose that it takes 10 minutes to test a pump that is in working condition and 30 minutes to test and repair a pump that does not work. Find the mean and variance of the total time it takes the maintenance worker to use her three repair kits.

SOLUTION

Let Y denote the number of the trial on which the third nonfunctioning pump is found. It follows that Y has a negative binomial distribution with $p = .2$. Thus, $E(Y) = 3/(.2) = 15$ and $V(Y) = 3(.8)/(.2)^2 = 60$. Because it takes an additional 20 minutes to repair each defective pump, the total time necessary to use the three kits is

$$T = 10Y + 3(20).$$

Using the result derived in Exercise 3.33, we see that

$$E(T) = 10E(Y) + 60 = 10(15) + 60 = 210$$

and

$$V(T) = 10^2 V(Y) = 100(60) = 6000.$$

- 4) Assume that the probability that a patient recovers from a disease is 0.8. Suppose 20 people have contracted this disease. What is the probability that
- (a) exactly 13 recover?
 - (b) at least 11 recover?
 - (c) at most 16 recover?

SOLUTION

(a)

Let's call X = the number of people who recover. We need to find $P(X=13)$ where $n = 20$ and $p = 0.8$. We use the following formula.

$$P(X = x) = b(x; n, p) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$P(X = 13) = b(x; n, p) = \binom{20}{13} (0.8)^{13} (0.2)^7$$

$$\begin{aligned} P(X = 13) &= \frac{20!}{13! * 7!} (0.8)^{13} (0.2)^7 \\ &= \frac{20 * 19 * 18 * 17 * 16 * 15 * 14}{7 * 6 * 5 * 4 * 3 * 2 * 1} (0.8)^{13} (0.2)^7 \\ &= 77,520 * 0.0549 * 0.0000128 = 0.0545 \end{aligned}$$

(b)

We are looking for

$$P(X \geq 11) = 1 - P(X \leq 10)$$

$$1 - P(X \leq 10) = 1 - B(10; 20, 0.8) = 1 - \sum_{y=0}^{10} b(y; 20, 0.8)$$

For this part of the question we can use the cumulative binomial probabilities table below.

d. $n = 20$

		p														
		0.01	0.05	0.10	0.20	0.25	0.30	0.40	0.50	0.60	0.70	0.75	0.80	0.90	0.95	0.99
x	0	.818	.358	.122	.012	.003	.001	.000	.000	.000	.000	.000	.000	.000	.000	.000
	1	.983	.736	.392	.069	.024	.008	.001	.000	.000	.000	.000	.000	.000	.000	.000
	2	.999	.925	.677	.206	.091	.035	.004	.000	.000	.000	.000	.000	.000	.000	.000
	3	1.000	.984	.867	.411	.225	.107	.016	.001	.000	.000	.000	.000	.000	.000	.000
	4	1.000	.997	.957	.630	.415	.238	.051	.006	.000	.000	.000	.000	.000	.000	.000
	5	1.000	1.000	.989	.804	.617	.416	.126	.021	.002	.000	.000	.000	.000	.000	.000
	6	1.000	1.000	.998	.913	.786	.608	.250	.058	.006	.000	.000	.000	.000	.000	.000
	7	1.000	1.000	1.000	.968	.898	.772	.416	.132	.021	.001	.000	.000	.000	.000	.000
	8	1.000	1.000	1.000	.990	.959	.887	.596	.252	.057	.005	.001	.000	.000	.000	.000
	9	1.000	1.000	1.000	.997	.986	.952	.755	.412	.128	.017	.004	.001	.000	.000	.000
	10	1.000	1.000	1.000	.999	.996	.983	.872	.588	.245	.048	.014	0.003	.000	.000	.000
	11	1.000	1.000	1.000	1.000	.999	.995	.943	.748	.404	.113	.041	.010	.000	.000	.000
	12	1.000	1.000	1.000	1.000	1.000	.999	.979	.868	.584	.228	.102	0.032	.000	.000	.000
	13	1.000	1.000	1.000	1.000	1.000	1.000	.994	.942	.750	.392	.214	.087	.002	.000	.000
	14	1.000	1.000	1.000	1.000	1.000	1.000	.998	.979	.874	.584	.383	.196	.011	.000	.000
	15	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.994	.949	.762	.585	.370	.043	.003	.000
	16	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.999	.984	.893	.775	0.589	.133	.016	.000
	17	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.996	.965	.909	.794	.323	.075	.001
	18	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.999	.992	.976	0.931	.608	.264	.017
	19	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.999	.997	.988	.878	.642	.182

$$P(X \geq 11) = 1 - P(X \leq 10) = 1 - 0.003 = 0.997$$

(c)

$$P(X \leq 16) = 0.589$$

5) The monthly worldwide average number of airplane crashes for commercial airplanes is 3.5. What is the probability that there will be

- (i) at most 1 accident in the next month?
- (ii) at least 2 such accidents in the next month?

SOLUTION

$$p(y) = \frac{\lambda^y e^{-\lambda}}{y!}, E(y) = \lambda \text{ and } Var(Y) = \sigma^2 = \lambda$$

(i) At most 1

$$P(X \leq 1) = P(X = 0) + P(X = 1)$$

$$P(X \leq 1) = \frac{e^{-3.5}(3.5)^0}{0!} + \frac{e^{-3.5}(3.5)^1}{1!} = 4.5e^{-3.5} = 0.1359$$

(ii) At least 2

$$P(X \geq 2) = 1 - P(X = 1) - P(X = 0)$$

$$P(X \geq 2) = 1 - \frac{e^{-3.5}(3.5)^1}{1!} - \frac{e^{-3.5}(3.5)^0}{0!} = 1 - 3.5e^{-3.5} - e^{-3.5} = 1 - 4.5e^{-3.5} = 0.8641$$

6) In southern California, a growing number of individuals pursuing teaching credentials are choosing paid internships over traditional student teaching programs. A group of eight candidates for three local teaching positions consisted of five who had enrolled in paid internships and three who enrolled in traditional student teaching programs. All eight candidates appear to be equally qualified, so three are randomly selected to fill the open positions. Let Y be the number of internship trained candidates who are hired.

- (i) Does Y have a binomial or hypergeometric distribution? Why?
- (ii) Find the probability that two or more internship trained candidates are hired.

SOLUTION

- (i) The random variable Y follows a hypergeometric distribution. The probability of being chosen on a trial is dependent on the outcome of previous trials.

$$(ii) \quad P(Y \geq 2) = P(Y = 2) + P(Y = 3) = \frac{\binom{5}{2}\binom{3}{1}}{\binom{8}{3}} + \frac{\binom{5}{3}}{\binom{8}{3}} = 0.5357 + 0.1786 = 0.7143$$

- 7) If 40% of the employees of a firm with a large number of employees have positive indications of asbestos in their lungs, find the probability that ten employees must be tested in order to find three positives.

SOLUTION

Let Y = # of employees tested until three positives are found. Then, Y is negative binomial with $r=3$ and $p=0.4$. $P(Y = 10) = \binom{9}{2} * (0.4)^3 * (0.6)^7 = 0.06$

- 8) The number of imperfections in the weave of a certain textile has a Poisson distribution with a mean of 4 per square yard. Find the probability that a 3-square-yard sample will contain at least one imperfection.

SOLUTION

Let W = # of imperfections in three-square yards of weave. Then, W is Poisson with $\lambda=12$.

$$P(W \geq 1) = 1 - P(W = 0) = 1 - \frac{e^{-12}(12)^0}{0!} = 1 - e^{-12}$$

- 9) One concern of a gambler is that she will go broke before achieving her first win. Suppose that she plays a game in which the probability of winning is 0.1 (and is unknown to her). It costs her \$10 to play and she receives \$80 for a win. If she commences with \$30, what is the probability that she wins exactly once before she loses her initial capital.

SOLUTION

The possible outcomes of interest are:

WLLLLLLLLLL, LWLLLLLLLLLL, LLWLLLLLLLLL

So the desired probability is $.1(.9)^{10} + .9(.1)(.9)^9 + (.9)^2(.1)(.9)^8 = 3(.1)(.9)^{10} = .104$.