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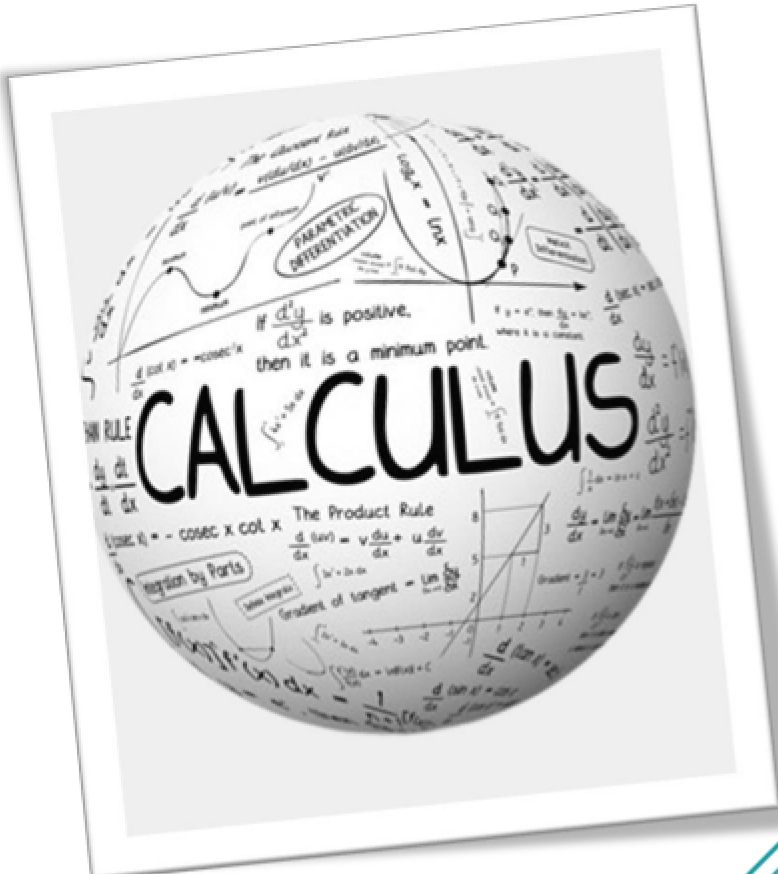
LECTURE NOTES (DERS NOTLARI)

[MATH 119 BOOKLET]



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RISHA BOOKLET
Road To Success

CALCULUS WITH ANALYTIC GEOMETRY

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INTRODUCTION

This booklet is for those who are taking Math119 (Calculus with Analytic Geometry).

What does the booklet offer?

1. Sufficient way to **read** the calculus subjects.
2. Organized way to **understand** calculus theorems and some proves as well as some necessary rules.
3. Sample examples to **understand** how to solve the related problems
4. Some suggested problems to **practice**
5. Appendix which contains functions and their graphs Geometric formulas
6. Short answer key of the suggested problems as well as full solution in the answer key booklet.

$$Read \dashrightarrow Understand \dashrightarrow Practice \dashrightarrow AA$$

What are the external booklets offered?

1. Notebook Booklet (This booklet)
2. Answer key Booklet (available via lms.rishabooklet.com website)
3. Sample midterms and Final exam Booklet (available via lms.rishabooklet.com website)
4. Brief Booklet (available via lms.rishabooklet.com website)

To have chance of having mock/sample exams as well as other online extra recourses and support, you may join MATH119 Course page via https://lms.rishabooklet.com

If you have any inquiries, please do not hesitate to contact me via

Email: *rishabooklet@gmail.com*

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I hope such a work helps you to reach AA letter grade.

Best Regards,

Risha

Booklet Sections (RISHA BOOKLET)	Book Chapters (A Complete Course Calculus)
Section 1: How to solve limit problems	1.2 Limits of Functions 1.3 Limits at Infinity and Infinite Limits
Section 2: Continuity and Differentiation	1.4 Continuity Ch 2: Differentiation
Section 3: Derivatives	3.1 Inverse Functions 3.2 Exponential and Logarithmic Functions 3.3 The Natural Logarithm and Exponential 3.5 The Inverse Trigonometric Functions 3.6 Hyperbolic Functions
Section 4: Tangent Lines and Their Slope	2.1 Tangent Lines and Their Slope
Section 5: Prove Rules with their conditions	1.5 The Formal Definition of Limit 2.8 The Mean-Value Theorem 1.4 Continuity (Intermediate value theorem)
Section 6: Related Rates	4.1 Related Rates
Section 7: Indeterminate Forms	4.3 Indeterminate Forms
Section 8: Curve Sketching Steps	4.4 Extreme Values 4.5 Concavity and Inflections 4.6 Sketching the Graph of a Function
Section 9: Extreme-Value Problem	4.8 Extreme-Value Problems
Section 10: Linear Approximations	4.9 Linear Approximations
Section 11: Sums and Sigma Notation	5.1 Sums and Sigma Notation 5.2 Areas as Limits of Sums
Section 12: The Definite Integral	5.3 The Definite Integral
Section 13: Properties of the Definite Integral	5.4 Properties of the Definite Integral
Section 14: The Fundamental Theorem of Calculus	5.5 The Fundamental Theorem of Calculus
Section 15.1: Integral Properties	5.3 The Definite Integral
Section 15.2: Substitution	5.6 The Method of Substitution
Section 15.3: Integration by Parts	6.1 Integration by Parts
Section 15.4: Partial Fractions	6.2 Integrals of Rational Functions
Section 15.5: Trig Substitutions	6.3 Inverse Substitutions
Section 16: Improper Integrals	6.5 Improper Integrals
Section 17: Areas of Plane Regions	5.7 Areas of Plane Regions
Section 18: Volumes by Slicing-Solids of Revolution	7.1 Volumes by Slicing-Solids of Revolution
Section 19: Arc Length and surface area	7.3 Arc Length and surface area
Section 20: Polar Coordinates and Polar Curves	8.5 Polar Coordinates and Polar Curves

NOTE 1.1

To compute $\lim\limits_{x\to a} f(x) = L \quad or \quad \lim\limits_{x\to a} g(x) = M$

<u>You need to know the limit rules:</u>

- Limit of a sum or difference: $\lim\limits_{x\to a} [f(x) \pm g(x)] = L \pm M$
- Limit of a product: $\lim\limits_{x\to a} [f(x)g(x)] = LM$
- Limit of a quotient: $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M} \qquad where\ \ M \neq 0$
- Limit of a power: $\lim\limits_{x\to a} [\,f(x)]^n = L^n \ \ ,n\ \in \mathbb{Z}\ (integer)$

NOTE 1.2

To compute $\lim\limits_{x\to a} f(x) = L \quad or \quad \lim\limits_{x\to a} g(x) = M$

<u>You need to know how to solve the limit:</u>

<u>Step 1:</u> substitute the value of $\boldsymbol{a}$ directly into the given function $f(x)\ or\ g(x)$.

- If the result of the limit is restricted to one specific value or <u>infinity</u>, then this determines the limit being sought.

<u>A limit confirmed to be infinity is not indeterminate since it has been determined to have a specific value (infinity)</u>

- However, in case the result of the limit is NOT restricted to one specific value, having the indeterminate form of $\left(\frac{0}{0},\ \frac{\infty}{\infty}, \infty - \infty, 0^0,\ 1^{\infty}, \infty^0,\ and\ 0 \times \infty\right)$, then you need to follow **<u>step 2</u>**

__<u>Remember that</u>__ L'Hôpital's theorem (given in section 7) should not be used at this part since it involves taking derivatives. In the exam, if you are NOT asked to use this theorem, then Instructors will NOT give any credit for any answer that includes

Example 1

Evaluate $\lim\limits_{x\to 1} \frac{x^2-x+2}{x+1}$

Answer

$$\lim_{x\to 1} \frac{x^2 - x + 2}{x+1} = \frac{2}{2} = 1$$

Example 2

Evaluate $\lim_{x \to 1} \frac{x^2-x+2}{x-1}$

Answer

$$\lim_{x \to 1} \frac{x^2 - x + 2}{x - 1} = \frac{2}{0} = \infty$$

NOTE 1.3

__Step 2:__ If the result of the limit is NOT restricted to one specific value, then you have two cases

- Type I: **Use Two-sided Limits** (Use of the absolute functions in limits)

$$\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x) \text{ or } \lim_{x \to a^-} g(x) \neq \lim_{x \to a^+} g(x)$$

  - for this case, the given limit __Does Not Exist__ (D.N.E)

- Type II: having the indeterminate form of $\left(\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0^0, 1^\infty, \infty^0, \text{ and } 0 \times \infty\right)$
  - then you need to simplify the expression. Simplification can involve any number of techniques including but certainly __not limited__ to the 6 given techniques in the following notes.

__Type I__: **Use Two-sided Limits** (Use of the absolute functions in limits) ***(chapter 1.2)***

NOTE 1.4

__Theorem__: A function $f(x)$ has limit L at $x = a$ if and only if it has both the right and left limits there and these one-sided limits are both equal to L:

$$\lim_{x \to a} f(x) = L \iff \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$$

Example 3

Evaluate $\lim_{x \to -2} \frac{x^2-4}{x\,|x+2|}$

Answer

$$\lim_{x \to -2} \frac{x^2 - 4}{x\,|x + 2|} = \begin{cases} \lim_{x \to -2^+} \frac{x^2 - 4}{x(x + 2)} = \lim_{x \to -2^+} \frac{(x - 2)\cancel{(x + 2)}}{x\cancel{(x + 2)}} = \lim_{x \to -2^+} \frac{(x - 2)}{x} = \frac{-4}{-2} = 2 & \to (1) \\ \lim_{x \to -2^-} \frac{x^2 - 4}{-x\,(x + 2)} = \lim_{x \to -2^-} \frac{(x - 2)\cancel{(x + 2)}}{-x\cancel{(x + 2)}} = \lim_{x \to -2^-} \frac{(x - 2)}{-x} = \frac{-4}{2} = -2 & \to (2) \end{cases}$$

From (1) and (2), we can conculde that $\lim_{x\to -2} \frac{x^2-4}{x\,|x+2|}$ ***Does Not Exist*** $(\boldsymbol{D.N.E})$ *Since both right and left side are NOT equal.*

NOTE 1.5

$\lim_{x\to\infty} \frac{1}{x} = 0$ $\qquad$ $\lim_{x\to 0^+} \frac{1}{x} = +\infty$ $\qquad$ $\lim_{x\to 0^-} \frac{1}{x} = -\infty$ $\qquad$ $\lim_{x\to 0} \frac{1}{x}$ ***Does Not Exist*** $(\boldsymbol{D.N.E})$

NOTE 1.6

For absolute limit questions, you need to know when you are going to use positive or negative sign.

$\lim_{x\to 2} f(x)$	Hints	$\lim_{x\to 2^+} f(x)$ $x>2$	$\lim_{x\to 2^-} f(x)$ $x<2$
$\lvert x-2\rvert$	$\begin{cases} x-2 & x>2 \\ -(x-2) & x<2 \end{cases} \Rightarrow x=2$	$x-2$	$-(x-2)$
$\lvert x-4\rvert$	$\begin{cases} x-4 & x>4 \\ -(x-4) & x<4 \end{cases} \Rightarrow x=4$	$-(x-4)$	$-(x-4)$
$\lvert x+2\rvert$	$\begin{cases} x+2 & x>-2 \\ -(x+2) & x<-2 \end{cases} \Rightarrow x=-2$	$x+2$	$x+2$

NOTE 1.7

<u>Type II:</u> having the indeterminate form of $\left(\frac{0}{0}, \frac{\infty}{\infty}, \infty-\infty, 0^0, 1^\infty, \infty^0, \text{ and } 0\times\infty\right)$

- then you need to simplify the expression. Simplification can involve any number of techniques including but certainly <u>not limited</u> to the 6 given techniques.

❖ **Technique 1:** Factoring *(chapter 1.2)*

Example 4

Evaluate $\lim_{x\to 1} \frac{x^2+2x-3}{x+1}$

Answer

$$\lim_{x\to 1} \frac{x^2+2x-3}{x-1} = \lim_{x\to 1} \frac{(x+3)\cancel{(x-1)}}{\cancel{x-1}} = \lim_{x\to 1} x+3 = 4$$

factoring the limit to get rid of $(x-1)$

❖ **Technique 2:** Using Squeeze Theorem *(chapter 1.2)*

NOTE 1.8

Theorem: suppose that $g(x) \leq f(x) \leq h(x)$ holds for all x in some open interval containing a, except possibly at $x = a$ itself. Suppose also that $\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$ Then, $\lim_{x \to a} f(x) = L$

Also, similar statements hold for left and right limits.

Remember that:

$$-1 \leq \sin(x) \leq 1 \qquad 0 \leq |\sin(x)| \leq 1 \qquad 0 \leq \sin^2(x) \leq 1$$

$$-1 \leq \cos(x) \leq 1 \qquad 0 \leq |\cos(x)| \leq 1 \qquad 0 \leq \cos^2(x) \leq 1$$

Example 5

Evaluate $\lim_{x \to \infty} \frac{2\sin(x^3)}{x^3+1}$

Answer

$$\lim_{x \to \infty} \frac{2\sin(x^3)}{x^3+1} \qquad since\ -1 \leq \sin(x^3) \leq 1 \qquad \therefore \quad \lim_{x \to \infty} \frac{-2}{x^3+1} \leq \lim_{x \to \infty} \frac{2\sin(x^3)}{x^3+1} \leq \lim_{x \to \infty} \frac{2}{x^3+1}$$

$$since \quad \lim_{x \to \infty} \frac{2}{x^3+1} = \lim_{x \to \infty} \frac{-2}{x^3+1} = 0 \qquad \therefore \lim_{x \to \infty} \frac{2\sin(x^3)}{x^3+1} = 0 \quad By\ Squeeze\ Theorem$$

NOTE 1.9

Squeeze theorem is applicable for $\lim_{x \to \infty} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} = 0$, Not for $\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to \infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} = 1$

❖ **Technique 3:** Use of the limit rules Finding a common denominator (chapter 1.3)

Example 6

Evaluate $\lim_{x\to\infty}\frac{2x+\sqrt[4]{81x^6+x^5-3x+7}}{5x+8+\sqrt{x^3}+x-2}$

Answer

Check the highest power in the denominator then factor it out from both denominator and nominator $\left(x^{\frac{3}{2}}\right)$

$$\lim_{x\to\infty}\frac{\frac{2x}{x^{\frac{3}{2}}}+\sqrt[4]{\frac{81x^6}{x^6}+\frac{x^5}{x^6}-\frac{3x}{x^6}+\frac{7}{x^6}}}{\frac{5x}{x^{\frac{3}{2}}}+\frac{8}{x^{\frac{3}{2}}}+\sqrt{\frac{x^3}{x^3}+\frac{x}{x^3}-\frac{2}{x^3}}}=\lim_{x\to\infty}\frac{\left(2\left(\frac{1}{x^{\frac{1}{2}}}\right)+\sqrt[4]{81+\frac{1}{x}-3\left(\frac{1}{x^5}\right)+7\left(\frac{1}{x^6}\right)}\right)}{\left(5\left(\frac{1}{x^{\frac{1}{2}}}\right)+8\left(\frac{1}{x^{\frac{3}{2}}}\right)+\sqrt{1+\frac{1}{x^2}-2\left(\frac{1}{x^3}\right)}\right)}$$

$$=\frac{\left(0+\sqrt[4]{81+0-3(0)+7(0)}\right)}{\left(0+8(0)+\sqrt{1+0-2(0)}\right)}=\frac{\sqrt[4]{81}}{1}=3 \qquad \left(where\ \lim_{x\to\infty}\frac{1}{x}=0\right)$$

NOTE 1.10

- The general formula of $\lim_{x\to\infty}\frac{1}{x}$ is $\lim_{x\to\infty}\frac{1}{x^p}=0 \quad where \quad 0<p<\infty$
- $\sqrt{x^2}=|x|\to\begin{cases} x & x\to\infty \\ -x & x\to-\infty\end{cases}$ $\qquad \sqrt{x^4}=|x^2|\to\begin{cases} x^2 & x\to\infty \\ -x^2 & x\to-\infty\end{cases}$

❖ **Technique 4:** Multiplying by the conjugate (chapter 1.3)

Example 7

Evaluate $\lim_{x\to\infty}\left(x-\sqrt{x^2+x+2}\right)$

Answer

$$\lim_{x\to\infty}\left(x-\sqrt{x^2+x+2}\right)\left(\frac{x+\sqrt{x^2+x+2}}{x+\sqrt{x^2+x+2}}\right)=\lim_{x\to\infty}\frac{x^2-(x^2+x+2)}{x+\sqrt{x^2+x+2}}$$

$$=\lim_{x\to\infty}\frac{-x-2}{x+\sqrt{x^2+x+2}}=\lim_{x\to\infty}\frac{-\frac{x}{x}-\frac{2}{x}}{\frac{x}{x}+\sqrt{\frac{x^2}{x^2}+\frac{x}{x^2}+\frac{2}{x^2}}}=\lim_{x\to\infty}\frac{-1-2\left(\frac{1}{x}\right)}{1+\sqrt{1+\frac{1}{x}+2\left(\frac{1}{x^2}\right)}}=\frac{-1}{2} \qquad \left(where\ \lim_{x\to\infty}\frac{1}{x}=0\right)$$

❖ **Technique 5:** Applying some memorized limit (chapter 2.5)

NOTE 1.11

$$\lim_{x \to \infty} \frac{\sin x}{x} = 0 \qquad \lim_{x \to \infty} \frac{\cos x}{x} = 0 \qquad \lim_{x \to 0} \frac{\sin x}{x} = 1$$

Example 8

Evaluate $\lim_{x\to 0} \frac{\tan 5x}{\sin 3x}$

Answer

$$\lim_{x\to 0} \frac{\tan 5x}{\sin 3x} = \left(\lim_{x\to 0} \frac{\sin 5x}{\cos 5x}\right)\left(\lim_{x\to 0} \frac{1}{\sin 3x}\right) = \left(\lim_{x\to 0} \frac{\sin 5x}{5x}\right)\left(\lim_{x\to 0} \frac{1}{\cos 5x}\right)\left(\lim_{x\to 0} \frac{3x}{\sin 3x}\right)\left(\frac{5}{3}\right) = (1)(1)(1)\left(\frac{5}{3}\right) = \frac{5}{3}$$

$$\Longrightarrow where \lim_{x \to 0} \frac{\sin x}{x} = 1$$

❖ **Technique 6:** Using the definition of the Derivative (chapter 2.7)

NOTE 1.12

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

Example 9

Evaluate $\lim_{x\to 0} \frac{\cos(x)-1}{x}$

Answer

$$\lim_{x\to 0} \frac{\cos(x) - 1}{x - 0} = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = f'(0)$$

$$where\ f(x) = \cos(x) \qquad \therefore f'(x) = -\sin x \qquad \therefore f'(0) = 0$$

NOTE 1.13

Some Continuous Functions

Partial list of continuous functions and the values of x for which they are continuous.

1. Polynomials for all x.
2. Rational function, except for x's that give division by zero.
3. $\sqrt[n]{x}$ (n odd) for all x.
4. $\sqrt[n]{x}$ (n even) for all $x \geq 0$.
5. e^x for all x.
6. $ln\, x$ for $x > 0$.
7. $\cos(x)$ and $\sin(x)$ for all x.
8. $\tan(x)$ and $\sec(x)$ provided $x \neq \cdots, \frac{-3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$
9. $\cot(x)$ and $\csc(x)$ provided $x \neq \cdots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$

NOTE 1.14

General Notes about limits:

I. Look how to take the factor out from the equation: (this is just one way of factorizing)

$$5x + 8 - \sqrt{x^3 + x - 2} = x^{\frac{3}{2}}\left(5x^{-\frac{1}{2}} + 8x^{-\frac{3}{2}} - \sqrt{1 + x^{-2} - 2x^{-3}}\right)$$

Before factoring	$5x$	8	$\sqrt{x^3 + \cdots - \cdots}$	$\sqrt{\ldots + x - \cdots}$	$\sqrt{\ldots + \cdots - 2}$
After factoring	$5x^{-\frac{1}{2}}$	$8x^{-\frac{3}{2}}$	$\sqrt{1 + \cdots - \cdots}$	$\sqrt{\ldots + x^{-2} - \cdots}$	$\sqrt{\ldots + \cdots - 2x^{-3}}$
Power of x after factoring	$1 - \frac{3}{2} = -\frac{1}{2}$	$0 - \frac{3}{2} = -\frac{3}{2}$	$3 - 3 = 0$	$1 - 3 = -2$	$0 - 3 = -3$

II. Polynomial Function Degree

$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \infty$	$f(x) > g(x)$	$ex\!: \lim_{x \to \infty} \frac{7x^5 + x^3 + 1}{5x^4 - 1} = \infty$	Large Degree Polynomial Function
$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L$ $0 < L < \infty$	$f(x) = g(x)$	$ex\!: \lim_{x \to \infty} \frac{7x^5 + x^3 + 1}{5x^5 - 1} = \frac{7}{5}$	Equal Degree Polynomial Functions
$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$	$f(x) < g(x)$	$ex\!: \lim_{x \to \infty} \frac{7x^4 + x^3 + 1}{5x^5 - 1} = 0$	Small Degree Polynomial Function

III. $a^{\infty} = \begin{cases} \infty & a > 1 \\ 0 & a < 1 \end{cases} \quad \Rightarrow \quad ex\!: 5^{\infty} = \infty, \quad \left(\frac{1}{5}\right)^{\infty} = 0$

$a^{-\infty} = \begin{cases} 0 & a > 1 \\ \infty & a < 1 \end{cases} \quad \Rightarrow \quad ex\!: 5^{-\infty} = 0, \quad \left(\frac{1}{5}\right)^{-\infty} = \infty$

IV. Remember that: $e \approx 2.7$ $\quad \ln 1 = 0 \quad \ln e = 1 \quad \ln 0 = -\infty \quad \ln \infty = \infty$

$e^{\infty} = \infty \qquad e^{-\infty} = 0$

NOTE 2.1

Continuity:

Rule: A function $f(x)$ is continuous at $x = a$ if and only if the following two conditions are satisfied:

1- $f(a)$ is defined

2- $\lim_{x \to a} f(x) = f(a)$

This is mainly used in the piecewise function shown in example 1

$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = f(a)$

This is mainly used in the piecewise function shown in example 2

Example 1

$$f(x) = \begin{cases} -x, & x \neq 0 \\ x^2, & x = 0 \end{cases}$$

show that the function f is continous at x = 0

Answer

$\because \lim_{x \to 0} -x = 0 \quad and \quad f(0) = (0)^2 = 0$

$\therefore$ *the function f is continous at x = 0*

Example 2

$$f(x) = \begin{cases} -x, & x < 0 \\ x^2, & x \geq 0 \end{cases}$$

show that the function f is continous at x = 0

Answer

$\because \lim_{x \to 0^+} x^2 = f(0) = 0 \quad and \quad \lim_{x \to 0^-} -x = 0$

$\therefore$ *the function f is continous at x = 0*

NOTE 2.2

Differentiation:

(Formal) Definition of the Derivative Does NOT require a point $(x = a)$ to use	Definition of the Derivative Requires a point $(x = a)$ to use
$f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$	$f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$
$If\ f(x)\ is\ differentiable\ at\ x = a\ then$ $f'(a) = \lim\limits_{h \to 0^+} \dfrac{f(a+h) - f(a)}{h} = \lim\limits_{h \to 0^-} \dfrac{f(a+h) - f(a)}{h}$	$If\ f(x)\ is\ differentiable\ at\ x = a\ then$ $f'(a) = \lim\limits_{x \to a^+} \dfrac{f(x) - f(a)}{x - a} = \lim\limits_{x \to a^-} \dfrac{f(x) - f(a)}{x - a}$
$If\ f(x)\ is\ differentiable\ at\ x = a\ then$ $f'(a) = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$	$If\ f(x)\ is\ differentiable\ at\ x = a\ then$ $f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$

Important: $If\ f(x)\ is\ differentiable\ at\ x = a\ \ Then,\ \ f(x)\ is\ continous\ at\ \ x = a$ (NOT the inverse)

Example 3

$$f(x) = \begin{cases} -x, & x < 0 \\ x^2, & x \geq 0 \end{cases} \qquad Find\ f'(0)\ if\ it\ exists$$

Answer

Answer using **(Formal) Definition of the Derivative**	Answer using **Definition of the Derivative**
$f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ $f'(0) = \lim\limits_{h \to 0} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0} \dfrac{f(h)}{h}$ $\lim\limits_{h \to 0^+} \frac{f(h)}{h} = \lim\limits_{h \to 0^+} \frac{h^2}{h} = h = 0$, $\lim\limits_{h \to 0^-} \frac{f(h)}{h} = \lim\limits_{h \to 0^-} \frac{-h}{h} = -1$ $\because \lim\limits_{h \to 0^+} \frac{f(h)}{h} \neq \lim\limits_{h \to 0^-} \frac{f(h)}{h}$ $\therefore the\ function\ f\ is\ NOT\ differentiable\ at\ x = 0$	$f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ $f'(0) = \lim\limits_{x \to 0} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0} \dfrac{f(x)}{x}$ $\lim\limits_{x \to 0^+} \frac{f(x)}{x} = \lim\limits_{x \to 0^+} \frac{x^2}{x} = x = 0$, $\lim\limits_{x \to 0^-} \frac{f(x)}{x} = \lim\limits_{x \to 0^-} \frac{-x}{x} = -1$ $\because \lim\limits_{x \to 0^+} \frac{f(x)}{x} \neq \lim\limits_{x \to 0^-} \frac{f(x)}{x}$ $\therefore the\ function\ f\ is\ NOT\ differentiable\ at\ x = 0$

Example 4

$$f(x) = \begin{cases} x^2, & x \neq 0 \\ 0, & x = 0 \end{cases} \qquad Find\ f'(0)\ \ if\ it\ exists$$

Answer

Answer using **(Formal) Definition of the Derivative**	Answer using **Definition of the Derivative**
$f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$	$f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$
$f'(0) = \lim\limits_{h \to 0} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0} \dfrac{f(h)}{h}$	$f'(0) = \lim\limits_{x \to 0} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0} \dfrac{f(x)}{x}$
$\lim\limits_{h \to 0} \dfrac{f(h)}{h} = \lim\limits_{h \to 0} \dfrac{h^2}{h} = h = 0$	$\lim\limits_{x \to 0} \frac{f(x)}{x} = \lim\limits_{x \to 0} \frac{x^2}{x} = x = 0$,
$\therefore$ *the function f is differentiable at $x = 0$*	$\therefore$ *the function f is differentiable at $x = 0$*

Example 5

$$f(x) = \begin{cases} -x, & x < 0 \\ x^2, & x \geq 0 \end{cases} \qquad Find\ f'(x)\ for\ x \neq 0$$

Answer

$$f'(x) = \begin{cases} -1, & x < 0 \\ 2x, & x > 0 \end{cases}$$

To solve this question, you need to be aware of Derivatives Rules (section 3)

NOTE 2.3

Note: if $f(g(x))$ is defined on interval containing a, and if $f(x)$ is continuous at L and $\lim\limits_{x \to a} f(g(x))$ then

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(L)$$

Example 6

$$\lim_{x\to 0} ln(cos(x))$$

Answer

$$\lim_{x\to 0} ln(cos(x)) \overset{\blacksquare}{=} \ln\left(\lim_{x\to 0} \cos(x)\right) = \ln(1) = 0$$

■ *since* $cos(x)$ *is continous at* $x = 0$

Section 3 Derivatives

NOTE 3.1

<u>The first derivative form:</u>

$$\frac{d}{dx} f(x) = f'(x) = y' = \frac{dy}{dx} = m = slope$$

<u>Basic Properties/Formulas</u>

I. $\left(f(x) \pm g(x)\right)' = f'(x) \pm g'(x)$

II. $\frac{d}{dx}\left(cf(x)\right) = c\, f'(x),\;\; c\ is\ any\ contant.$

NOTE 3.2

<u>Common Derivatives</u> [*where* $\boldsymbol{a}$ *is constant*]

Polynomials	Trig Functions	Inverse Trig Functions
$\frac{d}{dx}(c) = 0$	$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}(\sin^{-1} x) = \frac{d}{dx}(arcsin\, x) = \frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}(x) = 1$	$\frac{d}{dx}(\cos x) = -\sin x$	$\frac{d}{dx}(\cos^{-1} x) = \frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}(cx) = c$	$\frac{d}{dx}(\tan x) = \sec^2 x$	$\frac{d}{dx}(\tan^{-1} x) = \frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$
$\frac{d}{dx}(x^n) = nx^{n-1}$	$\frac{d}{dx}(\sec x) = \sec x \tan x$	
$\frac{d}{dx}(cx^n) = ncx^{n-1}$		

NOTE 3.3

Exponential/Logarithm Functions [*where $\boldsymbol{a}$ is constant*]

Derivatives	Properties
$\frac{d}{dx}(a^x) = a^x \ln(a)$	$\ln(xy) = \ln x + \ln y$
$\frac{d}{dx}(e^x) = e^x$ [*hint* $\boldsymbol{e} \approx 2.7$]	$\ln\left(\frac{x}{y}\right) = \ln x - \ln y$
$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$ $, x > 0$	$\ln\left(\frac{1}{x}\right) = -\ln x$
$\frac{d}{dx}(\log_a(x)) = \frac{1}{x \ln a}$ $, x > 0$	$\ln(x^r) = r \ln x$
$\frac{d}{dx}\left(a^{g(x)}\right) = a^{g(x)}\,(g'(x) \ln a)$	$\log_a(x) = \frac{\ln x}{\ln a}$
$\frac{d}{dx}(\ln g(x)) = \frac{g'(x)}{g(x)}$	$f(x) = e^{\ln f(x)}$
	$f(x)^{g(x)} = e^{\ln f(x)^{g(x)}} = e^{g(x) \ln f(x)}$

Some facts about Exponential/Logarithm Functions:

$$\ln(e) = 1 \; ; \; \ln(1) = 0 \; ; \; \ln(0) = -\infty \; ; \; \ln(\infty) = \infty$$

Hyperbolic Trig Functions: $\sinh x = \frac{e^x - e^{-x}}{2}$ $\cosh x = \frac{e^x + e^{-x}}{2}$ $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

The rules of Hyperbolic Trig Functions are the same as the Trig Functions in terms of inverse and derivatives. For example,

1) $\frac{d}{dx}(\sinh x) = \cosh x$ 2) $\frac{d}{dx}(\cosh x) = \sinh x$ (*except*)

NOTE 3.4

Inverse Trig Functions:

Function	Domain	Function	Domain
$\sin x$	$\mathbb{R}$	$\arcsin x = \sin^{-1} x$	$-1 \le x \le 1$
$\cos x$	$\mathbb{R}$	$\arccos x = \cos^{-1} x$	$-1 \le x \le 1$
$\tan x$	$\mathbb{R} - \left\{\frac{\pi}{2} + \pi k\right\}, k \in \mathbb{Z}$	$\arctan x = \tan^{-1} x$	$\mathbb{R}$
$\sin(\arcsin x) = x$	$-1 \le x \le 1$	$\arcsin(\sin x) = x$	$-\frac{\pi}{2} \le x \le \frac{\pi}{2}$
$\cos(\arccos x) = x$	$-1 \le x \le 1$	$\arccos(\cos x) = x$	$0 \le x \le \pi$
$\tan(\arctan x) = x$	$\mathbb{R}$	$\arctan(\tan x) = x$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$

Some Remarks about inverse Functions:

$f^{-1}(x) = y \ \rightarrow f(y) = x$!! $f^{-1}(x) \neq \frac{1}{f(x)}$

$\cos^{-1} x = y \ \rightarrow \cos y = x$!! $\cos^{-1} x \neq \frac{1}{\cos x}$

NOTE 3.5

One to One *Function / Inverse of a Function (**Uniqueness**)*

Condition 1: *If the function is **increasing** or **decreasing**, then it is invertible*

$\therefore$ *this is called **one to one** function*

Condition 2: $If\ \ f'(x) \geq 0\ \forall x \therefore f(x)\ is\ (strictly)\ increasing\ for\ all\ x$

$If\ \ f'(x) \leq 0\ \forall x \therefore f(x)\ is\ (strictly)\ decreasing\ for\ all\ x$

Note that: $f(x_1) = f(x_2)$ *and* $x_1 = x_2$ $\therefore f(x)$ *is invertible and it is called **one to one** function*

How to find the inverse derivate of a function:

$f^{-1\prime}(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(a)}$ *We assume* $f^{-1}(x) = a \quad \therefore f(a) = x$

Example 1

$f(x) = x^3 + 7x + 2 \qquad Find \quad f^{-1\prime}(10)$

Answer

$\because f'(x) = 3x^2 + 7 > 0 \quad \therefore f(x)\ is\ increasing\ for\ all\ x \quad \therefore f(x)$ is invertible $\quad \therefore f(x)$ is **one to one** function

$$f^{-1\prime}(10) = \frac{1}{f'(f^{-1}(10))} = \frac{1}{f'(a)} \quad where\ f^{-1}(10) = a \quad \therefore f(a) = 10$$

$$a^3 + 7a + 2 = 10 \quad \therefore a = 1 \quad thus\ f^{-1\prime}(10) = \frac{1}{f'(f^{-1}(10))} = \frac{1}{f'(1)} = \frac{1}{3(1)^2 + 7} = \frac{1}{10}$$

NOTE 3.6

Derivatives Rules:

(I) Product Rule: $(fg)' = f'g + fg'$ → *check example 2*

(II) Quotient Rule: $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ → *check example 3*

(III) Chain Rule: $\frac{d}{dx}\left(f\left(g(x)\right)\right) = f'\left(g(x)\right)\,g'(x)$ → *check example 4*

(IV) Implicit Differentiation: $\frac{d}{dx}(y) = y'$ → *check example 5 and 6*

(V) General Techniques: (1) $\frac{d}{dx}\left(\sqrt{f(x)}\right) = \frac{f'(x)}{2\sqrt{f(x)}}$ → *check example 7*

(2) $\frac{d}{dx}\left(e^{f(x)}\right) = e^{f(x)}(f'(x))$ → *check example 8*

Example 2

$$\frac{d}{dx}(x^2 \sin x)$$

Answer

$$\frac{d}{dx}(x^2 \sin x) = (x^2 \sin x)' = 2x \sin x + x^2 \cos x$$

Example 3

$$\frac{d}{dx}\left(\frac{x^2}{\sin x}\right)$$

Answer

$$\frac{d}{dx}\left(\frac{x^2}{\sin x}\right) = \frac{(\sin x)(2x) - (x^2)(\cos x)}{\sin^2 x}$$

Example 4

$$\frac{d}{dx}(\sin x^2)$$

Answer

$$\frac{d}{dx}(\sin x^2) = (\cos x^2)(2x) = 2x\cos x^2$$

Example 5

$$\frac{d}{dx}(2y^3)$$

Answer

$$\frac{d}{dx}(2y^3) = (6y^2)\, y'$$

Example 6

$$\frac{d}{dx}(y^2\ sin\ x)$$

Answer

$$\frac{d}{dx}(y^2\ sin\ x) = 2y\, y' \sin x + y^2 \cos x$$

Example 7

$$\frac{d}{dx}\left(\sqrt{x^2 \sin x}\right)$$

Answer

$$\frac{d}{dx}\left(\sqrt{x^2 \sin x}\right) = \frac{2x\sin x + x^2\cos x}{2\sqrt{x^2 \sin x}}$$

Example 8

$$\frac{d}{dx}\left(e^{x^2 \sin x}\right)$$

Answer

$$\frac{d}{dx}\left(e^{x^2 \sin x}\right) = e^{x^2 \sin x}(2x\sin x + x^2\cos x)$$

NOTE 3.7

(VI) Higher-Order Derivatives:

1st derivative	y'	$f'(x)$	$\frac{dy}{dx}$	$\frac{d}{dx}f(x)$
2nd derivative	y''	$f''(x)$	$\frac{d^2y}{dx^2}$	$\frac{d^2}{dx^2}f(x)$
3rd derivative	y'''	$f'''(x)$	$\frac{d^3y}{dx^3}$	$\frac{d^3}{dx^3}f(x)$
4th derivative	$y^{(4)}$	$f^{(4)}(x)$	$\frac{d^4y}{dx^4}$	$\frac{d^4}{dx^4}f(x)$
nth derivative	$y^{(n)}$	$f^{(n)}(x)$	$\frac{d^ny}{dx^n}$	$\frac{d^n}{dx^n}f(x)$

Example 9

$$f(x) = \frac{x^4}{4} + \frac{x^3}{9} + \frac{x^2}{3} + 3x + 2023$$

Answer

$$f'(x) = x^3 + \frac{x^2}{3} + \frac{2x}{3} + 3 \quad \curvearrowright \quad f''(x) = 3x^2 + \frac{2x}{3} + \frac{2}{3} \quad \curvearrowright \quad f'''(x) = 6x + \frac{2}{3} \quad \curvearrowright \quad f^{(4)}(x) = 6$$

$$\boldsymbol{Thus}: f^{(n)}(x) = 0 \quad for\ n \geq 5$$

Example 10

$y = (x)^{\sin x}$ $\qquad$ $Find\ \ y'$ $\qquad$ (Exponential/Logarithm Question)

Answer

Way I:

$$y = (x)^{\sin x} = e^{\ln(x)^{\sin x}} = e^{[\sin x\ \ln(x)]}$$

$$Then,\ y' = e^{[\sin x\ \ln(x)]}\left(\frac{d}{dx}[\sin x\ \ln(x)]\right)$$

$$where\ \frac{d}{dx}[\sin x\ \ln(x)] = \cos x\ \ \ln(x) + \sin x\ \left(\frac{1}{x}\right)$$

$$\therefore y` = e^{[\sin x\ \ln(x)]}\left[\cos x\ \ \ln(x)\ \ + \sin x\ \ \frac{1}{x}\right]$$

$$\therefore y` = (x)^{\sin x}\left[\cos x\ \ \ln(x)\ \ + \sin x\ \left(\frac{1}{x}\right)\right]$$

Way II:

$$y = (x)^{\sin x} \implies \ln y = \sin x\ \ln(x)$$

$$Then, take\ \frac{d}{dx}\ to\ both\ sides$$

$$\frac{y`}{y} = \cos x\ \ln(x)\ \ + \sin x\ \left(\frac{1}{x}\right)$$

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$\therefore y` = y\left[\cos x \;\; \ln(x) \;\; + \sin x \;\left(\frac{1}{x}\right)\right]$

$\therefore y` = (x)^{\sin x}\left[\cos x \;\; \ln(x) \;\; + \sin x \;\left(\frac{1}{x}\right)\right]$

Section 4 Tangent Lines and Their Slope

NOTE 4.1

Slope of tangent line is $\left(m = y' = f'(x) = \frac{dy}{dx}\right)$

Let ***Curve*** *equation is* $y = f(x)$, ***Tangent*** *equation of line* (L_1) *and,* ***Normal*** *equation of line* (L_2)

- ***Tangent*** *equation of line* (L_1) *that touches a curve* (y) *at one and only one point.*

Let this point is $(a, b) \in$ *curve and Tangent Line:*

- ***Tangent*** *line eqn is* $\frac{y-b}{(x-a)} = (\text{Slope of } \textbf{tangent}) = m_1$ *OR* $y = m_1(x-a) + b$

Normal *equation has the same form as the* ***Tangent*** *equation. The only difference is their slopes.*

- ***Normal*** *line eqn is* $\frac{y-b}{(x-a)} = (\text{Slope of } \textbf{normal}\,) = m_2$ *OR* $y = m_2(x-a) + b$

Remember: *Slope of tangent* $= \dfrac{-1}{\text{slope of normal}}$

❖ *Case I: when the point* $(a, b) \in$ *curve*

Example 1

Find an equation of both the **tangent** and **normal** lines to the curve $y = x^2$ at the point $(-1,1)$

Answer

By substituting the point in the curve equation, we have $1 = 1^2$

Thus, $(-1,1) \in$ *curve* $y = x^2$ *OR* $(-1,1)$ *is on the curve* $y = x^2$

$y' = 2x \quad \therefore y'|_{x=-1} = -2 = m$ (*Slope of* ***tangent***)

Tangent *line eqn is* $\dfrac{y-1}{(x-(-1))} = -2 \qquad \rightarrow y = -2x - 1$

Normal *line eqn is* $\dfrac{y-1}{(x-(-1))} = \dfrac{1}{2} \qquad \rightarrow y = \dfrac{1}{2}x + \dfrac{3}{2} = \dfrac{1}{2}(x+3)$

❖ *Case II: when the point $(a, b) \notin$ curve*

Example 2

Find an equation of the *tangent* line(s) to the curve $y = x^2 + x$ at the point $(1,1)$

Answer

By substituting the point in the curve equation, we have $1 \neq 1^2 + 1$

$Thus,\ (1,1) \notin curve\ y = x^2 + x \quad let\ x = a\ \ then\ the\ point\ is\ (a, f(a))\ \ where\ x \neq 1$

The **slope** of the **tangent line** with the point $(a, f(a))$ must be $y' = 2x + 1 \quad \therefore y'|_{x=a} = 2a + 1 \ \rightarrow (1)$

Because the line passes through the point $(1,1)$, the slope of the tangent line is **also**:

$\frac{f(a)-1}{a-1} = \frac{a^2+a-1}{a-1} \rightarrow (2) \quad \because (1) = (2) \therefore \frac{a^2+a-1}{a-1} = 2a + 1 \ \rightarrow\ a^2 + a - 1 = 2a^2 - a - 1 \ \rightarrow\ a^2 - 2a = 0$

$\rightarrow a(a-2) = 0 \ \therefore a = 0\ \ or\ \ a = 2 \qquad \therefore y'|_{x=0} = 1\ and\ \ y'|_{x=2} = 5$

$Tangent\ line\ eqn\ are\ \ y = (x-1) + 1\ \ and\ \ y = 5(x-1) + 1$

NOTE 4.2

!! *When the **slope** of line is **zero** this means the line is **horizontal***

$i.e.\ f`(x) = 0\ where\ y = f(x) = c\ \ \ where\ c\ is\ constant$

$Let\ the\ slope\ of\ the\ line\ (L_1)\ is\ m_1 \quad \& \quad the\ slope\ of\ the\ line\ (L_2)\ is\ m_2$

$If\ \ m_1 = m_2 \qquad then\ \ (L_1) // (L_2)\ \ \ where\ //\ \ is\ \ parallel$

- $ex: y = 2x + 1\ \ \&\ \ y = 2x + 3$

$If\ \ m_1 * m_2 = -1\ \ then\ \ (L_1) \perp (L_2)\ \ \ where\ \perp\ \ is\ \ perpendicular$

- $ex: y = 2x + 1\ \&\ \ y = -\frac{x}{2} + 3$

$If\ \ m_1 * m_2 = -1\ \ then\ \ (L_1)$ intersects with (L_2)

NOTE 5.1

Formal definition of the limit (Chapter 1.5)

Rule: $the\ formal\ definition\ (the\ \epsilon[Epsilon]-\delta[Delta]definition)\ of\ \lim_{x\to a} f(x)=L\ is$

$For\ every\ \epsilon>0,\quad there\ exists\ \delta>0\quad such\ that\ \ if\ 0<|x-a|<\delta,\quad then\ |f(x)-L|<\epsilon$

Example 1

$Show\ that\ \lim_{x\to 3}(2x+1)=7$

$by\ using\ the\ formal\ definition\ of\ the\ limit.$

Answer

$For\ every\ \epsilon>0,\ \ choose\ \delta=\frac{\epsilon}{2}$

$such\ that\ \ if\ 0<|x-3|<\delta=\frac{\epsilon}{2},$

$then\ |2x+1-7|=|2x-6|=2|x-3|<2\frac{\epsilon}{2}=\epsilon$

NOTE 5.2

Mean Value Theorem (MVT)

Rule: $Suppose\ f\ (x)\ is\ a\ function\ that\ satisfies\ both\ of\ the\ following:$

1. $f\ (x)\ is\ continuous\ on\ the\ closed\ interval\ [\ a\ ,b\ ]$
2. $f\ (x)\ is\ differentiable\ on\ the\ open\ interval\ (\ a\ ,b\ )$

$Then\ there\ is\ a\ number\ \ c\ \ such\ that\ a\ <\ c\ <\ b\qquad and\qquad f`(c)=\frac{f(b)-f(a)}{b-a}$

Example 2

$If\ f(0)=2\ and\ 3\le f'(x)\le 9\ \ for\ all\ x,\ \ Show\ that\ 5\le f(1)\le 11.$

Answer

Since f is differentiable for all $x\in\mathbb{R}$ we have, by MVT, $\frac{f(1)-f(0)}{1-0}=f'(c)\quad for\ some\ c\in(0,1\ )$

$Then\ \ 3\le f'(c)\ \le 9=3\le\frac{f(1)-2}{1-0}\le 9\qquad 3\le\frac{f(1)-2}{1-0}\le 9\qquad 5\le f(1)\le 11$

NOTE 5.3

Rolle's Theorem

Rule: Suppose $f(x)$ is a function that satisfies both of the following:

1. $f(x)$ *is continuous on the closed interval* $[a, b]$
2. $f(x)$ *is differentiable on the open interval* (a, b)
3. $f(a) = f(b)$

Then there is a number c *such that* $a < c < b$ *and* $f'(c) = 0$

Example 3

Let f be a differentiable function on $\mathbb{R}$ such that f'' exists everywhere and $f''(x) > 0$ for all x in $\mathbb{R}$. ***Show that f cannot have three roots.***

Answer

assume to the f has three roots, say $a < b < c$ are roots of f, that is, $f(a) = f(b) = f(c) = 0$

since f is differentiable, then by Rolles`s theorem, there exist $a < x_1 < b$ *and* $b < x_2 < c$ *such that*

$f'(x_1) = 0$ *and* $f'(x_2) = 0$. *Now we have two roots* x_1 *and* x_2

since f' *is diffenetiable and* $f'(x_1) = f'(x_2)$

By Rolles`s theorem, there exists x_3 *between* x_1 *and* x_2 *such that* $f''(x_3) = 0$,

but $f''(x) > 0$ *for all* x. ***Thus, f can not have three roots.***

NOTE 5.4

Intermediate Value Theorem (IVT)

Rule: Suppose that $f(x)$ is continuous on $[a, b]$ and let M be any number between $f(a)$ and $f(b)$

Then there exists a number c *such that* 1. $a < c < b$ 2. $f(c) = M$

NOTE 5.5

One to One *Function / Inverse of a Function (**Uniqueness**)*

Condition 1: $If\ the\ function\ is\ \textbf{increasing}\ or\ \textbf{decreasing}, then\ it\ is\ invertible$

$\therefore\ this\ is\ called\ \textbf{one to one}\ function$

Condition 2: $If\ \ f'(x) \geq 0\ \forall x \therefore f(x)\ is\ (strictly)\ increasing\ for\ all\ x$

$If\ \ f'(x) \leq 0\ \forall x \therefore f(x)\ is\ (strictly)\ decreasing\ for\ all\ x$

Note that: $f(x_1) = f(x_2)\ and\ x_1 = x_2\ \ \therefore f(x) is\ invertible\ \ and\ it\ is\ called\ \textbf{one to one}\ function$

Example 4

$Show\ that\ the\ equation\ x^2 + 6x + 3 = 0\ \ has\ a\ root, but\ no\ more\ than\ one.$

Answer

$Let\ f(x) = x^2 + 6x + 3\ \ then\ f(x)\ is\ cont.\ and\ diff.\ for\ all\ x$

$\begin{cases} f(0) = 3 > 0 \\ f(-1) = -2 < 0 \end{cases}$ $thus\ by\ IVT\ \ f(x)\ is\ cont. on\ [-1,0]\ and\ there\ is\ a\ point\ c_1 \in (-1,0)$

$such\ that\ f(c_1) = 0\ \ \ which\ means\ x = c_1\ \ is\ a\ root\ of\ f(x)$

$f'(x) = 2x + 6 \geq 0\ for\ all\ x \in (-1,0)\ so\ f(x)\ is\ a\ strictly\ increasing\ function\ and\ hence\ one\ to\ to\ one$

Therefore, the equation $f(x)$ has at most one solution. $f(x)\ has\ only\ one\ root\ which\ is\ c_1$

NOTE 6.1

Procedure for Solving Related Rates Problems:

1) Draw a picture (if one is not provided) and define the variables. Assign ALL numbers to variables. Remember, rates ALWAYS correspond to a derivative.

2) Determine and CLEARLY STATE goal of the problem, which is ALWAYS finding a rate/derivative.

3) Build your related rates equation. Usually, this involves the implicit differentiation of an equation from geometry. Note: If there's only ONE rate/derivative in your related rates equation, you did something wrong!

4) Isolate the goal rate in 2) and make sure you have the numbers for all the other variables and rates/derivatives. If not, you have to find the missing numbers, usually by solving another equation.

5) Plug in numbers and solve, making sure to INCLUDE UNITS. Think about the solution and its plausibility!

Example 1

Question: How fast is the surface area of a cube changing when the volume of cube is 64 cm^3 and is increasing at 2 cm^3/sec?

Answer

surface area $S(x) = 6x^2$ volume $V(x) = x^3$ $\frac{dV}{dt} = +2\ cm^3/sec$ & $V(x) = 64\ cm^3$

$$V(x) = x^3 = 64 \ \rightarrow x = 4\ cm \quad , \quad \frac{dV}{dt} = 3x^2\frac{dx}{dt} \ \rightarrow \ 2 = 3(4)^2\frac{dx}{dt} \quad , \quad \frac{dx}{dt} = \frac{1}{24}\ cm/sec$$

Thus, $\frac{dS}{dt} = 12x\frac{dx}{dt} = 12(4)\left(\frac{1}{24}\right) = 2\ cm^2/sec$ The surface area of a cube is increasing at rate 2 cm^2/sec

NOTE 7.1

L'Hospital's Rule:

Suppose that we have one of the following cases,

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0} \qquad OR \qquad \lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\pm\infty}{\pm\infty}$$

where ***a*** can be any real number, infinity, or negative infinity. In these cases, we have:

$$\lim_{x\to a}\frac{f(x)}{g(x)}\overset{L`H}{=}\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

NOTE 7.2

- In case you have the following types of indeterminate forms in a limit,

  $(0)(\pm\infty) \quad 1^{\infty} \quad 0^0 \quad \infty^0 \quad \infty-\infty$

Then you need to change the function of the limit to $\left(\frac{0}{0}\right)$ or $\left(\pm\frac{\infty}{\infty}\right)$ types.

(a) $\quad (0)(\pm\infty)=\dfrac{\pm\infty}{\frac{1}{0}}=\pm\dfrac{\infty}{\infty}$

(b) $\quad \ln\left(1^{\infty}\right)=\infty\ln 1=\dfrac{\ln 1}{\frac{1}{\infty}}=\dfrac{0}{0}$

(c) $\quad \ln\left(0^0\right)=0\ln 0=\dfrac{\ln 0}{\frac{1}{0}}=\dfrac{\infty}{\infty}$

(d) $\quad \ln\left(\infty^0\right)=0\ln\infty=\dfrac{\ln\infty}{\frac{1}{0}}=\dfrac{\infty}{\infty}$

- when you have the following indeterminate form types 1^{∞}, $0^0 and$ ∞^0 in your question, then apply the logaritem rule: (check **Example 1**)

Example 1

Evaluate $\lim\limits_{x\to 0^+}(1+4x)^{\frac{2}{x}}$

Answer

Way I:

$\left[let \lim\limits_{x\to 0^+}(1+4x)^{\frac{2}{x}} = \lim\limits_{x\to 0^+} y\right]$

By taking the logaritem to both sides:

$$\lim_{x\to 0^+} lny = \lim_{x\to 0^+} \frac{2}{x}\, ln(1+4x) = 2\lim_{x\to 0^+} \frac{ln(1+4x)}{x} \left(\frac{0}{0}\right) type$$

$L`H$

$$= 2\lim_{x\to 0^+} \frac{4}{1+4x} = 2*4 = 8$$

$\therefore \lim\limits_{x\to 0^+} lny = 8 \quad \therefore y = e^8 \quad \therefore \lim\limits_{x\to 0^+}(1+4x)^{\frac{2}{x}} = e^8$

Way II:

$\lim\limits_{x\to 0^+}(1+4x)^{\frac{2}{x}} \overset{*}{=} e^{\lim\limits_{x\to 0^+}\frac{2}{x}ln(1+4x)}$ $\quad$ * *where* e^x *is continues*

Solving the power of e

$$\lim_{x\to 0^+} \frac{2}{x}\, ln(1+4x) = 2\lim_{x\to 0^+} \frac{ln(1+4x)}{x} \left(\frac{0}{0}\right)$$

$L`H$

$$= 2\lim_{x\to 0^+} \frac{4}{1+4x} = 2*4 = 8$$

$$\lim_{x\to 0^+}(1+4x)^{\frac{2}{x}} = e^8$$

NOTE 7.3

For $(\infty-\infty)$ indeterminate forms, you should do the following steps to have $\left(\frac{0}{0}\right)$ *or* $\left(\frac{\pm\infty}{\pm\infty}\right)$ types)

- *1st: you neet to use the factor method*

$\lim\limits_{x\to\pm\infty} a^{f(x)} \pm b^{g(x)} = \lim\limits_{x\to\pm\infty} a^{f(x)}\left[1 \pm \frac{b^{g(x)}}{a^{f(x)}}\right]$, where (a & b can be **Constant or variable**)

- *2nd: you need to use the limit for both functions separately*

$$= \left[\lim_{x\to\pm\infty} a^{f(x)}\right].\left[1 \pm \lim_{x\to\pm\infty} \frac{b^{g(x)}}{a^{f(x)}}\right]$$

To solve $\left[\lim\limits_{x\to\pm\infty} a^{f(x)}\right]$, you should know that

- $\lim\limits_{x\to\infty} a^x = \infty,\ \lim\limits_{x\to-\infty} a^x = 0 \quad for\ any\ a > 1$
- $\lim\limits_{x\to\infty} a^x = 0,\ \lim\limits_{x\to-\infty} a^x = \infty \quad for\ any\ 0 < a < 1$

Example 2

Evaluate $\lim_{x\to\infty}(e^x - 2x)$

Answer

$$\lim_{x\to\infty}(e^x - 2x) = \lim_{x\to\infty} e^x\left(1 - \frac{2x}{e^x}\right) = \infty * 1 = \infty \qquad since \lim_{x\to\infty}\frac{2x}{e^x}\left(\frac{\infty}{\infty}\right) =^{(L`H)} \lim_{x\to\infty}\frac{2}{e^x} = \frac{2}{\infty} = 0$$

Example 3

Evaluate $\lim_{x\to\infty}\frac{3^x - 2^x}{3^{x+1}+2^x}$

Answer

$$\lim_{x\to\infty}\frac{3^x - 2^x}{3^{x+1}+2^x} = \lim_{x\to\infty}\frac{1-\frac{2^x}{3^x}}{3+\frac{2^x}{3^x}} = \lim_{x\to\infty}\frac{1-\left(\frac{2}{3}\right)^x}{3+\left(\frac{2}{3}\right)^x} = \frac{1}{3} \qquad where \lim_{x\to\infty}\left(\frac{2}{3}\right)^x = 0 \ and \ \frac{2}{3} < 1$$

NOTE 7.4

BE AWARE!

$$\ln(a^x - b^x) \neq \ln a^x - \ln b^x \qquad \text{since } \ln a^x - \ln b^x = \ln\left(\frac{a^x}{b^x}\right)$$

Section 8 Curve Sketching Steps

To sketch a function, you should check the following 6 steps as well as the following notes. Such notes will make you understand the properties of the function that you are going to sketch.

NOTE 8.1

Step 1: Determine the Domain

- **Definition**: The domain of a function $f(x)$ is the set of all input values (x - values) for the function.
- **Rule**: In rational function, which means $\left(\frac{numerator}{denominator}\right)$, the $denominator$ can not be zero ($denominator \neq 0$)
- **How to find the domain of the funcation** $\left(y = \frac{A(x)}{B(x)}\right)$: make the $denominator$ fucntion equal to zero to find what value of x that makes $B(x) = 0$
- **Recall**: $\sqrt{f(x)}\,, where\ f(x) \geq 0$ $\quad \ln(f(x))\,,\ where\ f(x) > 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},\ where\ \Delta = b^2 - 4ac$$

- **Δ < *0*** means the function has **no real roots**
- **Δ = *0*** means the function has **equal real roots**
- **Δ > *0*** means the function has **2 real roots**

NOTE 8.2

Step 2: Find both the $\boldsymbol{x}$ ***and*** $\boldsymbol{y}$ Intercepts

:: $\boldsymbol{x}$ − ***Intercepts*** ::	:: $\boldsymbol{y}$ − ***Intercepts***::
• Definition: • An $x - intercept$ of a function $f(x)$ is any point where the graph crosses the $x - axis$. • To find the $x - intercepts$, solve $f(x) = 0$. The $x - intercept\ point$ form is: $(x,\ 0)$. • Recall: there is no $x - intercept\ point$ when $\Delta = b^2 - 4ac < 0$	• Definition: • An $y - intercept$ of a function $f(x)$ is any point where the graph crosses the $y - axis$. • To find the $y - intercept,\ solve\ f(0)\ \ when\ x = 0$. The $y - intercept\ point$ form is: $(0,\ y) = (0,\ f(0))$. • Recall: there is no $y - intercept$ **point** when $f(0)\ is\ undefined$

NOTE 8.3

Step 3: Find Asymptote(s) (chapter 4.6)

:: Vertical Asymptote ::

- Definition: A Vertical Asymptote (V. A) for a function is a vertical line $x = a$ showing where the function becomes unbounded/ undefined.
- When we have $\lim_{x\to a^+} f(x) = \pm\infty$, or $\lim_{x\to a^-} f(x) = \pm\infty$ or $both$. Then, $x = a$ is Vertical Asymptote (V. A)
- Note: At $x = a$, the $f(x)$ is $undefined$, where "a" is constant. Therefore, If the domain is defined and continuous for all x, then there is no Vertical Asymptote (V. A)

:: Horizontal Asymptote ::

- Definition: A Horizontal Asymptote (H. A) for a function is a horizontal line $y = b$ that the graph of the function approaches as x approaches ∞ or $-\infty$.
- When we have $\lim_{x\to\infty} f(x) = b$ $(right)$, or $\lim_{x\to-\infty} f(x) = b$ $(left)$, or $both$. Then, $y = b$ is Horizontal Asymptote (H. A)
- Note: If we have $\lim_{x\to\infty} f(x) = \pm\infty$, then there is no Horizontal Asymptote (H. A) on the right. Similarly, if we have $\lim_{x\to-\infty} f(x) = \pm\infty$, then there is no Horizontal Asymptote (H. A) on the left.
- Recall: The function $\left(y = \frac{A(x)}{B(x)}\right)$ has no Horizontal Asymptote (H. A) when the degree of $A(x)$ is $greater$ the degree of $B(x)$, but it has oblique asymptote instead.

:: Oblique Asymptote ::

- Definition: An Oblique Asymptote for a function is slanted line $y = ax + b$ $(where$ $a \neq 0)$ that the function approaches as x approaches ∞ or $-\infty$.
- When we have $\lim_{x\to\infty} \big(f(x) - (ax + b)\big) = 0$ $(right)$, or $\lim_{x\to-\infty} \big(f(x) - (ax + b)\big) = 0$ $(left)$, or $both$.

Then, $y = ax + b$ is Oblique Asymptote

- Recall:The function $\left(y = \frac{A(x)}{B(x)}\right)$ has no Oblique Asymptote when the degree of $A(x)$ is $smaller$ $than$ or $equal$ to the degree of $B(x)$, but it has Horizontal Asymptote (H . A) instead.

Example 1

Determine all asymptotes of $f(x) = \dfrac{x^2+4}{2x}$.

Answer

$The\ domain\ of\ f(x)\ is\ \mathbb{R} - \{0\}$ $\quad hint: 2x \neq 0 \ \therefore x \neq 0$

$Vertical\ asymp.\quad \lim\limits_{x\to 0^+} f(x) = \infty \quad \& \quad \lim\limits_{x\to 0^-} f(x) = -\infty \quad \therefore x = 0\ is\ \ Vertical\ asymp.$

$Horizontal\ asymp.\quad \lim\limits_{x\to \infty} f(x) = \infty \quad \& \quad \lim\limits_{x\to -\infty} f(x) = -\infty \quad \therefore there\ is\ NO\ Horizontal\ asymp.$

Oblique $asymp.$ As $x \to \pm\infty$, $f(x) = \frac{x^2+4}{2x} = \frac{x}{2} + \frac{2}{x}$ approaches the line $\frac{x}{2}$, thus $\frac{x}{2}$ is an oblique asymptote

Note that $\lim\limits_{x\to\pm\infty} \left(f(x) - \dfrac{x}{2}\right) = \lim\limits_{x\to\pm\infty} \dfrac{2}{x} = 0$

Example 2

Determine all asymptotes of $f(x) = e^{2/x}$.

Answer

$The\ domain\ of\ f(x)\ is\ \mathbb{R} - \{0\}$ $\quad hint: 2x \neq 0 \ \therefore x \neq 0$

$Vertical\ asymp.\quad \lim\limits_{x\to 0^+} f(x) = \infty \quad \& \quad \lim\limits_{x\to 0^-} f(x) = \infty \quad \therefore x = 0\ is\ \ Vertical\ asymp.$

$Horizontal\ asymp.\quad \lim\limits_{x\to \infty} f(x) = 1 \quad \& \quad \lim\limits_{x\to -\infty} f(x) = 1 \quad \therefore y = 1\ is\ \ Horizontal\ asymp.$

There are no absolute extrema since f never takes on the value 0.

Example 3

Show that the $f(x) = \frac{x^2+1}{x}$ has a two-sided oblique asymptote at $y = x$

Answer

Consider the function $f(x) = \dfrac{x^2+1}{x} = x + \dfrac{1}{x}$. The straight line $y = x$ is a two $-$ sided oblique asymptote

of the graph of $f(x)$ because $\lim\limits_{x\to\pm\infty} (f(x) - x) = \lim\limits_{x\to\pm\infty} \dfrac{1}{x} = 0$

Example 4

Show that the graph of $y = \frac{x\,e^x}{1+e^x}$ has a horizontal asymptote $y = 0$ at the left and an oblique asymptote $y = x$ at the right.

Answer

$$\lim_{x\to-\infty} \frac{x\,e^x}{1+e^x} = \frac{0}{1} = 0 \quad \therefore y = 0 \text{ is } H.A \text{ of } y \text{ from the left.}$$

$$\lim_{x\to\infty} \left(\frac{x\,e^x}{1+e^x} - x\right) = \lim_{x\to\infty} \left(\frac{x\,e^x - x(1+e^x)}{1+e^x}\right) = \lim_{x\to\infty} \left(\frac{x(\,e^x - 1 - e^x)}{1+e^x}\right) = \lim_{x\to\infty} \left(\frac{-x}{1+e^x}\right) = 0$$

$\therefore y = x$ *is an oblique asymptote of* y *at the right*

NOTE 8.4

Suppose that f is **continuous** on [a, b] and **differentiable** on (a, b).

If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on [a, b].

If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on [a, b].

NOTE 8.5

Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself.

1. if f' changes from **negative to positive** at **c**, then f has a local **minimum** at **c**.

2. if f' changes from **positive to negative** at **c**, then f has a local **maximum** at **c**

3. if f' does not change sign at **c** (that is, is positive on both sides of **c** or negative on both sides), then f has no local extremum at **c**.

NOTE 8.6

Step 4: Determine the Intervals of Increase and Decrease and local extreme values (chapter 4.4)

- Check the continuity and differentiability of the function before taking the derivative
- Find $f`(x)$
- Find the values of x when the first derivative is zero [$f`(x) = 0$]
  - If $f`(a_1) = f`(a_2) = 0$, then the "$x = a_1$ and $x = a_2$" are called **the Critical Points (C.P)**
- The singular points (S.P)are the points x where f`(x) is not defined except the one(s) on domain.
- Draw a table with the values of x
  - [including: critical points, singular points, and undefined points of both f(x) , and f`(x)]
- From the table, the Intervals of Increase and Decrease can be determined as well as local extreme values (max/min)
  - If $f`(x) > 0$ on an interval, then f is increasing on that interval.
  - If $f`(x) < 0$ on an interval, then f is decreasing on that interval.
- Remark: If $f`(x)$ is postive (or negative) on both sides of a critical or singular point, the $f(x)$ has neither a maximum nor a minimum value at that point.

Example 5

Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the intervals on which f is increasing and on which f is decreasing.

Answer

The function f is everywhere continuous and differentiable.

$$f'(x) = 3x^2 - 12 = 3(x^2 - y) = 3(x+2)(x-2)$$

The first derivative is zero $(f`(x) = 0)$ at $x = -2$ and $x = 2$

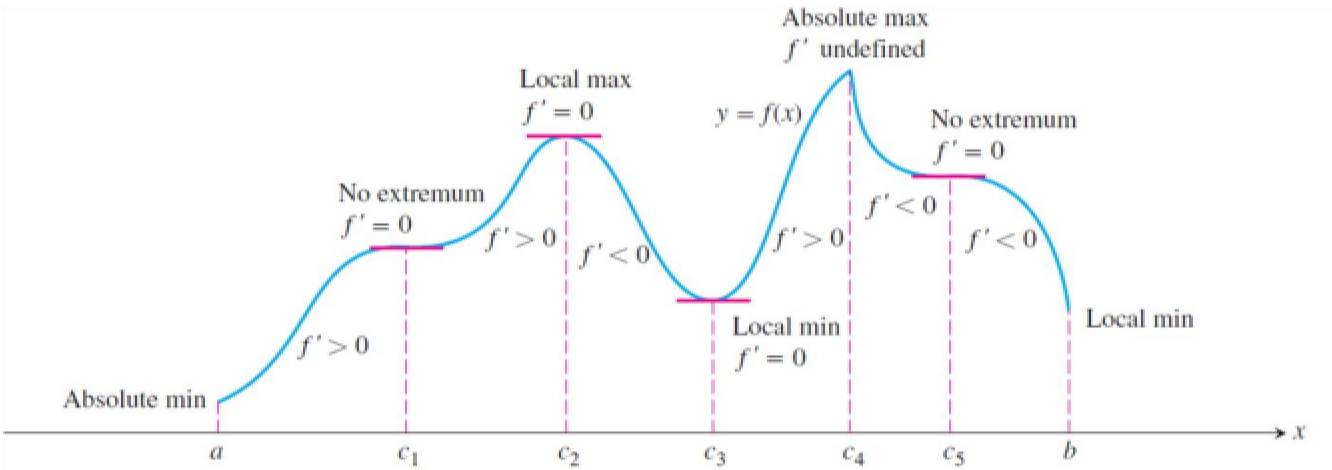
$f(x)$ *is increasing on* $(-\infty, -2)$ & $(2, \infty)$

$f(x)$ *is decreasing on* $(-2,2)$

x		-2		2	
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	↗		↘		↗

NOTE 8.7

The critical points of a function locate where it is increasing and where it is decreasing. The first derivative changes sign at a critical point where a local extremum occurs.



NOTE 8.8

Step 5: Determine the Intervals of concave up/down and all Inflection Points (chapter 4.5)

- Find $f``(x)$
- Find the values of x when the second derivative is zero [$f``(x) = 0$]
    - If $f``(a_1) = f``(a_2) = 0$, then the "$x = a_1$ and $x = a_2$" are called **the Inflection Points** (**C. P**)
- Draw a table with the values of x
    - [including: Inflection Points, and undefined points of both $f(x)$, and $f``(x)$]
- From the table, the Intervals of concave up/down can be determined.
    - If $f``(x) >$ 0 on an interval, then f is concave up on that interval.
    - If $f``(x)$< 0 on an interval, then f is concave down on that interval.

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NOTE 8.9

The Second Derivative Test

$(a)\ if\ f'(x_0) = 0\ and\ f''(x_0) < 0, then\ f\ has\ a\ local\ maximum\ value\ at\ x_0.$

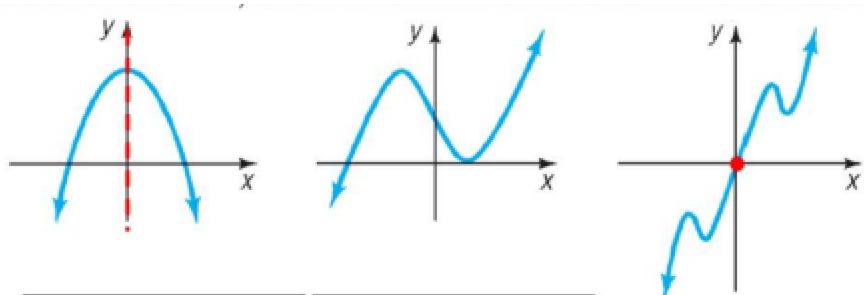
$(b)\ if\ f'(x_0) = 0\ and\ f''(x_0) > 0, then\ f\ has\ a\ local\ minimum\ value\ at\ x_0.$

$(a)\ if\ f'(x_0) = 0\ and\ f''(x_0) = 0, then\ no\ conclusion\ can\ be\ drawn.$

NOTE 8.10

Step 6: Draw the graph (chapter 4.6)

- using the previous step, you can draw the graph.
- Determine the symmetry of the graph:
  - **odd function**: symmetric about the origin, where $f(-x) = -f(x)$ for all x in the domain
  - **even function**: symmetric about the y −axis , where $f(-x) = f(x)$ for all x in the domain
  - neither odd nor ***even function***: no symmetry



Even function because it is symmetric with respect to the y-axis	Neither even nor odd because no symmetry with respect to the y-axis or the origin.	Odd function because it is symmetric with respect to the origin.

NOTE 8.11

For step 4 (NOTE 8.6), step5 (NOTE 8.8) and step6 (NOTE 8.10), you should construct the following tables which give you the properties of a function that you are going to sketch.

Step 4 $a_1 > a_2 > a_3 > a_4$

x		$\overset{C.P}{\widehat{a_1}}$		$\overset{undef}{\widehat{a_2}}$		$\overset{C.P}{\widehat{a_3}}$		$\overset{C.P}{\widehat{a_4}}$	
$f(x)$	$+$	0	$-$		$-$	0	$+$	0	$-$
$f'(x)$	↗		↘		↘		↗		↘
		Local max				Abs. Min		Local Max	

Step 5 $b_1 > a_2 > b_2 > b_3$

x		$\overset{Infl.\ P}{\widehat{b_1}}$		$\overset{undef}{\widehat{a_2}}$		$\overset{Infl.\ P}{\widehat{b_2}}$		$\overset{Infl.\ P}{\widehat{b_3}}$	
$f(x)$	$-$	0	$+$		$-$	0	$+$	0	$-$
$f''(x)$	∩		∪		∩		∪		∩
	Concave Down		Concave Up		Concave Down		Concave Up		Concave Down

Step 6 $a_1 > b_1 > a_2 > b_2 > a_3 > b_3 > a_4$

x		$\overset{C.P}{\widehat{a_1}}$		$\overset{Infl.\ P}{\widehat{b_1}}$		$\overset{undef}{\widehat{a_2}}$		$\overset{Infl.\ P}{\widehat{b_2}}$		$\overset{C.P}{\widehat{a_3}}$		$\overset{Infl.\ P}{\widehat{b_3}}$		$\overset{C.P}{\widehat{a_4}}$	
$f'(x)$	$+$	0	$-$		$-$		$-$		$-$	0	$+$		$+$	0	$-$
$f''(x)$	$-$		$-$	0	$+$		$-$	0	$+$		$+$	0	$-$		$-$
$f(x)$	↗		↘		↘		↘		↘		↗		↗		↘
$f(x)$	∩		∩		∪		∩		∪		∪		∩		∩
$f(x)$															

NOTE 8.11

Existence of extreme values on <u>closed</u> intervals *<u>(chapter 4.4)</u>*

If the domain of the function f is a ***<u>closed</u>***, finite interval or a union of finitely many such intervals, and if f is ***<u>continuous</u>*** on that domain, then f <u>must have an absolute maximum value and an absolute minimum value</u>.

NOTE 8.12

Existence of extreme values on <u>open</u> intervals *<u>(chapter 4.4)</u>*

If f is continuous on the open interval (a, b), and if

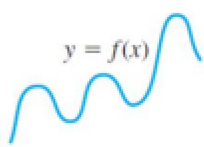
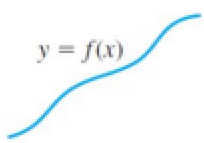
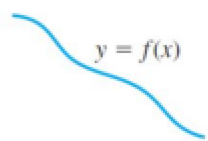
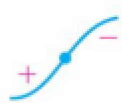
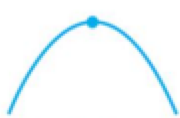
$\lim_{x\to a^+} f(x) = L$ *and* $\lim_{x\to b^-} f(x) = M$, then the following conclusions hold:

(i) *if* $f(u) > L$ *and* $f(u) > M$ *for some* u *in* (a, b), *then* f *has an absolute maximum value on* (a, b)

(i)*if* $f(v) < L$ *and* $f(v) < M$ *for some* v *in* (a, b), *then* f *has an absolute minimum value on* (a, b)

NOTE 8.13

The following table summarizes how the derivative and second derivative affect the shape of a graph.

 Differentiable $\Rightarrow$ smooth, connected; graph may rise and fall	 $y' > 0 \Rightarrow$ rises from left to right; may be wavy	 $y' < 0 \Rightarrow$ falls from left to right; may be wavy
 $y'' > 0 \Rightarrow$ concave up throughout; no waves; graph may rise or fall	 $y'' < 0 \Rightarrow$ concave down throughout; no waves; graph may rise or fall	 y'' changes sign at an inflection point
 y' changes sign $\Rightarrow$ graph has local maximum or local minimum	 $y' = 0$ and $y'' < 0$ at a point; graph has local maximum	 $y' = 0$ and $y'' > 0$ at a point; graph has local minimum

NOTE 8.14

Procedure for Graphing $y = f(x)$

1. Identify the domain of f and any symmetries the curve may have.
2. Identify any asymptotes that may exist (see).
3. Find the derivatives y' and y''
4. Find the critical points of f, if any, and identify the function's behavior at each one.
5. Find where the curve is increasing and where it is decreasing.
6. Find the points of inflection, if any occur, and determine the concavity of the curve.
7. Plot key points, such as the intercepts and the points found in Steps 4–6 and sketch the curve together with any asymptotes that exist.

Example 6

$$\text{let } f(x) = \frac{(3x+1)^2}{(x-1)^2}\text{ . You are given that } f'(x) = -\frac{8(3x+1)}{(x-1)^3} \text{ and } f''(x) = \frac{48(x+1)}{(x-1)^4}$$

Answer the following questions (A - F) using the given information

Answer

A. ***Find the domain of $f(x)$***

The domain of $f(x)$ *is* $\mathbb{R} - \{1\}$ *hint*: $(x-1)^2 \neq 0 \;\therefore\; x-1 \neq 0 \;\therefore\; x \neq 1$

B. ***Find both the x and y intercepts of $f(x)$***

$$x \text{ intercept: } \frac{(3x+1)^2}{(x-1)^2} = 0 \;\rightarrow\; (3x+1)^2 = 0 \;\rightarrow\; 3x+1 = 0 \;\rightarrow\; x = -\frac{1}{3} \qquad x \text{ intercept point is } \left(-\frac{1}{3}, 0\right)$$

$$y \text{ intercepts: } f(0) = \frac{(3(0)+1)^2}{((0)-1)^2} = 1 \;\rightarrow\; f(0) = 1 \qquad y \text{ intercept point is } (0,1)$$

C. ***Determine all asymptotes of $f(x)$.***

Vertical asymp. $\lim_{x\to1^+} f(x) = \infty$ & $\lim_{x\to1^-} f(x) = \infty$ $\therefore x = 1$ *is Vertical asymp.*

Horizontal asymp. $\lim_{x\to\infty} f(x) = 9$ & $\lim_{x\to-\infty} f(x) = 9$ $\therefore y = 9$ *is Horizontal asymp.*

There is no Oblique asymp.

D. Find all local maximum and local minimum of $f(x)$; and determine intervals of increase and decrease.

$f'(x) = -\frac{8(3x+1)}{(x-1)^3}$ *when* $f'(x) = 0$ $\therefore$ $-\frac{8(3x+1)}{(x-1)^3} = 0$ $\rightarrow$ $8(3x+1) = 0$ $\rightarrow$ *at* $x = -\frac{1}{3}$

$f(x)$ has a critical point

There is no singular point.

x		$-\frac{1}{3}$		1	
$f'(x)$	$-$	0	$+$		$-$
$f(x)$	↘		↗		↘

$f(x)$ is increasing on $\left(-\frac{1}{3}, 1\right)$

$f(x)$ is decreasing on $\left(-\infty, -\frac{1}{3}\right)$ & $(1, \infty)$

$f(x)$ has no local max point & $f(x)$ has local min point at $-\frac{1}{3}$ *The critical point is* $\left(-\frac{1}{3}, f\left(-\frac{1}{3}\right)\right)$

E. Determine all inflection point(s) and the intervals where $f(x)$ concave up and concave down.

$f''(x) = \dfrac{48(x+1)}{(x-1)^4}$ *when* $f''(x) = 0$ $\therefore$ $\dfrac{48(x+1)}{(x-1)^4} = 0$ $\rightarrow$ $48(x+1) = 0$

at $x = -1$ *$f(x)$ has an inflection point*

x		-1		1	
$f''(x)$	$-$	0	$+$		$+$
$f(x)$	$\cap$		$\cup$		$\cup$

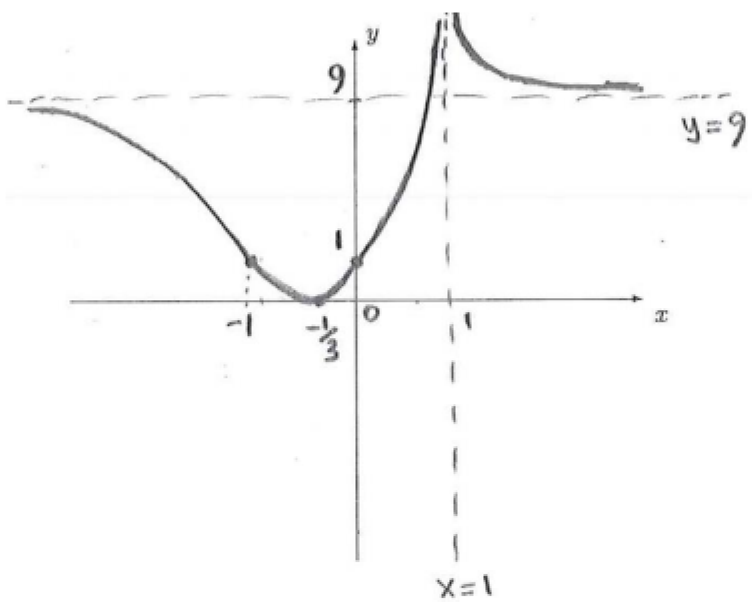
$f(x)$ is concaving up on $(-1,1)$ & $(1,\infty)$

$f(x)$ is concaving down on $(-\infty,-1)$

The inflection point is $(-1, f(-1))$

F. Using the parts a, b, c, d, and e above, sketch the graph $y = f(x)$ in the coordinate axes provided below.

x		-1		$-\frac{1}{3}$		1	
$f'(x)$	$-$		$-$	0	$+$		$-$
$f''(x)$	$\cap$	0	$\cup$		$\cup$		$\cup$
$f(x)$							



Section 9 Extreme-Value Problem

Example 1

Suppose a farmer wishes to enclose a rectangular field using 1000 m of fencing in such a way that the area of the field is maximized. Find the dimensions and the Total area of the field when the area of the field is **maximized**.

Answer

Let x and y be the dimensions of the field and let A be the area of the field. Then, $A = xy$

$1000 = 2x + 2y \quad \rightarrow \quad y = 500 - x.$ where 1000 m is the circumference of the rectangular field

$A = x(500 - x) = 500x - x^2$ The maximum value of A on the interval $x \in [0, 500]$

$\frac{dA}{dx} = 500 - 2x \quad when \ \frac{dA}{dx} = 0 \ \ then \ x = 250\ m \ \ and \ y = 250\ m \quad A|_{x=250} = (250)(250) = 62500\ m^2$

$The\ end\ points{:}\ \ A|_{x=0} = A|_{x=500} = 0 \qquad where\ A|_{x=250} > A|_{x=0} \ \ and \ \ A|_{x=250} > A|_{x=500}$

So A has a maximum value of 62500 m square when $x = 250\ m$ and $y = 500 - 250 = 250\ m$

Section 10 Linear Approximations

NOTE 10.1

The **linearization** of the function f about $\boldsymbol{a}$ is the function $\boldsymbol{L}$ defined by $L(x) = f(a) + f`(a)\,(x - a)$

We say that $f(x) \approx L(x) = f(a) + f`(a)\,(x - a)$ provids **linear approximations** for values of f near to $\boldsymbol{a}$.

Example 1

The equation $x^2y^3 - x^3y^2 = 4$ defines a funcation $y(x)$. **Find the lineriztion** $of\ y(x)\ at\ (1\,,2)$ and use it to **approximate** $y(1.01)$

Answer

using the lineriztion rule: $L(x) = f(a) + f`(a)\,(x - a) \qquad hint{:} \quad a = 1 \ \ and \ x = 1.01$

$from\ (1{,}2)\ since\ \ y = 2 \ \xrightarrow{so} y(a) = \ y(1) = 2$

$\frac{d}{dx}[\,x^2y^3 - x^3y^2\,] = \frac{d}{dx}[\,4\,] \quad \therefore 2xy^3 + 3x^2y^2y` - 3x^2y^2 - 2x^3yy` = 0$ using the point $(1\,,2)$

$2(1)(2)^3 + 3(1)^2(2)^2 y` - 3(1)^2(2)^2 - 2(1)^3(2) y` = 0 \xrightarrow{then} \mathbf{y`(1)} = y`_{at\ x=1} = \frac{-4}{8} = \frac{-1}{2}$

$\therefore$ the **lineriztion** of y at $a = 1 \overset{i.e}{\rightarrow} L(x) = y(1) + y`(1)\,(x-1) = 2 + \frac{-1}{2}(x-1)$

using the approximation rule, we have $f(1.01) \approx L(1.01) = 2 + \frac{-1}{2}(1.01 - 1) = 2 - \frac{1}{200} = \frac{399}{200}$

Section 11 Sums and Sigma Notation

NOTE 11.1

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \int_a^b f(x)\,dx \qquad note\ that \begin{Bmatrix} f(x)\ is\ continous\ on\ [a,b] \\ n\ is\ subinterval\ of\ width\ \Delta x \\ each\ interval\ is\ called\ x_i^* \\ p_n = (x_0, x_1, x_2, \dots\dots, x_n) \end{Bmatrix}$$

Example 1

Evaluate $\lim_{n\to\infty} \frac{\pi}{n} \sum_{i=1}^{n} cos\left(\frac{i\pi}{n}\right)$

Answer

$$\lim_{n\to\infty} \frac{\pi}{n} \sum_{i=1}^{n} cos\left(\frac{i\pi}{n}\right) = \lim_{n\to\infty} \Delta x \sum_{i=1}^{n} f(x_i^*) \overset{\blacksquare}{\Rightarrow} \int_0^{\pi} cos\ x\ dx = \sin x|_0^{\pi} = 0$$

$$\blacksquare \begin{Bmatrix} \Delta x = \frac{\pi}{n}(chosen) = \frac{b-a}{n}(rule),\ \ b - a = \pi \to (1) & \Rightarrow & let\ \ f(x_i^*) = cos\left(\frac{i\pi}{n}\right) \to (2) \\ Let\ x_i^* = \frac{i\pi}{n}(chosen) = a + i\,\Delta x\,(rule) = a + i\,\frac{\pi}{n} & \Rightarrow & \therefore\ a = 0\ \ \therefore b = \pi\ \ from\ eqn\ (1) \\ P_n = \left\{x_0^* = 0, =\ x_1^* = \frac{\pi}{n} \dots\dots.., x_n^* = \pi\right\} & \Rightarrow & x_0^* = 0 = a\,,\ \ x_n^* = \pi = b \\ from\ eqn\ (2), f(x_i^*) = cos\left(\frac{i\pi}{n}\right) = \cos(x_i^*) & \Rightarrow & let\ f(x) = cos\ x\ \ which\ is\ cont.\ on\ [0,\pi] \end{Bmatrix}$$

Example 2

For the funcation $f(x) = xe^{(x^2)}$

write a Riemann Sum obtained by dividing the interval $[1,2]$ into n equal length subintervals.

Answer

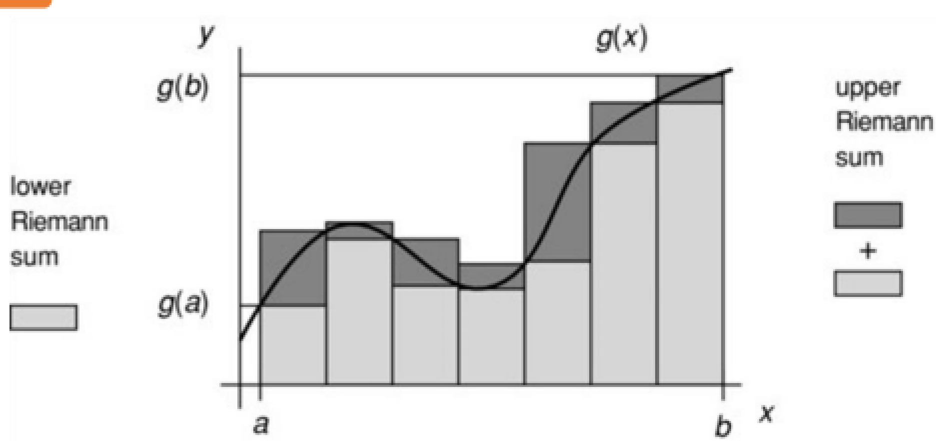
$$\text{we have } \int_1^2 xe^{(x^2)}\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\,\Delta x \;\Rightarrow (\text{using Riemann Sum})$$

$$\text{we have} \Rightarrow \left\{ \begin{matrix} \Delta x = \dfrac{b-a}{n} = \dfrac{2-1}{n} = \dfrac{1}{n}, \quad P_n = \{x_0^* = 1, \ldots\ldots, x_n^* = 2\}, \text{ then } x_i^* = 1 + \dfrac{i}{n} \text{ where } x_i^* = a + i\,\Delta x \\ \text{where } f(x) = xe^{(x^2)} \text{ which is cont. on } [1,2] \end{matrix} \right\}$$

$$\int_1^2 xe^{(x^2)}\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(1+\frac{i}{n}\right)\frac{1}{n} \Rightarrow (\text{using Riemann Sum}) \Rightarrow \lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{i}{n}\right)e^{\left(1+\frac{i}{n}\right)^2}\frac{1}{n}$$

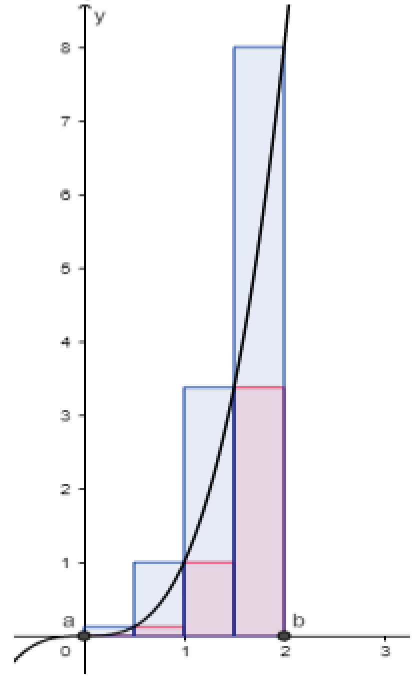
Section 12 The Definite Integral

NOTE 12.1



Example 1

calculate the upper and lower riemann sum for the funcation $f(x) = x^3$, corresponding to the partition P_n of $[0,2]$ into 4 subintervals of equal lenght.



Answer

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$$

$$L(f,p) = \Delta x\,[f(l_1) + f(l_2) + f(l_3) + f(l_4)] = \frac{1}{2}\left[f(0) + f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right)\right]$$

$$= \frac{1}{2}\left[0 + \left(\frac{1}{2}\right)^3 + 1^3 + \left(\frac{3}{2}\right)^3\right] = \frac{9}{4}$$

$$U(f,p) = \Delta x\,[f(u_1) + f(u_2) + f(u_3) + f(u_4)] = \frac{1}{2}\left[f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right) + f(2)\right]$$

$$= \frac{1}{2}\left[\left(\frac{1}{2}\right)^3 + 1^3 + \left(\frac{3}{2}\right)^3 + 2^3\right] = \frac{25}{4}$$

NOTE 12.2

Suppose there is exactly one number I such that for every partition P of $[a,b]$ we have $L(f,P) \le I \le U(f,P)$

Then we say that function f is integrable on $[a,b]$, and we call I the definite integral of function f on $[a,b]$.

The definite integral is denoted by the symbol $I = \int_a^b f(x)\,dx$.

Example 2

By using example 1, **Prove** that $f(x) = x^3$ is **integrable** on the interval $[0,2]$

Answer

Let $I = \int_0^2 x^3\,dx$. such that for every partition P of $[0,2]$ we have $L(f,P) \le I \le U(f,P)$ and

$\because\ \frac{9}{4} \le I \le \frac{25}{4}$ from example 1

Then the function $f(x)$ is integrable on $[0,2]$, and we call I the definite integral of function f on $[0,2]$.

Section 13 Properties of the Definite Integral

NOTE 13.1

If f is an **odd** function (i.e., $f(-x) = -f(x)$), then $\int_{-a}^{a} f(x)\,dx = 0$.

Example 1

Find $\int_{-\pi}^{\pi} \sin x \; dx$

Answer

$$\int_{-\pi}^{\pi} \sin x \; dx = 0 \quad since \quad \sin x \;\; is\ odd\ function \quad i.e \quad \sin(-x) = -\sin x$$

NOTE 13.2

If f is an **even** function (i.e., $f(-x) = f(x)$), then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

Example 2

Find $\int_{-\pi}^{\pi} \cos x \; dx$

Answer

$$\int_{-\pi}^{\pi} \cos x \; dx = 2\int_{0}^{\pi} \cos x \; dx = -2 \sin x|_{0}^{\pi} = -2(\sin \pi - \sin 0) = 0$$

$since \cos x \;\; is\ even\ function \quad i.e \quad \cos(-x) = \cos x$

NOTE 13.3

$$\int_{a}^{a} f(x)\,dx = 0$$

Example 3

Find $\int_{2}^{2} (2 + 5x)\,dx$

Answer

$$\int_{2}^{2} (2 + 5x)\,dx = 0$$

NOTE 14.1

Suppose that the function f is continuous on an interval I containing the point a.

PART I: Let the function F be defined on I by

$$F(x) = \int_a^x f(t)\,dt$$

Then F is differentiable on I, and $F'(x) = f(x)$ there. Thus, F is an antiderivative of f on I:

$$\boxed{\frac{d}{dx}\int_a^x f(t)\,dt = f(x)}$$

PART II: If $F(x)$ is any antiderivative of $f(x)$ on I, so that $F'(x) = f(x)$ on I, then for any b in I we have

$$\boxed{\int_a^b f(x)\,dx = F(x)|_a^b = F(b) - F(a)}$$

Example 1

$F(x) = \int_x^3 e^{-t^2}\,dt \quad find\ \frac{d}{dx}F(x)$

Answer

$$By\ F.T.C,\ \ we\ have\ \frac{d}{dx}F(x) = \left[\frac{d}{dx}(3)\right]\times e^{-(3)^2} - \left[\frac{d}{dx}(x)\right]\times e^{-(x)^2} = [0]\times e^{-6} - [1]\times e^{-x^2} = -e^{-x^2}$$

NOTE 15.1.1

$\int f(x) \pm g(x)\, dx = \int f(x)\, dx \pm \int g(x)\, dx$	$\int c\, f(x)\, dx = c \int f(x)\, dx, \quad c \text{ is constant}$
$\int f(x)\, dx = F(x) + c$ where $F(x)$is an anti-derivative of $f(x)$	$\int_a^b f(x)\, dx = F(x)\|_a^b = F(b) - F(a)$
$\int_a^b c\, dx = c(b-a)\,, \quad c \text{ is constant}$	$\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$

NOTE 15.1.2

Common Integrals

$$\int k\, dx = kx + c$$

$$\int x^n\, dx = \frac{1}{n+1} x^{n+1} + c, n \neq 1$$

$$\int x^{-1}\, dx = \int \frac{1}{x}\, dx = \ln|x| + c$$

$$\int \frac{1}{ax+b} dx = \frac{1}{a} \ln|ax+b| + c$$

$$\int e^x\, dx = e^x + c$$

$$\int \ln x\, dx = x \ln x - x + c$$

$$\int \cos x\, dx = \sin x + c$$

$$\int \sin x\, dx = -\cos x + c$$

$$\int \sec^2 x\, dx = \tan x + c$$

$$\int \sec x \tan x\, dx = \sec x + c$$

$$\int \csc x \cot x\, dx = -\csc x + c$$

$$\int \csc^2 x\, dx = -\cot x + c$$

$$\int \tan x\, dx = \ln|\sec x| + c$$

$$\int \sec x\, dx = \ln|\sec x + \tan x| + c$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$sin^2 x + cos^2 x = 1$$

$$sec^2 x - tan^2 x = 1$$

$$sin(2x) = 2\, sin(x)\, cos(x)$$

$$cos^2(x) = \frac{1}{2}(1 + cos(2x))$$

$$sin^2(x) = \frac{1}{2}(1 - cos(2x))$$

NOTE 15.2.1

The substitution $u = g(x)$ will convert $\int_a^b f(g(x))g'(x)\ dx = \int_{g(a)}^{g(b)} f(u)\ du\ using\ du = g'(x)\ dx$

Example 1

Evaluate $\int_1^2 5\,x^2 \cos(x^3)\,dx$

Answer

$$\int_1^2 5\,x^2 \cos(x^3)\,dx \overset{\blacksquare}{\Longrightarrow} \int_1^8 \frac{5}{3}\cos(u)\,du = \frac{5}{3}\sin(u)\Big|_1^8 = \frac{5}{3}(\sin(8) - \sin(1))$$

$$\blacksquare \begin{cases} u = x^3 \ \to du = 3x^2dx \ \to x^2dx = \dfrac{1}{3}du \\ x = 1 \ \to u = 1^3 = 1 \quad and \quad x = 2 \to u = 2^3 = 8 \end{cases}$$

NOTE 15.2.1

Products and (some) Quotients of Trig Functions

For $\int sin^n(x)\cos^m(x)\,dx$ we have the following:	For $\int tan^n(x)\sec^m(x)\,dx$ we have the following:
1. n odd. Use the substitution $u = cos\ x$	1. n odd. Use the substitution $u = sec\ x$
2. m odd. Use the substitution $u = sin\ x$	2. m even. Use the substitution $u = tan\ x$
(1) & (2) convert rest using $\sin^2 x + \cos^2 x = 1$	(1) & (2) convert rest using $sec^2\,x - tan^2\,x = 1$
3. n and m both even. Use Trig Formulas:	3. n odd and m even. Use either step 1. or 2.
$\sin(2x) = 2\sin(x)\,cos(x)$	4. n even and m odd. Each integral will be dealt with differently.
$\cos^2(x) = \frac{1}{2}(1 + cos(2x))$	
$\sin^2(x) = \frac{1}{2}(1 - cos(2x))$	

Example 2

Evaluate $\int \tan^3 x \sec^5 x dx$

Answer

$$\int \tan^3 x \sec^5 x dx = \int \tan^2 x \sec^4 x \tan x \sec x dx = \int (\sec^2 x - 1)\sec^4 x \tan x \sec x \, dx \qquad (u = \sec x) \rightarrow (du = \tan x \sec x \, dx)$$

$$= \int (u^2 - 1)u^4 du = \frac{1}{7}u^5 - \frac{1}{5}u^5 + c = \frac{1}{7}\sec^7 x - \frac{1}{5}\sec^5 x + c$$

Example 3

Evaluate $\int \frac{\mathrm{Sin}^5 x}{\cos^3 x} dx$

Answer

$$\int \frac{\mathrm{Sin}^5 x}{\cos^3 x} dx = \int \frac{\sin^4 x \sin x}{\cos^3 x} dx = \int \frac{(\sin^2 x)^2 \sin x}{\cos^{3x}} dx = \int \frac{(1 - \cos^2 x)^2 \sin x}{\cos^3 x} dx$$

$$(u = \cos x) \rightarrow (du = -\sin x \, dx)$$

$$= -\int \frac{(1 - u^2)^2}{u^3} du = -\int \frac{1 - 2u^2 + u^4}{u^3} du = -\int \frac{1}{u^3} du + \int \frac{2}{\mathrm{u}} du - \int \mathrm{u} \, du$$

$$= \frac{1}{2}\mathrm{u}^{-2} + 2\ln|\mathrm{u}| - \frac{1}{2}\mathrm{u}^2 + c = \frac{1}{2}\sec^2 x + 2\ln|\cos x| - \frac{1}{2}\cos^2 x + c$$

NOTE 15.3.1

Indefinite Integral	Definite Integral
$\int u\,dv = uv - \int v\,du$	$\int_a^b u\,dv = uv\vert_a^b - \int_a^b v\,du.$

Choose u *and* dv from the integral $\int u\,dv \Rightarrow \begin{cases} u & \xrightarrow{(u)'} & du \\ dv & \xrightarrow{\int dv} & u \end{cases}$

The choice of $u \rightarrow$ function to differentiate. The choice of $dv \rightarrow$ function easily to integrate

Example 1

Evaluate $\int x\,e^{-x}dx$

Answer

$u = x \qquad dv = e^{-x}dx$

$du = dx \qquad v = -e^{-x}$

$$\int x\,e^{-x}dx = -xe^{-x} + \int e^{-x}\,dx$$

$$= -xe^{-x} - e^{-x} + c$$

Example 2

Evaluate $\int_3^5 \ln x\ dx$

Answer

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x}dx \qquad v = x$

$$\int_3^5 \ln\ xdx = x\ln x\vert_3^5 - \int_3^5 d\,x = (x\ln(x) - x)\vert_3^5 = 5\ln(5) - 3\ln(3) - 2$$

NOTE 15.3.2

How to choose u *and* dv in Integrate by Parts Questions

Function	Example	u	dv
Log function	$\int \ln x\,dx$	$\ln x$	dx
Log times algebraic	$\int x^4 \ln x\,dx$	$\ln x$	$x^4 dx$
Log composed with algebraic	$\int \ln x^3\,dx$	$\ln x^3$	dx
Inverse trig forms	$\int \arcsin x\,dx$	$\arcsin x$	dx
Algebraic times sine	$\int x^2 \sin x\,dx$	x^2	$\sin x\,dx$
Algebraic times cosine	$\int 3x^5 \cos x\;dx$	$3x^5$	$\cos x\;dx$
Algebraic times exponential	$\int \frac{1}{2}x^2 e^{3x}\,dx$	$\frac{1}{2}x^2$	$e^{3x}\,dx$
Algebraic times integral	$\int x\left(\int \frac{1}{\sqrt{1+t^3}}\,dt\right)dx$	$\int \frac{1}{\sqrt{1+t^3}}\,dt$	$x\,dx$
Sine times exponential	$\int e^{\frac{x}{2}} \sin x\;dx$	$\sin x$	$e^{\frac{x}{2}}\,dx$
Cosine times exponential	$\int e^x \cos x\;dx$	$\cos x$	$e^x\,dx$

Order of choosing dv is (easiest: exponential $\Longrightarrow$ sine /cosine $\Longrightarrow$ algebraic) which have a simple form

NOTE 15.4.1

Partial Fractions: If integrating $\int \frac{f(x)}{g(x)}dx$ where the degree of $f(x)$ is smaller than the degree of $g(x)$. Both $f(x)$ and $g(x)$ are ONLY polynomial functions.

1. Factor denominator as completely as possible
2. Find the partial fraction decomposition (P.F.D.) of the rational expression.
3. Integrate the partial fraction decomposition (P.F.D.).
4. For each factor in the denominator, we get term(s) in the decomposition according to the following table.

Factor in $g(x)$	Term in P. F. D	Factor in $g(x)$	Term in P. F. D
$ax + b$	$\frac{A}{ax+b}$	$(ax + b)^k$	$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_k}{(ax+b)^k}$
$ax^2 + bx + c$	$\frac{Ax+B}{ax^2+bx+c}$	$(ax^2 + bx + c)^k$	$\frac{A_1x+B_1}{ax^2+bx+c}+\cdots+\frac{A_kx+B_k}{(ax^2+bx+c)^k}$

Example 1

Evaluate $\int \frac{7x^2+13x}{(x-1)(x^2+4)}dx$

Answer

$$\frac{7x^2+13x}{(x-1)(x^2+4)} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+4)} = \frac{A(x^2+4)+(Bx+C)(x-1)}{(x-1)(x^2+4)}$$

$$7x^2+13x = A(x^2+4)+(Bx+C)(x-1) = (A+B)x^2+(C-B)x+4A-C$$

$$\begin{cases} \text{the coefficient of } x^2 \\ \text{the coefficient of } x \\ \text{the coefficient of the constant} \end{cases} \Rightarrow \begin{cases} A+B=7 \\ C-B=13 \\ 4A-C=0 \end{cases} \quad A=4,\ \ B=3,\ \ C=16$$

$$\int \frac{7x^2+13x}{(x-1)(x^2+4)}dx = \int \frac{4}{(x-1)} + \frac{3x+16}{(x^2+4)}dx = \int \frac{4}{(x-1)} + \frac{3x}{(x^2+4)} + \frac{16}{(x^2+4)}dx$$

$$= 4\ln|x-1| + \frac{3}{2}\ln|x^2+4| + 8\tan^{-1}\left(\frac{x}{2}\right) + C$$

NOTE 15.5.1

Trig Substitutions: If the integral contains the **following roots,** then use the given substitution and formula to convert into an integral involving trig function. The form type with (▮) symbol needs to be at the **denominator** of the integrable function. However, the following roots can be shown at both the **denominator** (commonly) and **numerator** (rarely) of the integrable function.

$$\sqrt{\boldsymbol{a^2-b^2x^2}} \xrightarrow{\text{choose}} \sin\theta=\frac{\sqrt{b^2x^2}}{\sqrt{a^2}}=\frac{bx}{a}$$

$$\sqrt{\boldsymbol{b^2x^2-a^2}} \xrightarrow{\text{choose}} \sec\theta=\frac{\sqrt{b^2x^2}}{\sqrt{a^2}}=\frac{bx}{a}$$

$$\sqrt{\boldsymbol{a^2+b^2x^2}} \xrightarrow{\text{choose}} \tan\theta=\frac{\sqrt{b^2x^2}}{\sqrt{a^2}}=\frac{bx}{a}$$

$$(▮)\ \boldsymbol{a^2+b^2x^2} \xrightarrow{\text{choose}} \tan\theta=\frac{\sqrt{b^2x^2}}{\sqrt{a^2}}=\frac{bx}{a}$$

$$\sqrt{\boldsymbol{2^2-3^2x^2}} \xrightarrow{\text{choose}} \sin\theta=\frac{\sqrt{3^2x^2}}{\sqrt{2^2}}=\frac{3x}{2}$$

$$\sqrt{\boldsymbol{3^2x^2-2^2}} \xrightarrow{\text{choose}} \sec\theta=\frac{\sqrt{3^2x^2}}{\sqrt{2^2}}=\frac{3x}{2}$$

$$\sqrt{\boldsymbol{2^2+3^2x^2}} \xrightarrow{\text{choose}} \tan\theta=\frac{\sqrt{3^2x^2}}{\sqrt{2^2}}=\frac{3x}{2}$$

$$(▮)\ \boldsymbol{2^2+3^2x^2} \xrightarrow{\text{choose}} \tan\theta=\frac{\sqrt{3^2x^2}}{\sqrt{2^2}}=\frac{3x}{2}$$

Another form:

$$\sqrt{\boldsymbol{a^2-b^2(x+c)^2}} \xrightarrow{\text{choose}} \sin\theta=\frac{\sqrt{b^2(x+c)^2}}{\sqrt{a^2}}=\frac{b(x+c)}{a}$$

$$\sqrt{\boldsymbol{b^2(x+c)^2-a^2}} \xrightarrow{\text{choose}} \sec\theta=\frac{\sqrt{b^2(x+c)^2}}{\sqrt{a^2}}=\frac{b(x+c)}{a}$$

$$\sqrt{\boldsymbol{a^2+b^2(x+c)^2}} \xrightarrow{\text{choose}} \tan\theta=\frac{\sqrt{b^2(x+c)^2}}{\sqrt{a^2}}=\frac{b(x+c)}{a}$$

$$(▮)\ \boldsymbol{a^2+b^2(x+c)^2} \xrightarrow{\text{choose}} \tan\theta=\frac{\sqrt{b^2(x+c)^2}}{\sqrt{a^2}}=\frac{b(x+c)}{a}$$

$$\sqrt{\boldsymbol{2^2-3^2(x+1)^2}} \xrightarrow{\text{choose}} \sin\theta=\frac{\sqrt{3^2(x+1)^2}}{\sqrt{2^2}}=\frac{3(x+1)}{2}$$

$$\sqrt{\boldsymbol{3^2(x+1)^2-2^2}} \xrightarrow{\text{choose}} \sec\theta=\frac{\sqrt{3^2(x+1)^2}}{\sqrt{2^2}}=\frac{3(x+1)}{2}$$

$$\sqrt{\boldsymbol{2^2+3^2(x+1)^2}} \xrightarrow{\text{choose}} \tan\theta=\frac{\sqrt{3^2(x+1)^2}}{\sqrt{2^2}}=\frac{3(x+1)}{2}$$

$$(▮)\ \boldsymbol{2^2+3^2(x+1)^2} \xrightarrow{\text{choose}} \tan\theta=\frac{\sqrt{3^2(x+1)^2}}{\sqrt{2^2}}=\frac{3(x+1)}{2}$$

Example 1

Evaluate $\int \frac{16x}{\sqrt{4-9x^2}}dx$

Answer

let $\sin\theta = \frac{3x}{2} \rightarrow \quad x = \frac{2}{3}\sin\theta$ by taking the derivative for both sides, then we have $dx = \frac{2}{3}\cos\theta \; d\theta$

$$\sqrt{4-9x^2} = \sqrt{4-9\left(\frac{2}{3}\sin\theta\right)^2} = \sqrt{4-4\sin^2\theta} = \sqrt{4\cos^2\theta} = 2\cos\theta$$

$$\begin{cases} \text{replace } dx \text{ by } \frac{2}{3}\cos\theta \; d\theta \\ \text{replace } x \text{ by } \frac{2}{3}\sin\theta \\ \text{replace } \sqrt{4-9x^2} \text{ by } 2\cos\theta \end{cases}$$

$$\int \frac{16x}{\sqrt{4-9x^2}}dx = \int \frac{16\left(\frac{2}{3}\sin\theta\right)}{(2\cos\theta)}\left(\frac{2}{3}\cos\theta\right)d\theta = \frac{32}{9}\int \sin\theta \, d\theta = -\frac{32}{9}\cos\theta + C = -\frac{16}{9}\sqrt{4-9x^2} + C$$

Section 16 Improper Integrals

NOTE 16.1

Integral is called **convergent** if the limit <u>exists</u> and has a finite value and **divergent** if the limit <u>does NOT exist</u> or has infinite value.

NOTE 16.2

I. Direct test:

If f is continuous on the interval $(a,b]$, we define the improper integral $\int_a^b f(x)\;dx = \lim_{c\to a^+}\int_c^b f(x)\;dx$

If f is continuous on the interval $[a,b)$, we define the improper integral $\int_a^b f(x)\;dx = \lim_{c\to b^-}\int_a^c f(x)\;dx$

>>> **Important**: The following test is used <u>ONLY for positive functions</u>. <<<

$$P-test: \int_a^\infty \frac{1}{x^p}\,dx \rightarrow \begin{cases} p>1 & converges \\ p\le 1 & diverges \end{cases} \quad (\,0<a<\infty\,) \quad \int_0^a \frac{1}{x^p}\,dx \rightarrow \begin{cases} p<1 & converges \\ p\ge 1 & diverges \end{cases}$$

NOTE 16.3

II. Comparison test:

Suppose $f(x)$ and $g(x)$ are two **positive** functions with $0 \le f(x) \le g(x)$. Then we have the following statements:

$$If \int_a^\infty g(x)\ dx\ \ converges\ then\ so\ does \int_a^\infty f(x)\ dx.\ \ 0 \le f(x) \le g(x)$$

$$If\ 0 \le g(x) \le f(x) \int_a^\infty g(x)\ dx\ \ diverges\ then\ so\ does \int_a^\infty f(x)\ dx.$$

Note: the inverse statements are NOT applicable.

NOTE 16.4

III. The Limit Comparison Test (LCT):

Suppose $f(x)$ and $g(x)$ are two **positive** functions and $I_1 = \int_a^\infty f(x)\ dx\ \ and\ \ I_2 = \int_a^\infty g(x)\ dx.$

Further suppose that the 3 different conditions are 1. $\lim\limits_{x\to\infty}\frac{f(x)}{g(x)} = c$ 2. $\lim\limits_{x\to 0^+}\frac{f(x)}{g(x)} = c$ 3. $\lim\limits_{x\to a}\frac{f(x)}{g(x)} = c$

$I_1\ is\ improper$ 1. $at\ x = \infty$ 2. $at\ x = 0$ 3. $at\ x = a$ $where\ 0 < a < \infty$

A. If $c \ne 0$ and $c \ne \infty$ then both integrals converge of both divergent.

B. If $c = 0$ then $0 \le f(x) \le g(x)$. Check NOTE 16.3 Comparison test

C. If $c = \infty$ then $0 \le g(x) \le f(x)$. Check NOTE 16.3 Comparison test

Example 1

This problem has 3 unrelated parts about improper integrals:

(a) Evaluate the improper integral $I = \int_0^\infty xe^{-x^2}\ dx$

(b) Determine whether the integral $\int_0^\infty \frac{1}{(x+e^x)^{1/3}}\ \ dx$ is convergent or divergent.

(c) Find the values of p for which the integral $\int_0^1 \frac{1}{(\tan x)^p}$ is convergent.

Answer

(a) Evaluate the improper integral $I = \int_0^\infty xe^{-x^2}\, dx$

This integral is improper at $x = \infty$ Using Direct test, we have $\lim\limits_{R\to\infty} \int_0^R xe^{-x^2}\, dx$

$let\ I_R = \int_0^R xe^{-x^2}\, dx = -\frac{1}{2}e^{-x^2}\Big|_0^R = -\frac{1}{2}\left(e^{-R^2} - 1\right) = \frac{1}{2}\left(1 - e^{-R^2}\right)$ Hence, $\frac{1}{2}\lim\limits_{R\to\infty}\left(1 - e^{-R^2}\right) = \frac{1}{2}$

(b) Determine whether the integral $\int_0^\infty \frac{1}{(x+e^x)^{1/3}}\, dx$ is convergent or divergent.

This integral is improper at $x = \infty$ For $x \geq 0$, $x + e^x \geq e^x \;\rightarrow\; \frac{1}{x+e^x} \leq \frac{1}{e^x}$

$$0 \leq \frac{1}{(x+e^x)^{\frac{1}{3}}} \leq \frac{1}{e^{\frac{x}{3}}} \quad where \int_0^\infty \frac{1}{e^{\frac{x}{3}}}\, dx = \lim_{R\to\infty}\int_0^R \frac{1}{e^{\frac{x}{3}}}\, dx = \lim_{R\to\infty}\int_0^R e^{-\frac{x}{3}}\, dx = -3\lim_{R\to\infty}\left(e^{-\frac{R}{3}} - e^0\right) = 3$$

$\int_0^\infty \frac{1}{(x+e^x)^{\frac{1}{3}}}\, dx\ \ is\ convergent\ by\ comparison\ test.$

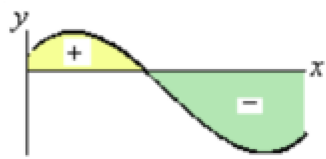
(c) Find the values of p for which the integral $\int_0^1 \frac{1}{(\tan x)^p}$ is convergent.

$$\lim_{x\to 0^+}\frac{\frac{1}{(\tan x)^p}}{\frac{1}{(x)^p}} = \lim_{x\to 0^+}\frac{x^p}{(\cos x)^p}\cdot(\cos x)^p = 1 \quad for\ all\ p.\ Recall \int_0^1 \frac{1}{x^p}\, dx \;\rightarrow p < 1\ \ converges$$

So, by the Limit Comparison Test, $\int_0^1 \frac{1}{(\tan x)^p}\ \ is\ \ converges\ \ for\ \ p < 1$

NOTE 17.1

Net Area: $\int_a^b f(x)\ dx$ represents the net area between $f(x)$ and the $x-axis$ is with area above $x-axis$ is positive and area below $x-axis$ negative.



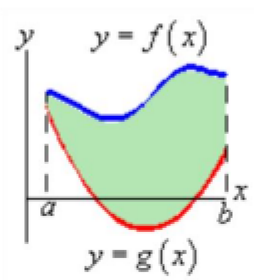
Area Between Curves: The general formulas for the two main cases for each are,

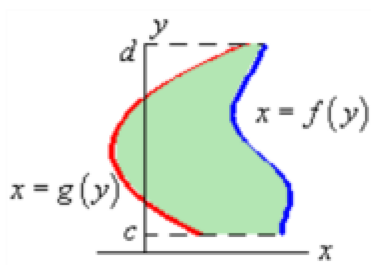
$$y = f(x) \Rightarrow A = \int_a^b [upper\ function] - [lower\ function]\ dx$$

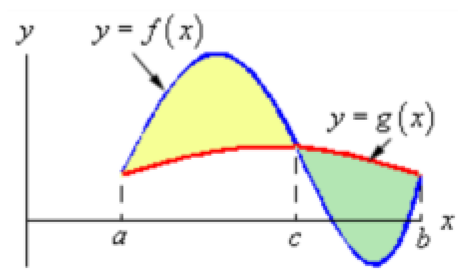
$$x = f(y) \Rightarrow A = \int_c^d [right\ function] - [left\ function]\ dy$$

NOTE 17.2

If the curves intersect then the area of each portion must be found individually. Here are some sketches of a couple possible situations and formulas for a couple of possible cases.







$A = \int_a^b f(x) - g(x)\ dx$	$A = \int_c^d f(y) - g(y)\ dy$	$A = \int_a^c f(x) - g(x)\ dx + \int_c^b g(x) - f(x)\ dx$

Example 1

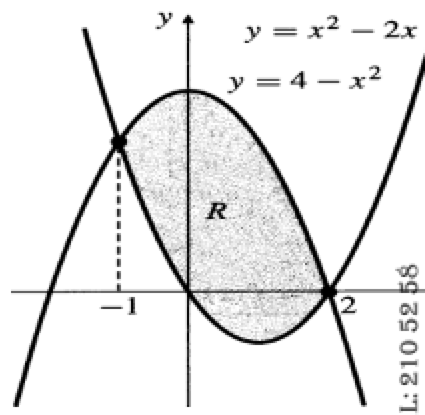
Find the area of the bounded, plane region R lying between the curves $y = x^2 - 2x$ and $y = 4 - x^2$.

Answer

First, we must find the intersections of the curves, so we solve the equations simultaneously:

$x^2 - 2x = y = 4 - x^2 \qquad 2x^2 - 2x - 4 = 0 \quad 2(x-2)(x+1) = 0 \;\; so\; x = 2 \; or \; x = 1$

Since $4 - x^2 \geq x^2 - 2x \;\; for - 1 \leq x \leq 2, the\ area\ A\ of\ R\ is\ given\ by$



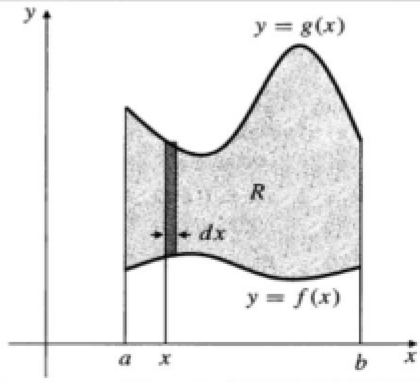
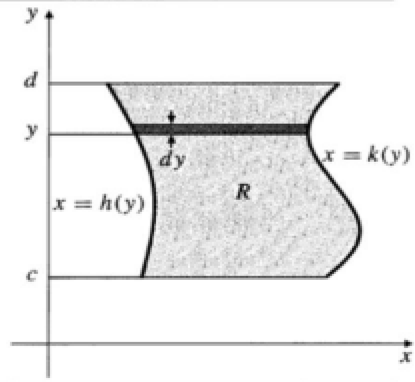
$$A = \int_{-1}^{2} \big((4 - x^2) - (x^2 - 2x)\big)\, dx$$

$$= \int_{-1}^{2} (4 - 2x^2 + 2x)\, dx = \left(4x - \frac{2}{3}x^3 + x^2\right)\Big|_{-1}^{2}$$

$$= 4(2) - \frac{2}{3}(8) + 4 - \left(-4 + \frac{2}{3} + 1\right) = 9 \;\; square\ unites.$$

NOTE 18.1

Table 1. Volumes of solids of revolution

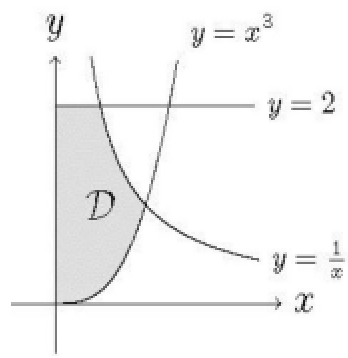
If region R ⟶ is rotated about ↓		
the x-axis	use plane slices $V = \pi \int_a^b \left((g(x))^2 - (f(x))^2\right) dx$	use cylindrical shells $V = 2\pi \int_c^d y\,(k(y) - h(y))\, dy$
the y-axis	use cylindrical shells $V = 2\pi \int_a^b x\,(g(x) - f(x))\, dx$	use plane slices $V = \pi \int_c^d \left((k(y))^2 - (h(y))^2\right) dy$

Example 1

let D be the region in the plane described by all four inequalities

$$x \geq 0, \qquad y \leq 2, \qquad y \geq x^3, y \leq \frac{1}{x}$$

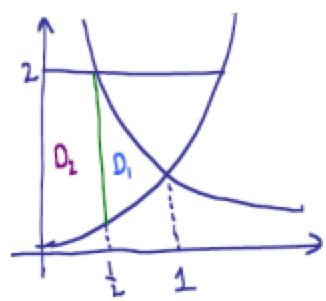
(See the figure on the right). Answer the questions in bold form.



Answer

a) Let S be the solid of the revolution obtained by rotating the region D about y-axis.

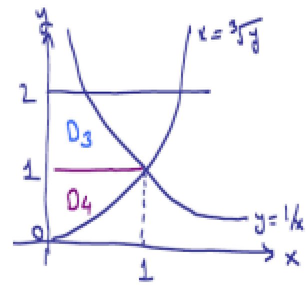
(I) Set up the integral(s) which give the volume of S using cylindrical shells method (DO NOT EVALUATE).



Divide D into two regions D_1 *and* D_2 using $v = \int_a^b 2\pi r h\, dx$ cylindrical shells method.

Thus, $v = \int_0^{1/2} 2\pi x(2 - x^3)\, dx + \int_{1/2}^{1} 2\pi x\left(\frac{1}{x} - x^3\right)\, dx$

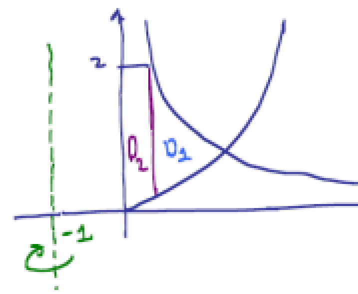
(II) Set up the integral(s) which give the volume of S using washer (slicing, disc) method (DO NOT EVALUATE).



using $v = \int_a^b \pi(R^2 - r^2)\, dx$ washer method

In D_4: $r = 0,\ R = \frac{1}{y}$ In D_4: $r = 0, R = \sqrt[3]{y}$ $\therefore v = \int_0^1 \pi\left(\sqrt[3]{y}\right)^2 dy + \int_1^2 \pi \left(\frac{1}{y}\right)^2 dy$

b) Let T be the solid of the revolution obtained by rotating the region D about the line $x = -1$. Set up but (DO NOT EVALUATE) the integral(s) which give the volume of T using cylindrical shells method.



using $v = \int_a^b 2\pi r h\, dx$ cylindrical shells method.

Thus, $v = \int_0^{1/2} 2\pi(x+1)(2 - x^3)\, dx + \int_{1/2}^{1} 2\pi(x+1)\left(\frac{1}{x} - x^3\right)\, dx$

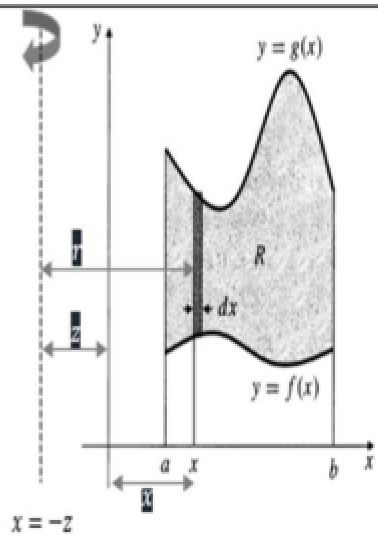
NOTE 18.2

$r = x + z \quad as\ x < 0$

$h = g(x) - f(x)$

$At\ z = 0\ (y - axis)$

$\therefore r = x$

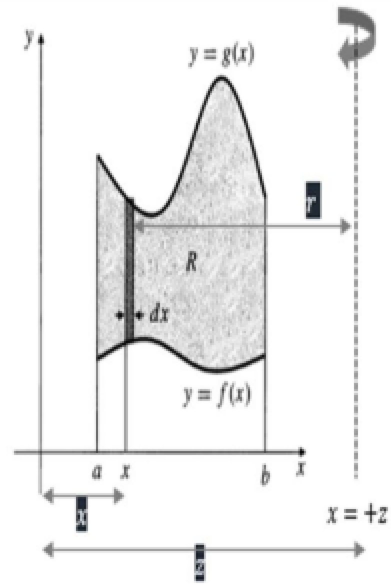


$r = z - x \quad as\ x > 0$

$h = g(x) - f(x)$

$At\ z = 0\ (y - axis)$

$\therefore r = x$

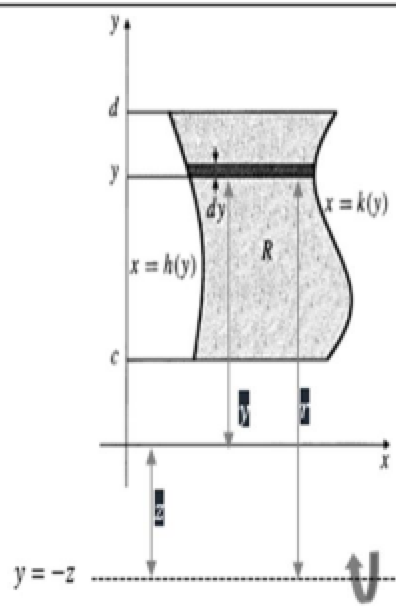


$r = y + z \quad as\ y < 0$

$h = k(y) - h(y)$

$At\ z = 0\ (x - axis)$

$\therefore r = y$

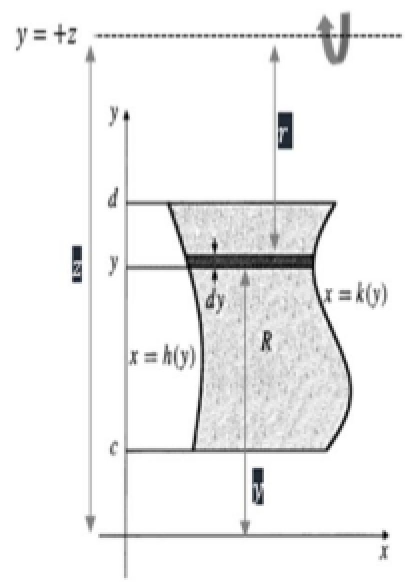


$r = z - y \quad as\ y > 0$

$h = k(y) - h(y)$

$At\ z = 0\ (x - axis)$

$\therefore r = y$



For the cylindrical shell method, the representative rectangle is always parallel to the axis of revolution (rotation)

$$V = 2\pi \int_a^b r\,h\,dx \quad (\text{Vertical Rectangle Element}) \qquad \leftrightarrow \qquad V = 2\pi \int_c^d r\,h\,dy \quad (\text{Horizontal Rectangle Element})$$

r: Distance between the axis of revolution and rectangle element. h: height of rectangle element

In a horizontal rectangle element: y: Distance between the x-axis and rectangle element (fixed). dy is the width of the rectangle element

In a vertical rectangle element: x: Distance between the y-axis and rectangle element (fixed). dx is the width of the rectangle element

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NOTE 18.3

As $x<0$

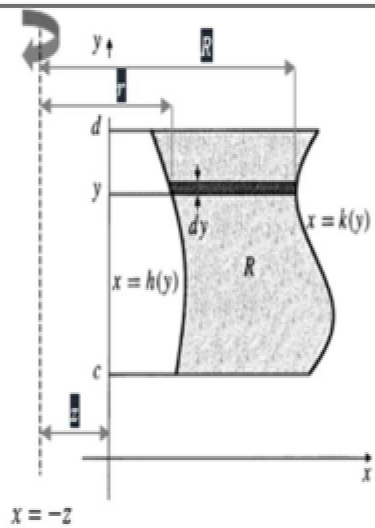
$R=k(y)+z$

$r=h(y)+z$

At $z=0$ ($y-axis$)

$\therefore R=k(y)$

$\therefore r=h(y)$



As $x>0$

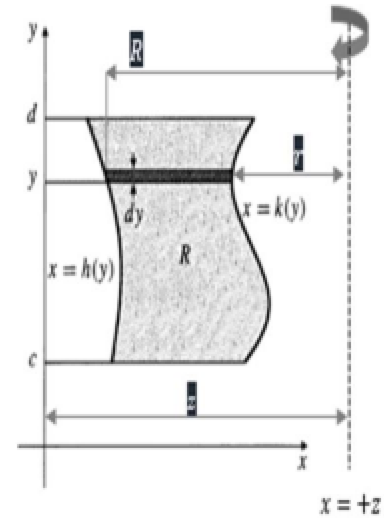
$R=z-h(y)$

$r=z-k(y)$

At $z=0$ ($y-axis$)

$\therefore R=k(y)$

$\therefore r=h(y)$



As $y<0$

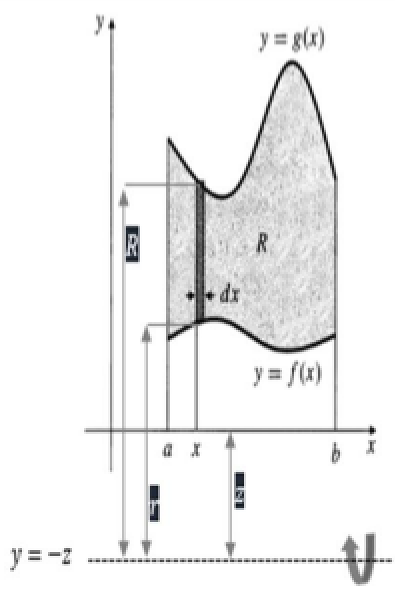
$\therefore R=g(x)+z$

$\therefore r=f(x)+z$

At $z=0$ ($x-axis$)

$\therefore R=g(x)$

$\therefore r=f(x)$



As $y>0$

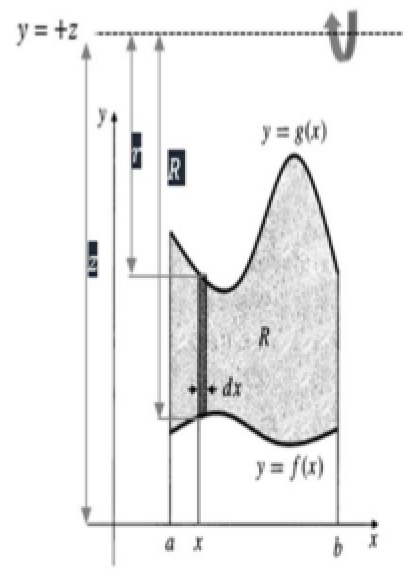
$\therefore R=z-f(x)$

$\therefore r=z-g(x)$

At $z=0$ ($x-axis$)

$\therefore R=g(x)$

$\therefore r=f(x)$



For the disc (slicing, washer) method, the representative rectangle is always perpendicular to the axis of revolution (rotation)

$$V=\pi\int_a^b (R^2-r^2)\,dx \quad \text{(Vertical Rectangle Element)} \quad \leftrightarrow \quad V=\pi\int_c^d (R^2-r^2)\,dy \quad \text{(Horizontal Rectangle Element)}$$

R (Outer Radius): Distance between the axis of revolution and the ending (furthest) edge of the rectangle element.

r (Inner Radius): Distance between the axis of revolution and the starting (nearest) edge of the rectangle element.

In a horizontal rectangle element: dy is the width of the rectangle element

In a vertical rectangle element: dx is the width of the rectangle element

NOTE 19.1

arclengh is $l = \int_{x=a}^{x=b} ds$ where $ds = \sqrt{1 + \left(f'(x)\right)^2}dx \quad \leftrightarrow \quad l = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}dx$

Example 1

Write a difinite integral which is equal to the ***arclengh*** *of the curve* $y = e^x$ *between the point* $(0,1)$ *and* $(1, e)$. *DO NOT EVALUATE.*

Answer

$$l = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}dx \text{ where} \begin{cases} f'(x) = e^x \\ a = 0\,, \quad b = 1 \end{cases} \qquad \therefore \text{arclengh } l = \int_0^1 \sqrt{1 + e^{2x}}dx$$

NOTE 19.2

Area of a surface of revaluation

If $f'(x)$ is continuous on $[a, b]$ and the curve $y = f(x)$ is rotated about the

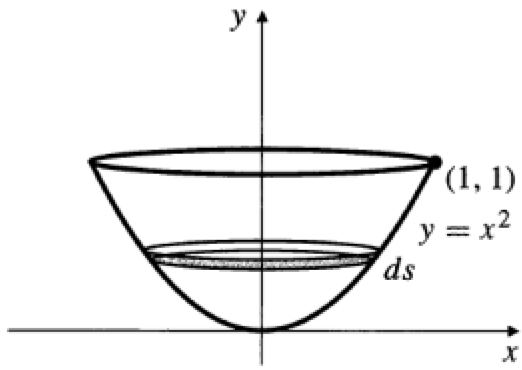
x –axis	y –axis
The surface area is $S = 2\pi \int_{x=a}^{x=b} \lvert y \rvert \, ds$ $\boxed{S = 2\pi \int_{x=a}^{x=b} \lvert f(x) \rvert \sqrt{1 + \left(f'(x)\right)^2}dx}$	The surface area is $S = 2\pi \int_{x=a}^{x=b} \lvert x \rvert \, ds$ $\boxed{S = 2\pi \int_{x=a}^{x=b} \lvert x \rvert \sqrt{1 + \left(f'(x)\right)^2}dx}$

If $g'(x)$ is continuous on $[c, d]$ and the curve $x = g(y)$ is rotated about the

x –axis	y –axis
The surface area is $S = 2\pi \int_{y=c}^{y=d} \lvert y \rvert \, ds$ $\boxed{S = 2\pi \int_{y=c}^{y=d} \lvert y \rvert \sqrt{1 + \left(g'(y)\right)^2}dy}$	The surface area is $S = 2\pi \int_{y=c}^{y=d} \lvert x \rvert \, ds$ $\boxed{S = 2\pi \int_{y=c}^{y=d} \lvert g(y) \rvert \sqrt{1 + \left(g'(y)\right)^2}dy}$

Example 2

Find the surface area of a parabolic reflector whose shape is obtained by rotating the parabolic arc $y = x^2$, $(0 \leq x \leq 1)$, about the y —axis, as illustrated in Figure below.



Answer

The arc length element for the parabola $y = x^2$ is $ds = \sqrt{1 + yx^2}\, dx$, so the required surface area is

$$s = 2\pi \int_0^1 x\sqrt{1 + 4x^2}\, dx \overset{\blacksquare}{=} \frac{\pi}{4}\int_1^5 u^{1/2}\, du = \frac{\pi}{6} u^{3/2}\Big|_1^5 = \frac{\pi}{6}\left(5\sqrt{5} - 1\right) \text{ square units}$$

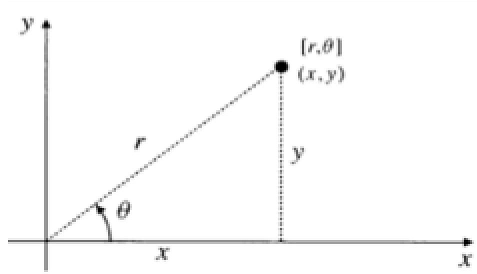
$$\blacksquare \begin{cases} \text{Let } u = 1 + 4x^2 \quad \rightarrow \quad du = 8x\, dx \\ x = 0 \rightarrow u = 1 \quad and \quad x = 1 \rightarrow u = 5 \end{cases}$$

Section 20 Polar Coordinates and Polar Curves

NOTE 20.1

Polar-rectangular conversion

$x = r\cos\theta \qquad y = r\sin\theta \qquad x^2 + y^2 = r^2 \qquad \tan\theta = \frac{y}{x}$



Example 1

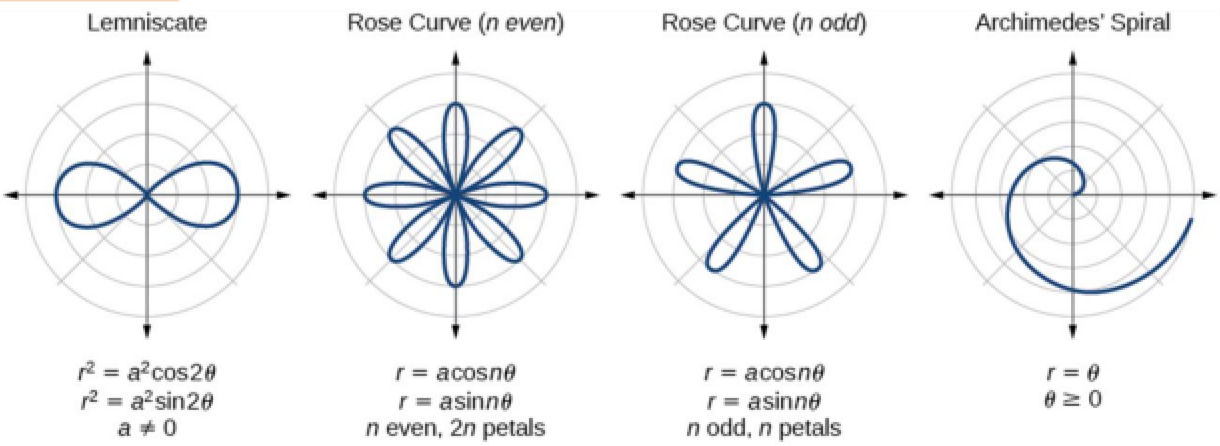
Find the Cartesian equation of the curve represented by the polar eqation $r = \sin 2\,\theta$

Answer

$$r = \sin 2\,\theta \; = 2 \sin\theta \cos\theta$$

$$r^3 \; = 2\,(r \sin\theta)(r\cos\theta) \;\; \rightarrow \;\; \sqrt[3]{x^2 + y^2} = 2xy$$

NOTE 20.2



ALGEBRA

Arithmetic Operations

$a(b+c) = ab + ac$

$\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad+bc}{bd}$

$\dfrac{a+c}{b} = \dfrac{a}{b} + \dfrac{c}{b}$

$\dfrac{\frac{a}{b}}{\frac{c}{d}} = \dfrac{a}{b} \times \dfrac{d}{c} = \dfrac{ad}{bc}$

Exponents and Radicals

$x^m x^n = x^{m+n}$

$\dfrac{x^m}{x^n} = x^{m-n}$

$(x^m)^n = x^{mn}$

$x^{-n} = \dfrac{1}{x^n}$

$(xy)^n = x^n y^n$

$\left(\dfrac{x}{y}\right)^n = \dfrac{x^n}{y^n}$

$x^{1/n} = \sqrt[n]{x}$

$x^{m/n} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m$

$\sqrt[n]{xy} = \sqrt[n]{x}\sqrt[n]{y}$

$\sqrt[n]{\dfrac{x}{y}} = \dfrac{\sqrt[n]{x}}{\sqrt[n]{y}}$

Factoring Special Polynomials

$x^2 - y^2 = (x+y)(x-y)$

$x^3 + y^3 = (x+y)(x^2 - xy + y^2)$

$x^3 - y^3 = (x-y)(x^2 + xy + y^2)$

Binomial Theorem

$(x+y)^2 = x^2 + 2xy + y^2$ $\qquad (x-y)^2 = x^2 - 2xy + y^2$

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2}x^{n-2}y^2$$

$$+ \cdots + \binom{n}{k}x^{n-k}y^k + \cdots + nxy^{n-1} + y^n$$

where $\dbinom{n}{k} = \dfrac{n(n-1)\cdots(n-k+1)}{1\cdot 2\cdot 3\cdot \cdots \cdot k}$

Quadratic Formula

If $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Inequalities and Absolute Value

If $a < b$ and $b < c$, then $a < c$.

If $a < b$, then $a + c < b + c$.

If $a < b$ and $c > 0$, then $ca < cb$.

If $a < b$ and $c < 0$, then $ca > cb$.

If $a > 0$, then

$|x| = a$ means $x = a$ or $x = -a$

$|x| < a$ means $-a < x < a$

$|x| > a$ means $x > a$ or $x < -a$

GEOMETRY

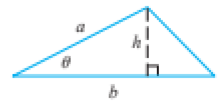
Geometric Formulas

Formulas for area A, circumference C, and volume V:

Triangle

$A = \frac{1}{2}bh$

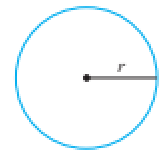
$= \frac{1}{2}ab\sin\theta$



Circle

$A = \pi r^2$

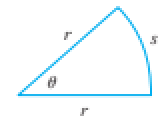
$C = 2\pi r$



Sector of Circle

$A = \frac{1}{2}r^2\theta$

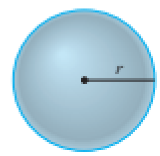
$s = r\theta$ (θ in radians)



Sphere

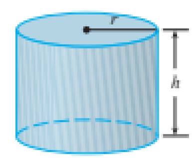
$V = \frac{4}{3}\pi r^3$

$A = 4\pi r^2$



Cylinder

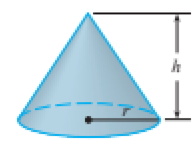
$V = \pi r^2 h$



Cone

$V = \frac{1}{3}\pi r^2 h$

$A = \pi r\sqrt{r^2 + h^2}$



Distance and Midpoint Formulas

Distance between $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Midpoint of $\overline{P_1P_2}$: $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$

Lines

Slope of line through $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Point-slope equation of line through $P_1(x_1, y_1)$ with slope m:

$$y - y_1 = m(x - x_1)$$

Slope-intercept equation of line with slope m and y-intercept b:

$$y = mx + b$$

Circles

Equation of the circle with center (h, k) and radius r:

$$(x-h)^2 + (y-k)^2 = r^2$$

TRIGONOMETRY

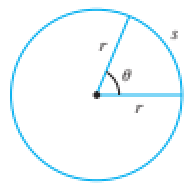
Angle Measurement

π radians $= 180°$

$1° = \dfrac{\pi}{180}$ rad $\qquad 1$ rad $= \dfrac{180°}{\pi}$

$s = r\theta$

(θ in radians)

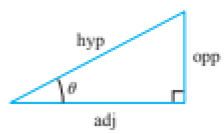


Right Angle Trigonometry

$\sin\theta = \dfrac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \dfrac{\text{hyp}}{\text{opp}}$

$\cos\theta = \dfrac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \dfrac{\text{hyp}}{\text{adj}}$

$\tan\theta = \dfrac{\text{opp}}{\text{adj}} \qquad \cot\theta = \dfrac{\text{adj}}{\text{opp}}$

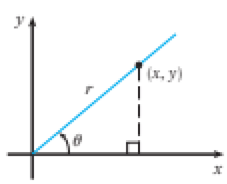


Trigonometric Functions

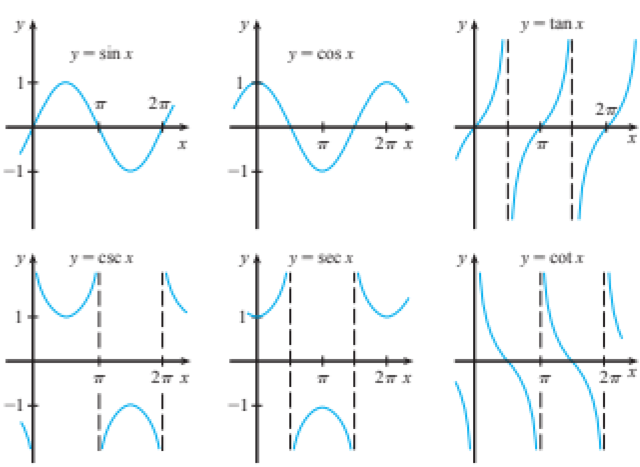
$\sin\theta = \dfrac{y}{r} \qquad \csc\theta = \dfrac{r}{y}$

$\cos\theta = \dfrac{x}{r} \qquad \sec\theta = \dfrac{r}{x}$

$\tan\theta = \dfrac{y}{x} \qquad \cot\theta = \dfrac{x}{y}$



Graphs of Trigonometric Functions



Trigonometric Functions of Important Angles

θ	radians	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0	0	1	0
30°	$\pi/6$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$
45°	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
60°	$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
90°	$\pi/2$	1	0	—

Fundamental Identities

$\csc\theta = \dfrac{1}{\sin\theta} \qquad \sec\theta = \dfrac{1}{\cos\theta}$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta} \qquad \cot\theta = \dfrac{\cos\theta}{\sin\theta}$

$\cot\theta = \dfrac{1}{\tan\theta} \qquad \sin^2\theta + \cos^2\theta = 1$

$1 + \tan^2\theta = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$

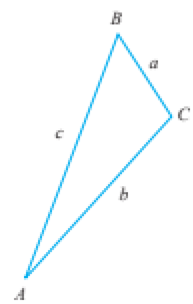
$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta \qquad \sin\left(\dfrac{\pi}{2} - \theta\right) = \cos\theta$

$\cos\left(\dfrac{\pi}{2} - \theta\right) = \sin\theta \qquad \tan\left(\dfrac{\pi}{2} - \theta\right) = \cot\theta$

The Law of Sines

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



The Law of Cosines

$a^2 = b^2 + c^2 - 2bc\cos A$

$b^2 = a^2 + c^2 - 2ac\cos B$

$c^2 = a^2 + b^2 - 2ab\cos C$

Addition and Subtraction Formulas

$\sin(x + y) = \sin x\cos y + \cos x\sin y$

$\sin(x - y) = \sin x\cos y - \cos x\sin y$

$\cos(x + y) = \cos x\cos y - \sin x\sin y$

$\cos(x - y) = \cos x\cos y + \sin x\sin y$

$\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x\tan y}$

$\tan(x - y) = \dfrac{\tan x - \tan y}{1 + \tan x\tan y}$

Double-Angle Formulas

$\sin 2x = 2\sin x\cos x$

$\cos 2x = \cos^2x - \sin^2x = 2\cos^2x - 1 = 1 - 2\sin^2x$

$\tan 2x = \dfrac{2\tan x}{1 - \tan^2x}$

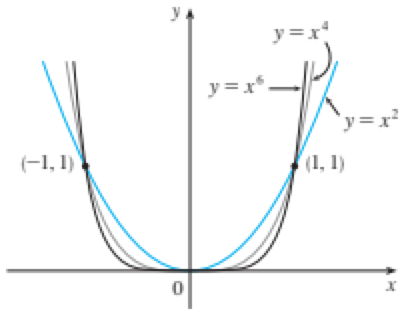
Half-Angle Formulas

$\sin^2x = \dfrac{1 - \cos 2x}{2} \qquad \cos^2x = \dfrac{1 + \cos 2x}{2}$

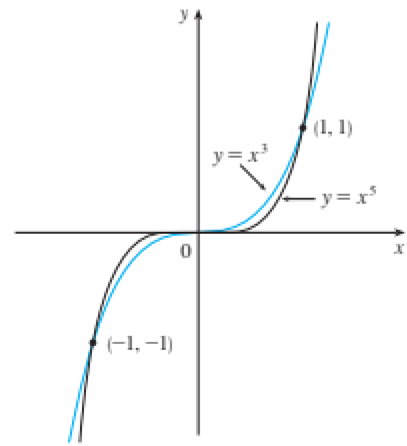
SPECIAL FUNCTIONS

Power Functions $f(x) = x^a$

(i) $f(x) = x^n$, n a positive integer

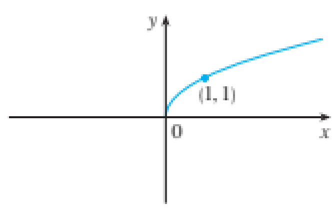


n even

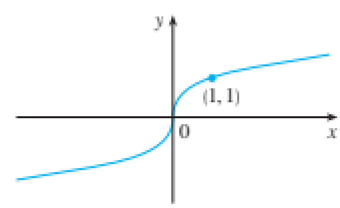


n odd

(ii) $f(x) = x^{1/n} = \sqrt[n]{x}$, n a positive integer

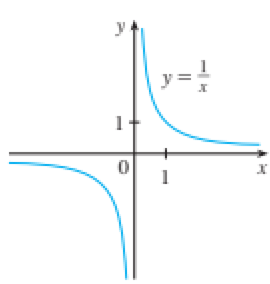


$f(x) = \sqrt{x}$



$f(x) = \sqrt[3]{x}$

(iii) $f(x) = x^{-1} = \dfrac{1}{x}$

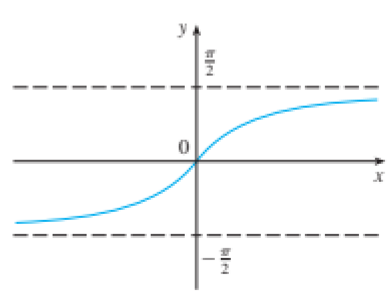


Inverse Trigonometric Functions

$$\arcsin x = \sin^{-1}x = y \iff \sin y = x \quad \text{and} \quad -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$$

$$\arccos x = \cos^{-1}x = y \iff \cos y = x \quad \text{and} \quad 0 \leqslant y \leqslant \pi$$

$$\arctan x = \tan^{-1}x = y \iff \tan y = x \quad \text{and} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$



$y = \tan^{-1}x = \arctan x$

$$\lim_{x \to -\infty} \tan^{-1}x = -\frac{\pi}{2}$$

$$\lim_{x \to \infty} \tan^{-1}x = \frac{\pi}{2}$$

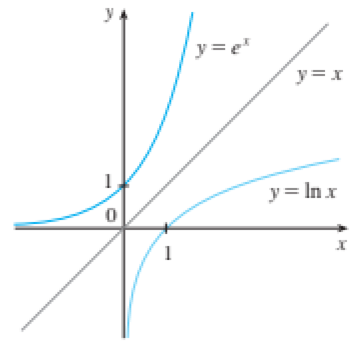
SPECIAL FUNCTIONS

Exponential and Logarithmic Functions

$\log_a x = y \iff a^y = x$

$\ln x = \log_e x, \quad \text{where} \quad \ln e = 1$

$\ln x = y \iff e^y = x$



$\lim_{x \to -\infty} e^x = 0 \qquad \lim_{x \to \infty} e^x = \infty$

$\lim_{x \to 0^+} \ln x = -\infty \qquad \lim_{x \to \infty} \ln x = \infty$

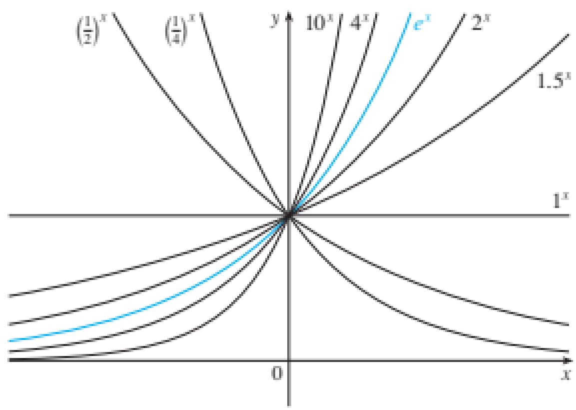
Cancellation Equations

$\log_a(a^x) = x \qquad a^{\log_a x} = x$

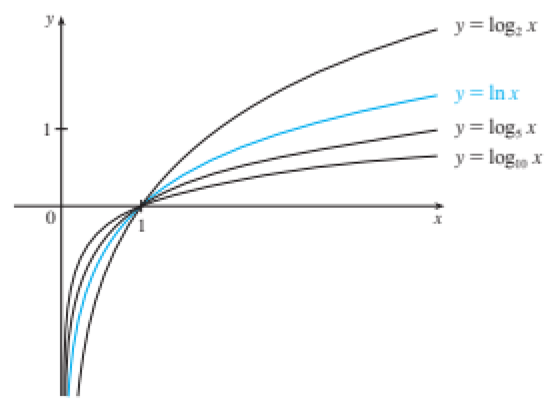
$\ln(e^x) = x \qquad e^{\ln x} = x$

Laws of Logarithms

1. $\log_a(xy) = \log_a x + \log_a y$
2. $\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$
3. $\log_a(x^r) = r \log_a x$



Exponential functions



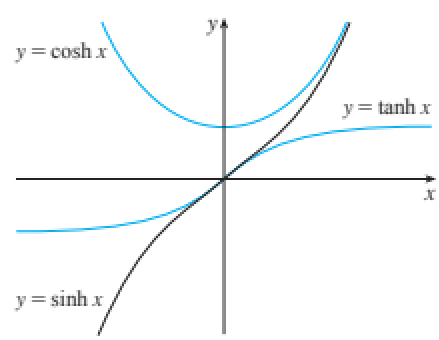
Logarithmic functions

Hyperbolic Functions

$\sinh x = \dfrac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \dfrac{1}{\sinh x}$

$\cosh x = \dfrac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \dfrac{1}{\cosh x}$

$\tanh x = \dfrac{\sinh x}{\cosh x} \qquad \coth x = \dfrac{\cosh x}{\sinh x}$



Inverse Hyperbolic Functions

$y = \sinh^{-1} x \iff \sinh y = x \qquad \sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$

$y = \cosh^{-1} x \iff \cosh y = x \text{ and } y \geq 0 \qquad \cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right)$

$y = \tanh^{-1} x \iff \tanh y = x \qquad \tanh^{-1} x = \frac{1}{2} \ln\left(\dfrac{1 + x}{1 - x}\right)$