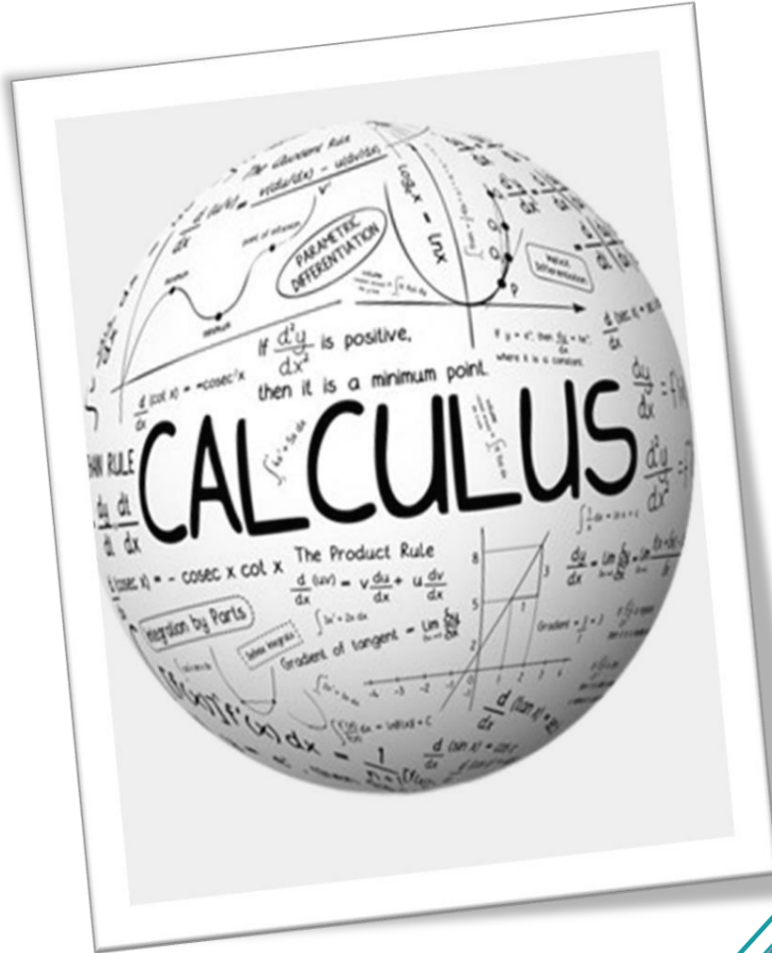


# LECTURE NOTES (DERS NOTLARI)

## [MATH 120 BOOKLET]



<https://lms.rishabooklet.com>



**RISHA  
BOOKLET**  
Road To Success

### CALCULUS OF FUNCTIONS OF SEVERAL VARIABLES

Prepared by: Risha

Last Updated: 2021 - 2022

Makina Müh. Bölümü B Blok No:105

TEL: 0312 210 52 58

WhatsApp İletişim: 0539 499 0 117

## INTRODUCTION

This booklet is for those who are taking Math120 (Calculus of Functions of Several Variables).

What does the booklet offer?

1. Sufficient way to **read** the calculus subjects.
2. Organized way to **understand** calculus theorems and some proves as well as some necessary rules.
3. Sample examples to **understand** how to solve the related problems
4. Some suggested problems to **practice**
5. Appendix which contains functions and their graphs Geometric formulas
6. Short answer key of the suggested problems as well as full solution in the answer key booklet.

*Read → Understand → Practice → AA*

What are the sources used on this booklet?

1. Calculus a Complete Course Textbook
2. James Stewart's CALCULUS Textbook
3. Thomas' Calculus Book by George B. Thomas Textbook

To have chance of having mock/sample exams as well as other online extra recourses and support, you may join MATH120 Course page via <https://lms.rishabooklet.com>

If you have any inquiries, please do not hesitate to contact me via

Email: [rishabooklet@gmail.com](mailto:rishabooklet@gmail.com)

WhatsApp: <https://wa.me/00201553122454>

I hope such a work helps you to reach AA letter grade.

Best Regards,

*Risha*



| Booklet Sections (RISHA BOOKLET)                                       | Book Chapters (A Complete Course Calculus)                                |
|------------------------------------------------------------------------|---------------------------------------------------------------------------|
| Section 1.1: Sequences and Convergence                                 | 9.1 Sequences and Convergence                                             |
| Section 1.2: Infinite Series                                           | 9.2 Infinite Series                                                       |
| Section 1.3: Convergence Tests for Positive Series                     | 9.3 Convergence Tests for Positive Series                                 |
| Section 1.4: Absolute and Conditional Convergence                      | 9.4 Absolute and Conditional Convergence                                  |
| Section 1.5: Power Series                                              | 9.5 Power Series                                                          |
| Section 1.6: Taylor and Maclaurin Series                               | 9.6 Taylor and Maclaurin Series                                           |
| Section 1.7: Applications of Taylor and Maclaurin Series               | 9.7 Applications of Taylor and Maclaurin Series                           |
| Section 2.1: Analytic Geometry in Three Dimensions                     | 10.1 Analytic Geometry in Three Dimensions                                |
| Section 2.2: Vectors                                                   | 10.2 Vectors                                                              |
| Section 2.3: The Cross Product in 3-Space                              | 10.3 The Cross Product in 3-Space                                         |
| Section 2.4: Planes and Lines                                          | 10.4 Planes and Lines                                                     |
| Section 2.5: Quadric Surfaces                                          | 10.5 Quadric Surfaces                                                     |
| Section 3.1: Functions of Several Variables                            | 12.1 Functions of Several Variables                                       |
| Section 3.2: Limits and Continuity                                     | 12.2 Limits and Continuity                                                |
| Section 3.3: Partial Derivatives                                       | 12.3 Partial Derivatives                                                  |
| Section 3.4: Higher-Order Derivatives                                  | 12.4 Higher-Order Derivatives                                             |
| Section 3.5: The Chain Rule                                            | 12.5 The Chain Rule                                                       |
| Section 3.6: Linear Approximations                                     | 12.6 Linear Approximations                                                |
| Section 3.7: Gradients and Directional Derivatives                     | 12.7 Gradients and Directional Derivatives                                |
| Section 3.8: Implicit Functions                                        | 12.8 Implicit Functions ( <i>"Systems of Equations" is not included</i> ) |
| Section 4.1: Extreme Values                                            | 13.1 Extreme Values                                                       |
| Section 4.2: Extreme Values of Functions Defined on Restricted Domains | 13.2 Extreme Values of Functions Defined on Restricted Domains            |
| Section 4.3: Lagrange Multipliers                                      | 13.3 Lagrange Multipliers                                                 |
| Section 5.1: Double Integrals                                          | 14.1 Double Integrals                                                     |
| Section 5.2: Iteration of Double Integrals in Cartesian Coordinates    | 14.2 Iteration of Double Integrals in Cartesian Coordinates               |
| Section 5.3: Double Integrals in Polar Coordinates                     | 14.4 Double Integrals in Polar Coordinates                                |
| Section 5.4: Triple Integrals                                          | 14.5 Triple Integrals                                                     |
| Section 5.5: Change of Variables in Triple Integrals                   | 14.6 Change of Variables in Triple Integrals                              |
| Section 6.1: Vector Functions of One Variable                          | 11.1 Vector Functions of One Variable                                     |
| Section 6.2: Curves and Parametrizations                               | 11.3 Curves and Parametrizations                                          |
| Section 7.1: Line Integrals                                            | 15.3 Line Integrals                                                       |
| Section 7.2: Vector and Scalar Fields                                  | 15.1 Vector and Scalar Fields                                             |
| Section 7.3: Gradient, Divergence, and Curl                            | 16.1 Gradient, Divergence, and Curl                                       |
| Section 7.4: Conservative Fields                                       | 15.2 Conservative Fields                                                  |
| Section 7.5: Line Integrals of Vector Fields                           | 15.4 Line Integrals of Vector Fields                                      |
| Section 8.1: Green's Theorem in the Plane                              | 16.3 Green's Theorem in the Plane                                         |

## NOTE 1.1.1

A sequence  $\{a_n\}$  is a set of terms  $\{a_1, a_2, a_3, a_4, \dots, a_{n-1}, a_n\}$ , where “ $n$ ” is a positive integer ( $\mathbb{Z}^+$ ).

Sequence can be denoted by  $\{a_n\}_{n=k}^{\infty}$

In case the initial value ( $k$ ) is Not shown, then start the sequence with  $n = 1$

$a_1$  is the first term and  $a_n$  is the  $n$ th term of the sequence.

## Example 1

Find the terms of the following sequence.

$$\{a_n\} = \left\{ \frac{(-1)^{n+1} 2^n}{n+1} \right\}$$

## Answer

$$n = 1 \Rightarrow a_1 = 1$$

$$n = 2 \Rightarrow a_2 = -\frac{4}{3}$$

$$n = 3 \Rightarrow a_3 = 2$$

$$n = 4 \Rightarrow a_4 = -\frac{16}{5}$$

$$n = 5 \Rightarrow a_5 = \frac{16}{3}$$

$$\{a_n\} = \left\{ \frac{(-1)^{n+1} 2^n}{n+1} \right\} = \left\{ 1, -\frac{4}{3}, 2, -\frac{16}{5}, \frac{16}{3}, \dots \right\}$$

## Example 2

Find the terms of the following sequence.

$$\{a_n\} = \{(-1)^n \sqrt{n-2}\}_{n=2}^6$$

## Answer

$$n = 2 \Rightarrow a_2 = 0$$

$$n = 3 \Rightarrow a_3 = -1$$

$$n = 4 \Rightarrow a_4 = \sqrt{2}$$

$$n = 5 \Rightarrow a_5 = -\sqrt{3}$$

$$n = 6 \Rightarrow a_6 = 2$$

$$\{a_n\} = \{(-1)^n \sqrt{n-2}\}_{n=2}^6 = \{0, -1, \sqrt{2}, -\sqrt{3}, 2\}$$

**How to Find  $a_n$** **Example 3**

Find the general form of the sequence related terms to “ $n$ ”.

$$\left\{-1, \frac{1}{2}, -\frac{1}{6}, \frac{1}{24}, -\frac{1}{120}, \dots\right\}$$

**Answer**

$$n = 1 \Rightarrow a_1 = -1 = \frac{-1}{1!}$$

$$n = 2 \Rightarrow a_2 = \frac{1}{2} = \frac{1}{2!}$$

$$n = 3 \Rightarrow a_3 = -\frac{1}{6} = -\frac{1}{3!}$$

$$n = 4 \Rightarrow a_4 = \frac{1}{24} = \frac{1}{4!}$$

$$n = 5 \Rightarrow a_5 = -\frac{1}{120} = \frac{1}{5!}$$

$$\{a_n\} = \left\{\frac{(-1)^n}{n!}\right\}$$

**Example 4**

Find the general form of the sequence related terms to “ $n$ ”.

$$\left\{2, \frac{3}{\sqrt{2}}, \frac{4}{\sqrt{3}}, \frac{5}{2}, \dots\right\}$$

**Answer**

$$n = 1 \Rightarrow a_1 = 2 = \frac{2}{\sqrt{1}} = \frac{1+1}{\sqrt{1}}$$

$$n = 2 \Rightarrow a_2 = \frac{3}{\sqrt{2}} = \frac{2+1}{\sqrt{2}}$$

$$n = 3 \Rightarrow a_3 = \frac{4}{\sqrt{3}} = \frac{3+1}{\sqrt{3}}$$

$$n = 4 \Rightarrow a_4 = \frac{5}{2} = \frac{4+1}{\sqrt{4}}$$

$$\{a_n\} = \left\{\frac{n+1}{\sqrt{n}}\right\}$$

**NOTE 1.1.2**

A **recursive formula** is a formula that defines each term of a sequence using preceding term(s). Recursive formulas must always state the initial term(s) of the sequence.

**Example 5**

Suppose

$$a_1 = 2$$

$$a_2 = 4$$

$$a_{n+1} = 2a_n - a_{n-1}$$

Find the first 5 nonzero terms.

**Answer**

This is a **recursive sequence**.

$$a_3 = 2a_2 - a_1 = 2(4) - 2 = 6$$

$$a_4 = 2a_3 - a_2 = 2(6) - 4 = 8$$

$$a_5 = 2a_4 - a_3 = 2(8) - 6 = 10$$

The **recursive sequence** is  $\{2, 4, 6, 8, 10, \dots\}$

## NOTE 1.1.3

The limit of a function is equivalent (same as) to the limit of a sequence. Just the notation is different.

$$\text{If } \lim_{x \rightarrow \infty} f(x) = L \text{ and } a_n = f(n) \quad \text{then } \lim_{n \rightarrow \infty} a_n = L$$

## NOTE 1.1.4

Sequence in the form of limit:

If  $\lim_{n \rightarrow \infty} a_n = L$  then  $\{a_n\}$  is called **convergent** if the limit exists and has a finite value and **divergent** if the limit does not exist or has infinite value.

## Example 6

What is the difference between the following sequences?

- I.  $\left\{\frac{1}{n}\right\}$
- II.  $\{(e)^n\}$

## Answer

- I.  $\left\{\frac{1}{n}\right\}$  is convergent (exist) since  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ .
- II.  $\{(e)^n\}$  is divergent (Does Not Exist) since  $\lim_{n \rightarrow \infty} (e)^n = \infty$ .

## Example 7

Evaluate  $\lim_{n \rightarrow \infty} \frac{(-1)^n}{n}$

## Answer

$$\lim_{n \rightarrow \infty} \left| \frac{(-1)^n}{n} \right| = \lim_{n \rightarrow \infty} \frac{1}{n} = 0 \text{ thus, } \lim_{n \rightarrow \infty} \frac{(-1)^n}{n} = 0$$

$$a_n a_{n+1} = \frac{(-1)^n}{n} \frac{(-1)^{n+1}}{n+1} < 0 \quad \therefore \left\{ \frac{(-1)^n}{n} \right\} \text{ is alternating}$$

## Example 8

Show whether the following sequences are convergent or divergent.

I.  $\{(-1)^n\}$

III.  $a_n = \frac{2+(-1)^n}{n}$

V.  $a_n = \sqrt{n+1} - \sqrt{n}$

II.  $a_n = \frac{2n}{n+1}$

IV.  $a_n = \frac{\ln n}{n}$

## NOTE 1.1.5

If  $\{a_n\}$  and  $\{b_n\}$  converge  $n \geq 1$ , then.

- $\lim_{n \rightarrow \infty} (a_n \pm b_n) = \lim_{n \rightarrow \infty} a_n \pm \lim_{n \rightarrow \infty} b_n$
- $\lim_{n \rightarrow \infty} c a_n = c \lim_{n \rightarrow \infty} a_n$
- $\lim_{n \rightarrow \infty} (a_n b_n) = \left( \lim_{n \rightarrow \infty} a_n \right) \left( \lim_{n \rightarrow \infty} b_n \right)$
- $\lim_{n \rightarrow \infty} \left( \frac{a_n}{b_n} \right) = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n}$  where  $\lim_{n \rightarrow \infty} a_n \neq 0$
- $\lim_{n \rightarrow \infty} (a_n)^p = \left( \lim_{n \rightarrow \infty} a_n \right)^p$  where  $a_n > 0$  and  $p > 0$
- If  $a_n \leq b_n$  for all  $n \geq 1$  then  $\lim_{n \rightarrow \infty} a_n \leq \lim_{n \rightarrow \infty} b_n$

## NOTE 1.1.6

$\{a_n\}$  is alternating if  $a_n a_{n+1} < 0$  for  $n \geq 1$

That is, if any consecutive terms have opposite sign.

## Answer

- I.  $\{(-1)^n\} = -1, 1, -1, 1, -1, \dots \rightarrow$  divergent.
- II.  $a_n = \frac{2n}{n+1} \rightarrow \lim_{n \rightarrow \infty} \frac{2n}{n+1} = \lim_{n \rightarrow \infty} \frac{\frac{2n}{n}}{\frac{n+1}{n}} = \frac{2}{1+0} = 2 \rightarrow$  convergent.
- III.  $a_n = \frac{2+(-1)^n}{n} \rightarrow \lim_{n \rightarrow \infty} \frac{2+(-1)^n}{n} = \lim_{n \rightarrow \infty} \frac{\frac{2}{n} + \frac{(-1)^n}{n}}{\frac{n}{n}} = \frac{0+0}{1} = 0 \rightarrow$  convergent.
- IV.  $a_n = \frac{\ln n}{n} \rightarrow \lim_{n \rightarrow \infty} \frac{\ln n}{n} \left( \frac{\infty}{\infty} \right) \xrightarrow{L'H} \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{1} = 0 \rightarrow$  convergent.
- V.  $a_n = \sqrt{n+1} - \sqrt{n} = \lim_{n \rightarrow \infty} (\sqrt{n+1} - \sqrt{n}) \left( \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \right)$   
 $= \lim_{n \rightarrow \infty} \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} = 0 \rightarrow$  convergent.

## NOTE 1.1.7

**Squeeze Theorem**

If  $a_n \leq b_n \leq c_n$  and  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$  Then  
 $\lim_{n \rightarrow \infty} b_n = L$

## NOTE 1.1.8

$n! < n^n$  as

$n! = n(n-1)(n-2) \dots$  are called  $n$ -terms.

$n^n = n \cdot n \cdot n \dots$  are called  $n$ -terms.

$n! < n^{n-1}$

$n^{n-3} < n^{n-2} < n^{n-1} < n^n$

The last term of

$n! = n(n-1)(n-2) \dots$  is 1

$n^n = n \cdot n \cdot n \dots$  is  $n$

Thus,  $n^n > n!$

## NOTE 1.1.9

If  $f(x)$  is continuous and  $\lim_{n \rightarrow \infty} a_n = L$ , then  $\lim_{n \rightarrow \infty} f(a_n) = f\left(\lim_{n \rightarrow \infty} (a_n)\right) = f(L)$

## Example 10

$$a_n = e^{\sin\left(\frac{1}{n}\right)}$$

Convergent or divergent?

## Answer

$a_n = e^{\sin\left(\frac{1}{n}\right)}$  since  $e^x$  is continuous  
 $\left\{ \lim_{n \rightarrow \infty} \sin\left(\frac{1}{n}\right) = 0 \right\}$  exists (convergent)  
 $e^{\lim_{n \rightarrow \infty} \sin\left(\frac{1}{n}\right)} = e^0 = 1$  Thus,  $a_n = e^{\sin\left(\frac{1}{n}\right)}$  is convergent.

## Example 9

$$a_n = \frac{n!}{n^n} \quad \text{Convergent or divergent?}$$

## Answer

$$a_1 = \frac{1}{1}$$

$$a_2 = \frac{2 \cdot 1}{2 \cdot 2} \leq \frac{2}{2 \cdot 2}$$

$$a_3 = \frac{3 \cdot 2 \cdot 1}{3 \cdot 3 \cdot 3} \leq \frac{3 \cdot 3}{3 \cdot 3 \cdot 3}$$

$$a_4 = \frac{4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 4 \cdot 4 \cdot 4} \leq \frac{4 \cdot 4 \cdot 4}{4 \cdot 4 \cdot 4 \cdot 4}$$

$$a_n = \frac{n!}{n^n} \leq \frac{n^{n-1}}{n^n}$$

$$0 \leq a_n \leq \frac{n^{n-1}}{n^n}$$

$$\lim_{n \rightarrow \infty} \frac{n^{n-1}}{n^n} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0. \text{ Then, } \lim_{n \rightarrow \infty} a_n = 0$$

Thus,  $a_n$  is convergent by squeeze theorem.



**Example 11**

$$a_n = \tan^{-1} n^2$$

Convergent or divergent?

**Answer**

$$a_n = \tan^{-1} n^2 \left\{ \begin{array}{l} \text{since } \tan^{-1} x \text{ is continuous} \\ \lim_{n \rightarrow \infty} n^2 = \infty \end{array} \right.$$

$$\tan^{-1} \left( \lim_{n \rightarrow \infty} n^2 \right) = \tan^{-1} (\infty) = \frac{\pi}{2}$$

Thus,  $a_n = \tan^{-1} n^2$  is convergent

**NOTE 1.1.10**

A **sequence** that is called Monotonic:

if the sequence is always (strictly) **increasing** like  $a_1 < a_2 < a_3 < \dots \Leftrightarrow a_{n+1} \geq a_n$

or the sequence is always (strictly) **decreasing** like  $a_1 > a_2 > a_3 > \dots \Leftrightarrow a_{n+1} \leq a_n$

**Example 13**

Show that  $\left\{ \frac{n}{n+1} \right\}$  is monotonic.

**Answer**

$$\text{Let } f(x) = \frac{x}{x+1}$$

$$f'(x) = \frac{(x+1)-x}{(x+1)^2} = \frac{1}{(x+1)^2} \geq 0 \text{ for } \forall x \geq 1$$

Thus,  $\left\{ \frac{n}{n+1} \right\}$  is monotonic since it is strictly increasing.

**Example 14**

Show that  $\left\{ \frac{n}{e^n} \right\}$  is monotonic.

**Answer**

$$\text{Let } f(x) = \frac{x}{e^x}$$

$$f'(x) = \frac{e^x - xe^x}{e^{2x}} = \frac{e^x(1-x)}{e^{2x}} = \frac{1-x}{e^x} \leq 0 \text{ for } \forall x \geq 1$$

Thus,  $\left\{ \frac{n}{e^n} \right\}$  is monotonic since it is strictly decreasing.

**NOTE 1.1.11**

If  $a_n \leq M$ , then the sequence is called **Bounded Above**.

If  $a_n \geq m$ , then the sequence is called **Bounded Below**.

If  $m \leq a_n \leq M$ , then the sequence is called **Bounded**.

**Every Bounded sequence is called a monotonic sequence.**

**NOTE 1.1.12****THEOREM**

If  $\{a_n\}$  is bounded and monotonic then  $\{a_n\}$  is convergent.

If  $\{a_n\}$  is convergent then  $\{a_n\}$  is bounded.

**!! If  $\{a_n\}$  is bounded, then  $\{a_n\}$  is convergent. (WRONG)**

For example:  $\{(-1)^n\}$  is bounded but not convergent.

**NOTE 1.1.13****THEOREM**

If sequence  $\{a_n\}$  is bounded **above** and is (strictly) **increasing**, then it **converges**.

If sequence  $\{a_n\}$  is bounded **below** and is (strictly) **decreasing**, then it **converges**.

## NOTE 1.1.14

$$\lim_{n \rightarrow \infty} r^n = \begin{cases} 0, & -1 < r < 1 \\ 1, & r = 1 \end{cases}$$

## NOTE 1.1.5

If  $\lim_{n \rightarrow \infty} |a_n| = 0$ , then  $\lim_{n \rightarrow \infty} a_n = 0$

$\therefore \{a_n\}$  is convergent.

If  $\lim_{n \rightarrow \infty} |a_n| \neq 0$ , then  $\lim_{n \rightarrow \infty} a_n = DNE$  (Does Not Exist)

$\therefore \{a_n\}$  is divergent.

## Example 15

Determine whether the following sequences are convergent or divergent.

I.  $a_n = \frac{(2n)!}{(n!)^2 4^n}$

## Answer

$$a_n = \frac{(2n)!}{(n!)^2 4^n} > 0 \quad \forall n > 1$$

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{(2n+2)!}{((n+1)!)^2 4^{n+1}} \cdot \frac{(n!)^2 4^n}{(2n)!} \\ &= \frac{(2n+2)(2n+1)(2n)!}{(n+1)^2 (n!)^2 \cdot 4 \cdot 4^n} \cdot \frac{(n!)^2 \cdot 4^n}{(2n)!} \\ &= \frac{2(n+1)(2n+1)}{4(n+1)^2} = \frac{(2n+1)}{2(n+1)} = \frac{2n+1}{2n+2} < 1 \quad \forall n \geq 1 \end{aligned}$$

Then,  $a_n$  is decreasing and it is bounded from below by 0.

Hence,  $\{a_n\}$  is a convergent sequence.

II.  $\left\{ \frac{(-1)^n n^3}{2^n} \right\}_{n=1}^{\infty}$

## Answer

$$\left| \frac{(-1)^n n^3}{2^n} \right| = \frac{n^3}{2^n}$$

$$\lim_{x \rightarrow \infty} \frac{x^3}{2^x} \left[ \frac{\infty}{\infty} \right] \stackrel{L'H}{=} \lim_{x \rightarrow \infty} \frac{3x^2}{2^x \ln 2} \left[ \frac{\infty}{\infty} \right] \stackrel{L'H}{=} \lim_{x \rightarrow \infty} \frac{6x}{2^x (\ln 2)^2} \left[ \frac{\infty}{\infty} \right]$$

$$\stackrel{L'H}{=} \lim_{x \rightarrow \infty} \frac{6}{2^x (\ln 2)^3} = 0$$

$$\text{So } \lim_{n \rightarrow \infty} \frac{n^3}{2^n} = 0 \text{ and hence } \lim_{n \rightarrow \infty} \frac{(-1)^n n^3}{2^n} = 0$$

Hence,  $\{a_n\}$  is a convergent sequence.

## NOTE 1.1.16

**THEOREM:** The following six sequences **converges** to the limits listed below:

1.  $\lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$

2.  $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$

3.  $\lim_{n \rightarrow \infty} x^{\frac{1}{n}} = 1 \quad (x > 0)$

4.  $\lim_{n \rightarrow \infty} x^n = 0 \quad (|x| < 1)$

5.  $\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x \quad (\text{any } x)$

6.  $\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0 \quad (\text{any } x)$

In formulas (3) through (6),  $x$  remains fixed (constant value) as  $n \rightarrow \infty$ .

### NOTE 1.2.1

#### Definition:

A **series** is a formal sum of infinitely ordered list of terms (numbers), for example the series of positive integers:

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + \dots \leftrightarrow \text{They are the terms of a series, which equal to } \sum_{n=1}^{\infty} n$$

Sometimes a series starts with  $a_0$  (index zero) instead of  $a_1$ .

### NOTE 1.2.2

#### Notation:

The series is denoted by  $s = \sum_{n=1}^{\infty} a_n$  which equal to  $a_1 + a_2 + a_3 + a_4 + \dots + a_{n-1} + a_n + a_{n+1} + \dots$

However,  $n$ th partial sum is  $s_n = \sum_{i=1}^n a_i = a_1 + a_2 + a_3 + a_4 + \dots + a_n$

### NOTE 1.2.3

#### Series in the form of limit:

$$\text{Note: } \sum_{n=1}^{\infty} a_n = \lim_{n \rightarrow \infty} \sum_{i=1}^n a_i \quad \leftrightarrow \quad s = \lim_{n \rightarrow \infty} s_n$$

If  $\lim_{n \rightarrow \infty} s_n = s$  exists, then the series is called *convergent*. Otherwise, the series is called *divergent*.

### NOTE 1.2.4

#### DO NOT CONFUSE YOURSELF!

$a_n$  is the  $n$ th term

$s_n$  is the sum of the  $n$ th term

$a_{n+1}$  is the  $(n + 1)$ th term

$s$  is the sum of all terms

### NOTE 1.2.5

#### Geometric Series in two main forms:

$$\sum_{n=1}^{\infty} a r^{n-1} = a + ar + ar^2 + \dots, \text{ whose } n\text{th term is } a_n = a r^{n-1}$$

$$\sum_{n=0}^{\infty} a r^n = a + ar + ar^2 + \dots, \text{ whose } n\text{th term is } a_{n+1} = a r^n$$

So, for  $|r| < 1$  the series is convergent, and its sum is  $\sum_{n=1}^{\infty} a r^{n-1} = \sum_{n=0}^{\infty} a r^n = \frac{a}{1-r}$

So, for  $|r| \geq 1$  the series is divergent, thus  $s = \lim_{n \rightarrow \infty} s_n$  does not exists.

**Example 1**

Find the sum of  $\sum_{n=1}^{\infty} 3\left(-\frac{1}{2}\right)^{n-1}$

**Answer**

The type of the series is Geometric

$$a = 3 \quad r = -\frac{1}{2}$$

$$\text{Since } |r| = \left|-\frac{1}{2}\right| = \frac{1}{2} < 1$$

Then series  $\sum_{n=1}^{\infty} 3\left(-\frac{1}{2}\right)^{n-1}$  converges.

$$\text{Its sum is } \frac{3}{1 - \left(-\frac{1}{2}\right)} = \frac{3}{\frac{1}{2}} = 2$$

**Example 3**

Find the sum of  $\sum_{n=1}^{\infty} a_n = \left(\frac{5}{3} - \frac{5}{9} + \frac{5}{27} - \frac{5}{81} + \dots\right)$

**Answer**

$$\sum_{n=1}^{\infty} a_n = \left(\frac{5}{3} - \frac{5}{9} + \frac{5}{27} - \frac{5}{81} + \dots\right)$$

$$= \frac{5}{3} \left(\frac{1}{1} - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \dots\right)$$

$$= \frac{5}{3} \left(\frac{1}{3^0} - \frac{1}{3^1} + \frac{1}{3^2} - \frac{1}{3^3} + \dots\right)$$

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{5}{3} \left(-\frac{1}{3}\right)^{n-1}$$

The type of the series is Geometric

$$a = \frac{5}{3} \quad r = -\frac{1}{3} \quad \text{Since } |r| = \left|-\frac{1}{3}\right| = \frac{1}{3} < 1$$

Then series  $\sum_{n=1}^{\infty} \frac{5}{3} \left(-\frac{1}{3}\right)^{n-1}$  converges.

$$\text{Its sum is } \frac{\frac{5}{3}}{1 - \left(-\frac{1}{3}\right)} = \frac{\frac{5}{3}}{\frac{4}{3}} = \frac{5}{4}$$

**Example 2**

Find the sum of  $\sum_{n=1}^{\infty} 3\left(-\frac{4}{3}\right)^{n-1}$

**Answer**

The type of the series is Geometric

$$a = 3 \quad r = -\frac{4}{3}$$

$$\text{Since } |r| = \left|-\frac{4}{3}\right| = \frac{4}{3} > 1$$

Then series  $\sum_{n=1}^{\infty} 3\left(-\frac{4}{3}\right)^{n-1}$  is divergent.

Its sum does not exist.

**Example 4**

Find the sum of  $\sum_{n=1}^{\infty} \frac{e^n}{3^{n+1}}$

**Answer**

$$\sum_{n=1}^{\infty} \frac{e^n}{3^{n+1}} = \sum_{n=1}^{\infty} \frac{e}{3^2} \frac{e^{n-1}}{3^{n-1}} = \sum_{n=1}^{\infty} \frac{e}{9} \left(\frac{e}{3}\right)^{n-1}$$

The type of the series is Geometric

$$a = \frac{e}{9} \quad r = \frac{e}{3}$$

$$\text{Since } |r| = \left|\frac{e}{3}\right| = \frac{e}{3} < 1$$

Then series  $\sum_{n=1}^{\infty} \frac{e^n}{3^{n+1}}$  converges.

$$\text{Its sum is } \frac{\frac{e}{9}}{1 - \left(\frac{e}{3}\right)} = \frac{\frac{e}{9}}{\frac{3-e}{3}} = \frac{e}{3(3-e)}$$

## NOTE 1.2.6

## Telescopic Series

A telescopic series is a series whose terms can be rewritten so that most of them cancel out.

## Example 5

Find the sum of  $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = s$

## Answer

Since  $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

So, the  $n$ th for the sum is

$$s_n = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{n-1} - \frac{1}{n} + \frac{1}{n} - \frac{1}{n+1}$$

After the terms are cancelled, then the remaining terms are

$$s_n = 1 - \frac{1}{n+1}$$

Hence, the sum of the series is

$$s = \lim_{n \rightarrow \infty} (s_n) = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+1}\right) = 1$$

## Example 7

Find the sum of  $\sum_{n=1}^{\infty} \frac{4}{4n^2-1}$

## Answer

$$\frac{4}{4n^2-1} = \frac{4}{(2n-1)(2n+1)} = \frac{A}{2n-1} + \frac{B}{2n+1}$$

$$= \frac{A(2n+1)}{(2n-1)(2n+1)} + \frac{B(2n-1)}{(2n-1)(2n+1)}$$

$$4 = A(2n+1) + B(2n-1)$$

$$4 = (2A + 2B)n + A - B$$

$$\text{we have } \begin{cases} 2A + 2B = 0 \rightarrow (1) \\ A - B = 4 \rightarrow (2) \end{cases}$$

From equation (1) & (2),

we have  $A = 2$  and  $B = -2$

Thus, the series is convergent to 1

## Example 6

Find the sum of  $\sum_{n=1}^{\infty} \left(\frac{1}{2n+3} - \frac{1}{2n+1}\right)$

## Answer

$$\begin{aligned} s_n &= \left(\frac{1}{5} - \frac{1}{3}\right) + \left(\frac{1}{7} - \frac{1}{5}\right) + \left(\frac{1}{9} - \frac{1}{7}\right) + \dots \\ &+ \left(\frac{1}{2(n-2)+3} - \frac{1}{2(n-2)+1}\right) + \left(\frac{1}{2(n-1)+3} - \frac{1}{2(n-1)+1}\right) \\ &+ \left(\frac{1}{2n+3} - \frac{1}{2n+1}\right) \end{aligned}$$

$$s_n = -\frac{1}{3} + \frac{1}{2n+3}$$

Hence, the sum of the series is

$$s = \lim_{n \rightarrow \infty} (s_n) = \lim_{n \rightarrow \infty} \left(-\frac{1}{3} + \frac{1}{2n+3}\right) = -\frac{1}{3}$$

Thus, the series is convergent to  $-\frac{1}{3}$

$$\text{Thus, } s = \sum_{n=1}^{\infty} \frac{4}{4n^2-1} = \sum_{n=1}^{\infty} \frac{2}{2n-1} - \frac{2}{2n+1}$$

$$\begin{aligned} s_n &= \left(\frac{2}{1} - \frac{2}{3}\right) + \left(\frac{2}{3} - \frac{2}{5}\right) + \left(\frac{2}{5} - \frac{2}{7}\right) + \dots \\ &+ \left(\frac{2}{2n-3} - \frac{2}{2n-1}\right) + \left(\frac{2}{2n-1} - \frac{2}{2n+1}\right) \\ &= 2 - \frac{2}{2n+1} \end{aligned}$$

Hence, the sum of the series is

$$s = \lim_{n \rightarrow \infty} (s_n) = \lim_{n \rightarrow \infty} \left(2 - \frac{2}{2n+1}\right) = 2$$

Thus, the series is convergent to 2

### NOTE 1.2.7

The **harmonic series** is the divergent infinite series

The general formula is  $\sum_{n=1}^{\infty} \frac{1}{an+b}$ , where  $a \neq 0$  and  $b$  are real number ( $\mathbb{R}$ )

For example,  $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$  is divergent

### NOTE 1.2.8

**Theorem:** if the series  $s = \sum_{n=1}^{\infty} a_n$  is convergent, then  $\lim_{n \rightarrow \infty} a_n = 0$  (The converse is false)

The converse is false since  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ , however  $\sum_{n=1}^{\infty} \frac{1}{n}$  is divergent (harmonic series)

The  $n$ th term of divergence test, if  $\lim_{n \rightarrow \infty} a_n \neq 0$ , thus the series  $s = \sum_{n=1}^{\infty} a_n$  is divergent.

Determine whether the following series are convergent or divergent.

#### Example 8

$$\sum_{n=1}^{\infty} \frac{n}{2n-1}$$

#### Answer

This series is divergent because  $\lim_{n \rightarrow \infty} \frac{n}{2n-1} = \frac{1}{2} \neq 0$

#### Example 9

$$\sum_{n=1}^{\infty} (-1)^n n \sin(1/n)$$

#### Answer

This series is divergent because  $\lim_{n \rightarrow \infty} \left| (-1)^n n \sin\left(\frac{1}{n}\right) \right| = \lim_{n \rightarrow \infty} \frac{\sin(1/n)}{1/n} = \lim_{x \rightarrow 0^+} \frac{\sin(x)}{x} = 1 \neq 0$

\* let  $x = \frac{1}{n}$  then  $x \rightarrow 0^+$

### NOTE 1.2.9

**Theorem:** the series  $s = \sum_{n=1}^{\infty} a_n$  is convergent, if and only if  $\sum_{n=1}^{\infty} a_n$  converges for any integers  $N \geq 1$

Operations with Series.

If  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  are convergent series and  $c$  is a constant, then the following series are also convergent and:

1-  $\sum_{n=1}^{\infty} ca_n = c \sum_{n=1}^{\infty} a_n$

2-  $\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$

3- if  $a_n \leq b_n$  for all  $n = 1, 2, 3, \dots$ , then  $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$



**Example 10**

Find the sum of the series  $\sum_{n=1}^{\infty} \frac{1+2^{n+1}}{3^n}$

**Answer**

The given series is the sum of two geometric series,

$$\sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} \frac{1}{3} \left(\frac{1}{3}\right)^{n-1} = \frac{1/3}{1-(1/3)} = \frac{1}{2}$$

$$\sum_{n=1}^{\infty} \frac{2^{n+1}}{3^n} = \sum_{n=1}^{\infty} \frac{4}{3} \left(\frac{2}{3}\right)^{n-1} = \frac{4/3}{1-(2/3)} = 4.$$

Thus, its sum is  $\frac{1}{2} + 4 = \frac{9}{2}$

**Section 1.3** Convergence Tests for Positive Series**NOTE 1.3.1**

**Integral Test:** If  $f$  is a continuous, positive, and decreasing function where  $f(n) = a_n$  on the interval  $[N, \infty)$  for some positive integer  $N$ ,

then the improper integral  $\int_N^{\infty} f(x) dx$  and the infinite series  $\sum_{n=N}^{\infty} a_n$

either both converge or both diverge to infinity.

Determine whether the following series are convergent or divergent.

**Example 1**

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$$

**Answer**

If  $f(x) = \frac{1}{x^2+1}$  is a continuous, positive, and decreasing on  $[1, \infty)$ , then we can apply integral test.

$$\int_1^{\infty} \frac{1}{x^2+1} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x^2+1} dx = \lim_{b \rightarrow \infty} [\tan^{-1} x]_1^b$$

$$= \lim_{b \rightarrow \infty} [\tan^{-1} b - \tan^{-1} 1] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Thus, both  $\int_1^{\infty} \frac{1}{x^2+1} dx$  and  $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$  are convergent.

**Example 2**

$$\sum_{n=1}^{\infty} \frac{3}{2n-1}$$

**Answer**

If  $f(x) = \frac{3}{2x-1}$  is a continuous, positive, and decreasing on  $[1, \infty)$ , then we can apply integral test.

$$\int_1^{\infty} \frac{3}{2x-1} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{3}{2x-1} dx = \lim_{b \rightarrow \infty} \frac{3}{2} \int_1^{2b-1} \frac{1}{u} du$$

$$= \lim_{b \rightarrow \infty} [\ln u]_1^{2b-1} = \lim_{b \rightarrow \infty} [\ln(2b-1) - \ln 1] = \infty - 0 = \infty$$

\* let  $u = 2x - 1 \rightarrow du = 2 dx$

as  $x = b \rightarrow u = 2b - 1$  and as  $x = 1 \rightarrow u = 1$

Thus, both  $\int_1^{\infty} \frac{3}{2x-1} dx$  and  $\sum_{n=1}^{\infty} \frac{3}{2n-1}$  are divergent.

## Example 3

$$\sum_{n=1}^{\infty} \frac{\ln n}{n}$$

## Answer

If  $f(x) = \frac{\ln x}{x}$  is a continuous, positive, and decreasing on  $[3, \infty)$ , then we can apply integral test.

$$* f'(x) = \frac{x \cdot \frac{1}{x} - \ln x}{x^2} = \frac{1 - \ln x}{x^2} \leq 0 \text{ where } 1 - \ln x \leq 0$$

$\rightarrow 1 \leq \ln x \rightarrow e^1 \leq e^{\ln x} \rightarrow e \leq x$  which means  $f(x)$  is decreasing for  $3 \leq x$

$$\begin{aligned} \int_3^{\infty} \frac{\ln x}{x} dx &= \lim_{b \rightarrow \infty} \int_3^b \frac{\ln x}{x} dx = \lim_{b \rightarrow \infty} \int_{\ln 3}^{\ln b} u du \\ &= \lim_{b \rightarrow \infty} \left[ \frac{1}{2} u^2 \right]_{\ln 3}^{\ln b} = \lim_{b \rightarrow \infty} \left[ \frac{1}{2} (\ln b)^2 - \frac{1}{2} (\ln 3)^2 \right] = \infty \end{aligned}$$

$$* \text{ let } u = \ln x \rightarrow du = \frac{1}{x} dx$$

$$\text{as } x = b \rightarrow u = \ln b \text{ and } \text{as } x = 3 \rightarrow u = \ln 3$$

Thus, since both  $\int_3^{\infty} \frac{\ln x}{x} dx$  and  $\sum_{n=3}^{\infty} \frac{\ln n}{n}$  are divergent, then  $\sum_{n=1}^{\infty} \frac{\ln n}{n}$  is divergent.

## Example 4

$$\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n^2}$$

## Answer

If  $f(x) = \frac{e^{\frac{1}{x}}}{x^2}$  is a continuous, positive, and decreasing on  $[1, \infty)$ , then we can apply integral test.

$$* f'(x) = \frac{x^2 \left( -\frac{1}{x^2} e^{\frac{1}{x}} \right) - (2x) e^{\frac{1}{x}}}{x^4} = \frac{-e^{\frac{1}{x}} - 2x e^{\frac{1}{x}}}{x^4} = \frac{-e^{\frac{1}{x}} (1 + 2x)}{x^4} \leq 0$$

$$\text{where } -e^{\frac{1}{x}} (1 + 2x) \leq 0 \rightarrow 1 + 2x \geq 0 \rightarrow x \geq -\frac{1}{2}$$

which means  $f(x)$  is also decreasing for  $x \geq 1$

$$\begin{aligned} \int_1^{\infty} \frac{e^{\frac{1}{x}}}{x^2} dx &= \lim_{b \rightarrow \infty} \int_1^b \frac{e^{\frac{1}{x}}}{x^2} dx = \lim_{b \rightarrow \infty} \int_1^{\frac{1}{b}} -e^u du \\ &= \lim_{b \rightarrow \infty} [-e^u]_1^{\frac{1}{b}} = \lim_{b \rightarrow \infty} \left[ -e^{\frac{1}{b}} - (-e^1) \right] = -1 + e \end{aligned}$$

$$* \text{ let } u = \frac{1}{x} \rightarrow du = -\frac{1}{x^2} dx$$

$$\text{as } x = b \rightarrow u = \frac{1}{b} \text{ and } \text{as } x = 1 \rightarrow u = 1$$

Thus, both  $\int_1^{\infty} \frac{e^{\frac{1}{x}}}{x^2} dx$  and  $\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n^2}$  are convergent.

## NOTE 1.3.2

The Comparison Test

Let  $\sum a_n$ ,  $\sum b_n$ , and  $\sum c_n$  be series with nonnegative terms. Suppose that for some integer  $N$

$$b_n \leq a_n \leq c_n \text{ for all } n > N.$$

(a) If  $\sum c_n$  (chosen) converges, then  $\sum a_n$  (given) also converges.

(b) If  $\sum b_n$  (chosen) diverges, then  $\sum a_n$  (given) also diverges.

Recall: For positive integer  $N$ ,  $\sum_{n=N}^{\infty} \frac{1}{n^p}$  is **convergent** for  $p > 1$  and **divergent** for  $p \leq 1$

Determine whether the following series are convergent or divergent.

### Example 5

$$\sum_{n=1}^{\infty} \frac{1}{2^n + 1}$$

### Answer

Since  $0 < \frac{1}{2^{n+1}} < \frac{1}{2^n}$  for  $n \geq 1$

Since  $\sum_{n=1}^{\infty} \frac{1}{2^n}$  is convergent geometric series ( $|r| = \frac{1}{2} < 1$ ),

Then  $\sum_{n=1}^{\infty} \frac{1}{2^{n+1}}$  is convergent by the comparison test.

### Example 6

$$\sum_{n=1}^{\infty} \frac{3n+1}{n^3+1}$$

### Answer

Since  $0 < \frac{3n+1}{n^3+1} = \frac{3n}{n^3+1} + \frac{1}{n^3+1} < \frac{3}{n^2} + \frac{1}{n^2} = \frac{4}{n^2}$  for  $n \geq 1$

Since  $\sum_{n=1}^{\infty} \frac{4}{n^2}$  is convergent

$p$ -series ( $p = 2 > 1$ ).

Then,  $\sum_{n=1}^{\infty} \frac{3n+1}{n^3+1}$  is convergent by the comparison test.

### Example 7

$$\sum_{n=2}^{\infty} \frac{1}{\ln n}$$

### Answer

We have  $0 < \ln n < n$ , for  $n \geq 2$

$\frac{1}{\ln n} > \frac{1}{n} > 0$ , for  $n \geq 2$

since  $\sum_{n=2}^{\infty} \frac{1}{\ln n}$  diverges to infinity because it is a harmonic series.

Then,  $\sum_{n=2}^{\infty} \frac{1}{\ln n}$  is divergent by the comparison test.

### NOTE 1.3.3

#### Limit Comparison Test

Suppose that  $a_n > 0$  and  $b_n > 0$  for all  $n > N$  ( $N$  an integer).

1. If  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L > 0$  then  $\sum a_n$  (given) and  $\sum b_n$  (chosen) both converge or both diverge.
2. If  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$  and  $\sum b_n$  (chosen) converges, then  $\sum a_n$  (given) converges.
3. If  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty$  and  $\sum b_n$  (chosen) diverges, then  $\sum a_n$  (given) diverges.

Determine whether the following series are convergent or divergent.

### Example 8

$$\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$$

### Answer

Since  $0 < \frac{1}{\sqrt{n}}$  for  $n \geq 1$

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{1+\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{1+\sqrt{n}} = 1$$

Since  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$  is divergent  $p$ -

series ( $p = \frac{1}{2} < 1$ ), then

$\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$  is divergent by the limit comparison test.

### Example 9

$$\sum_{n=1}^{\infty} \frac{n+5}{n^3-2n+3}$$

### Answer

Since  $0 < \frac{1}{n^2}$  for  $n \geq 1$

$$\lim_{n \rightarrow \infty} \frac{\frac{n+5}{n^3-2n+3}}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{(n+5) \cdot n^2}{n^3-2n+3}$$

$$= \lim_{n \rightarrow \infty} \frac{n^3+5n^2}{n^3-2n+3}$$

$$= \lim_{n \rightarrow \infty} \frac{n^3+5n^2}{n^3-2n+3} = 1$$

Since  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  is convergent  $p$ -

series ( $p = 2 > 1$ ), then

$\sum_{n=1}^{\infty} \frac{n+5}{n^3-2n+3}$  is convergent by the limit comparison test.

### Example 10

$$\sum_{n=1}^{\infty} \frac{1+\sin n}{n^2}$$

### Answer

Since  $0 < \frac{1}{n^2}$  for  $n \geq 1$

$$\lim_{n \rightarrow \infty} \frac{\frac{1+\sin n}{n^2}}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} 1+\sin n = \infty$$

Thus, the limit comparison test fails. We can apply comparison test instead

Since  $0 \leq \sin n \leq 1$ ,

then  $0 \leq \frac{1+\sin n}{n^2} \leq \frac{2}{n^2}$  for  $n \geq 1$

Since  $\sum_{n=1}^{\infty} \frac{2}{n^2}$  is convergent by  $p$ -series ( $p = 2 > 1$ )

Then,  $\sum_{n=1}^{\infty} \frac{1+\sin n}{n^2}$  is convergent by the comparison test.

### NOTE 1.3.4

#### The Ratio Test

Let  $\sum a_n$  be a series with positive terms and suppose that

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \rho$$

Then (a) the series converges if  $\rho < 1$

(b) the series diverges if  $\rho > 1$  or is infinite,

(c) the test is inconclusive if  $\rho = 1$ .

Determine whether the following series are convergent or divergent.

### Example 11

$$\sum_{n=1}^{\infty} \frac{99^n}{n!}$$

#### Answer

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{99^{n+1}}{\frac{(n+1)!}{n!}} &= \lim_{n \rightarrow \infty} \frac{99^{n+1}}{(n+1)!} \cdot \frac{n!}{99^n} \\ &= \lim_{n \rightarrow \infty} \frac{99 \cdot \cancel{99^n}}{(n+1) \cdot \cancel{n!}} \cdot \frac{\cancel{n!}}{99^n} = \lim_{n \rightarrow \infty} \frac{99}{(n+1)} = 0 < 1 \end{aligned}$$

Thus,  $\sum_{n=1}^{\infty} \frac{99^n}{n!}$  is convergent by ratio test.

### Example 13

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

#### Answer

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\frac{(n+1)!}{n!}}{\frac{n!}{n^n}} &= \lim_{n \rightarrow \infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} \\ &= \lim_{n \rightarrow \infty} \frac{(n+1)(n)!}{(n+1)(n+1)^n} \cdot \frac{n^n}{n!} = \lim_{n \rightarrow \infty} \frac{n^n}{(n+1)^n} \\ &= \lim_{n \rightarrow \infty} \left( \frac{n}{n+1} \right)^n = \lim_{n \rightarrow \infty} \left( \frac{n}{n+1} \right)^n = \lim_{n \rightarrow \infty} \frac{1}{\left( \frac{n+1}{n} \right)^n} \\ \lim_{n \rightarrow \infty} \frac{1}{\left( 1 + \frac{1}{n} \right)^n} &= \frac{1}{e} < 1 \end{aligned}$$

Thus,  $\sum_{n=1}^{\infty} \frac{n!}{n^n}$  is convergent by ratio test.

### Example 12

$$\sum_{n=1}^{\infty} \frac{n^5}{2^n}$$

#### Answer

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\frac{(n+1)^5}{2^{n+1}}}{\frac{n^5}{2^n}} &= \lim_{n \rightarrow \infty} \frac{(n+1)^5}{2^{n+1}} \cdot \frac{2^n}{n^5} \\ &= \lim_{n \rightarrow \infty} \left( \frac{n+1}{n} \right)^5 \cdot \lim_{n \rightarrow \infty} \frac{2^n}{2^{n+1}} = 1 \cdot \frac{1}{2} = \frac{1}{2} < 1 \end{aligned}$$

Thus,  $\sum_{n=1}^{\infty} \frac{n^5}{2^n}$  is convergent by ratio test.

### Example 14

$$\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$$

#### Answer

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\frac{(2(n+1))!}{((n+1)!)^2}}{\frac{(2n)!}{(n!)^2}} &= \lim_{n \rightarrow \infty} \frac{(2(n+1))!}{((n+1)!)^2} \cdot \frac{(n!)^2}{(2n)!} \\ &= \lim_{n \rightarrow \infty} \frac{(2n+2)!}{((n+1)!)^2} \cdot \frac{(n!)^2}{(2n)!} \\ &= \lim_{n \rightarrow \infty} \frac{(2n+2) \cdot (2n+1) \cdot (2n)!}{((n+1)(n!))^2} \cdot \frac{(n!)^2}{(2n)!} \\ &= \lim_{n \rightarrow \infty} \frac{(2n+2) \cdot (2n+1)}{(n+1)^2} = \lim_{n \rightarrow \infty} \frac{4n+2}{n+1} = 4 > 1 \end{aligned}$$

Thus,  $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$  is divergent by ratio test.

### NOTE 1.3.5

#### The Root Test

Let  $\sum a_n$  be a series with positive terms and suppose that

$$\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \rho$$

Then (a) the series converges if  $\rho < 1$

(b) the series diverges if  $\rho > 1$  or is infinite,

(c) the test is inconclusive if  $\rho = 1$ .

#### Example 15

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{n+3}}{(n+1)^n}$$

#### Answer

$$\lim_{n \rightarrow \infty} \sqrt[n]{\left| (-1)^{n-1} \frac{2^{n+3}}{(n+1)^n} \right|} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{2^{n+3}}{(n+1)^n}} = \lim_{n \rightarrow \infty} \sqrt[n]{2^3 \left( \frac{2}{n+1} \right)^n}$$

$$= \lim_{n \rightarrow \infty} \sqrt[n]{2^3} \cdot \sqrt[n]{\left( \frac{2}{n+1} \right)^n} = \lim_{n \rightarrow \infty} 2^{\frac{3}{n}} \cdot \left( \frac{2}{n+1} \right) = 1 \cdot 0 = 0 < 1$$

Thus,  $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{n+3}}{(n+1)^n}$  is convergent by root test.

## Section 1.4 Absolute and Conditional Convergence

### NOTE 1.4.1

#### Absolute convergence

The series  $\sum_{n=1}^{\infty} a_n$  is said to be **absolutely convergent** if  $\sum_{n=1}^{\infty} |a_n|$  converges

**Theorem:** if a series converges absolutely, then it converges.

**Example:**  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$  is absolutely convergent.

since  $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$  is convergent by  $p$ -series ( $p = 2 > 1$ )

## NOTE 1.4.2

**Conditional Convergence**

If  $\sum_{n=1}^{\infty} a_n$  is convergent, but not absolutely convergent, then we can say that it is **conditionally convergent** or that it **converges conditionally**.

Test the following series for absolute convergence:

**Example 1**

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$$

**Answer**

$$\lim_{n \rightarrow \infty} \frac{\left| \frac{(-1)^{n-1}}{2n-1} \right|}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n}{2n-1} = \frac{1}{2} > 0$$

Since the harmonic series  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges to infinity, the limit comparison test says that  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$  does not converge absolutely.

**Example 2**

$$\sum_{n=1}^{\infty} \frac{n \cos(n\pi)}{2^n}$$

**Answer**

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{n \cos(n\pi)}{2^n} &= \sum_{n=1}^{\infty} \frac{n (-1)^n}{2^n} \\ \lim_{n \rightarrow \infty} \left| \frac{(n+1)(-1)^{n+1}}{2^{n+1}} \cdot \frac{2^n}{n(-1)^n} \right| &= \lim_{n \rightarrow \infty} \frac{(n+1)}{2^{n+1}} \cdot \frac{2^n}{n} \\ &= \frac{1}{2} \lim_{n \rightarrow \infty} \frac{n+1}{n} = \frac{1}{2} < 1 \end{aligned}$$

Thus,  $\sum_{n=1}^{\infty} \frac{n \cos(n\pi)}{2^n}$  converges absolutely.

## NOTE 1.4.3

**The alternating series test**

Let  $a_n > 0$  for all  $n \in \mathbb{Z}^+$ . The alternating series

$$\sum_{n=1}^{\infty} (-1)^n a_n \quad \text{and} \quad \sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converge by Alternating Series Test (A.S.T) when the two conditions listed below met.

1.  $a_{n+1} \leq a_n$  for all  $n \in \mathbb{Z}^+ \quad \leftrightarrow \quad f'(x) < 0$
2.  $\lim_{n \rightarrow \infty} a_n = 0$



Test the following series for absolutely or conditionally convergence:

### Example 3

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

### Answer

$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$  is divergent because it is a harmonic series.

Thus,  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$  does not converge absolutely.

Let  $a_n = \frac{1}{n} > 0$  for  $n \geq 1$

1.  $\frac{1}{n+1} \leq \frac{1}{n}$  since  $a_n$  is decreasing
2.  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$  is conditionally convergent by Alternating Series Test (A.S.T).

### Example 4

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\ln n}$$

### Answer

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\ln n} =$$

$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\ln n} \right| = \sum_{n=1}^{\infty} \frac{1}{\ln n}$  is divergent by comparison test  $\left( \frac{1}{\ln n} > \frac{1}{n} > 0 \right)$ .

Thus,  $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\ln n}$  does not converge absolutely.

Let  $a_n = \frac{1}{\ln n} > 0$  for  $n \geq 1$

1.  $\frac{1}{\ln(n+1)} \leq \frac{1}{\ln n}$  since  $a_n$  is decreasing
2.  $\lim_{n \rightarrow \infty} \frac{1}{\ln n} = 0$

$\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\ln n}$  is conditionally convergent by Alternating Series Test (A.S.T).

### Example 5

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^4}$$

### Answer

$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^4} \right| = \sum_{n=1}^{\infty} \frac{1}{n^4}$  is convergent by  $p$ -series ( $p = 4 > 1$ ).

Thus,  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^4}$  converges absolutely.

## NOTE 1.4.4

**Important! For Absolute and Conditional Convergence Questions**

To check whether the alternating series  $\sum_{n=1}^{\infty} a_n$  is **absolutely /conditionally convergent or divergent**, then you need to do the following:

Take the absolute value  $\sum_{n=1}^{\infty} |a_n|$

- If  $\sum_{n=1}^{\infty} |a_n|$  is **convergent** by any test, then we can state that  $\sum_{n=1}^{\infty} a_n$  is **absolutely convergent**.
- If  $\sum_{n=1}^{\infty} |a_n|$  is **divergent** by any test, then we need to do the following:
  - if  $\sum_{n=1}^{\infty} |a_n|$  is **convergent** by Alternating Series Test (A.S.T), then we can say that  $\sum_{n=1}^{\infty} a_n$  is **conditionally convergent**.
  - If the Alternating Series Test (A.S.T) fails, then we can say that  $\sum_{n=1}^{\infty} a_n$  is **divergent**.

**Example 6**

For what values of  $x$  does the series  $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n 2^n}$  converge absolutely? Converge conditionally? Diverge?

**Answer**

Firstly, we may apply the ratio test

$$\lim_{n \rightarrow \infty} \left| \frac{(x-5)^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n 2^n}{(x-5)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x-5)^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n 2^n}{(x-5)^n} \right| = \lim_{n \rightarrow \infty} \frac{n}{n+1} \left| \frac{x-5}{2} \right| = \left| \frac{x-5}{2} \right|.$$

The series converges absolutely if  $\left| \frac{x-5}{2} \right| < 1$ . That is  $|x-5| < 2 \rightarrow 3 < x < 7 \rightarrow (3,7)$

At  $x = 3 \rightarrow \sum_{n=1}^{\infty} \frac{(-2)^n}{n 2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$  is convergent by Alternating Series Test (A.S.T).

At  $x = 7 \rightarrow \sum_{n=1}^{\infty} \frac{(2)^n}{n 2^n} = \sum_{n=1}^{\infty} \frac{1}{n}$  is divergent because it is a harmonic series.

The series diverges if  $\left| \frac{x-5}{2} \right| > 1$ . That is  $|x-5| > 2 \rightarrow 3 > x \text{ or } 7 < x \rightarrow (-\infty, 3) \cup (7, \infty)$

Thus, the series  $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n 2^n}$

- converges absolutely on  $(3,7)$
- converges conditionally at  $x = 3$

diverges everywhere else  $\leftrightarrow (-\infty, 3) \cup [7, \infty)$ .

## NOTE 1.4.5

### Error Estimate for Alternating Series ONLY

If  $\sum_{n=1}^{\infty} a_n$  converges, then the limit of the partial sum  $s_n$  is equal to  $S$ .

**Recall:**  $s = \lim_{n \rightarrow \infty} s_n$

If  $\lim_{n \rightarrow \infty} s_n = s$  exists, then the series is called *convergent*. Otherwise, the series is called *divergent*.

$$\text{So, } \lim_{n \rightarrow \infty} (s - s_n) = 0$$

Therefore, if  $n$  does not approach infinity, there will be some error between  $S$  &  $S_n$

Error =  $S - S_n$  = The Whole Sum – The Partial Sum.

$$\underbrace{a_1 - a_2 + a_3 - a_4 + \cdots - a_n}_{S_n} + \underbrace{a_{n+1} - a_{n+2} + a_{n+3} \cdots}_{\text{Error}} \rightarrow |Error| = |S - S_n| \leq a_{n+1} = |\text{first omitted term}|$$

**Important:** in order to find the error estimation, you need to prove that the series is convergent by Alternating Series Test (A.S.T).

#### Example 7

How many terms of the series  $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$  are needed to compute the sum of the series with error less than 0.0005 ?

#### Answer

Let  $a_n = \frac{1}{n!} > 0$  for  $n \geq 0$

1.  $\frac{1}{(n+1)!} \leq \frac{1}{n!}$  since  $a_n$  is decreasing
2.  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$

$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$  is convergent by Alternating Series Test (A.S.T).

$$|Error| = |S - S_n| \leq a_{n+1} = \frac{1}{(n+1)!}$$

To compute the sum of the series with error less than 0.0005, i.e.,  $|Error| < 0.0005$

Then we have  $\frac{1}{(n+1)!} < 0.0005$

$$\rightarrow (n+1)! > \frac{1}{0.0005} = 2000$$

$$(n+1)! > 2000 \rightarrow n = 6$$

#### Example 8

How many terms of the series  $\sum_{n=1}^{\infty} \frac{(-1)^n}{1+2^n}$  are needed to compute the sum of the series with error less than 0.001 ?

#### Answer

Let  $\frac{1}{1+2^n} > 0$  for  $n \geq 1$

1.  $\frac{1}{1+2^{n+1}} \leq \frac{1}{1+2^n}$  since  $a_n$  is decreasing
2.  $\lim_{n \rightarrow \infty} \frac{1}{1+2^n} = 0$

$\sum_{n=0}^{\infty} \frac{(-1)^n}{1+2^n}$  is convergent by Alternating Series Test (A.S.T).

$$|Error| = |S - S_n| \leq a_{n+1} = \frac{1}{1+2^{n+1}}$$

To compute the sum of the series with error less than 0.0005, i.e.,  $|Error| < 0.0005$

Then we have  $\frac{1}{1+2^{n+1}} < 0.001$

$$\rightarrow 1 + 2^{n+1} > \frac{1}{0.001} = 1000$$

$$2^{n+1} > 999 \rightarrow 2^{10} = 1024 \rightarrow n+1 = 10 \therefore n = 9$$

**SUMMARY OF TESTS FOR SERIES**

1. If  $\lim_{n \rightarrow \infty} a_n \neq 0$ ,  $\sum_{n=N}^{\infty} a_n \rightarrow$  For positive integer  $N$
2. **By the type:**
  - a. **Geometric:**  $\sum_{n=1}^{\infty} a_n$ 
    - i. converges if  $|r| < 1$  and its sum  $\frac{a}{1-r}$
    - ii. diverges if  $|r| \geq 1$
  - b. **telescoping:** partial fractions to get  $S_n$  then find  $\lim_{n \rightarrow \infty} S_n$
  - c. **p-series:**  $N \in \mathbb{Z}^+$ ,  $\sum_{n=N}^{\infty} \frac{1}{n^p}$  is **convergent** for  $p > 1$  and **divergent** for  $p \leq 1$
3. **Integral Test:** (easy to integrate) If  $f$  is a continuous, positive, and decreasing function where  $f(n) = a_n$  on the interval  $[N, \infty)$  for some positive integer  $N$ ,

then the improper integral  $\int_N^{\infty} f(x) dx$  and the infinite series  $\sum_{n=N}^{\infty} a_n$  either both **converge**, or both **diverge** to infinity.

**4. The Comparison Test**

Let  $\sum a_n$ ,  $\sum b_n$ , and  $\sum c_n$  be series with nonnegative terms. Suppose that for some integer  $N$

$$b_n \leq a_n \leq c_n \text{ for all } n > N.$$

(a) If  $\sum c_n$  (chosen) converges, then  $\sum a_n$  (given) also converges.

(b) If  $\sum b_n$  (chosen) diverges, then  $\sum a_n$  (given) also diverges.

**5. Limit Comparison Test**

Suppose that  $a_n > 0$  and  $b_n > 0$  for all  $n > N$  ( $N$  an integer).

If  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L > 0$  then  $\sum a_n$  (given) and  $\sum b_n$  (chosen) both converge or both diverge.

**6. The Ratio Test** (factorials &  $n^{\text{th}}$  powers)

Let  $\sum a_n$  be a series with positive terms and suppose that

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \rho$$

Then (a) the series converges if  $\rho < 1$

(b) the series diverges if  $\rho > 1$  or is infinite,

(c) the test is inconclusive if  $\rho = 1$ .

**7. The Root Test** ( $n^{\text{th}}$  powers)

Let  $\sum a_n$  be a series with positive terms and suppose that

$$\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \rho$$

Then (a) the series converges if  $\rho < 1$

(b) the series diverges if  $\rho > 1$  or is infinite,

(c) the test is inconclusive if  $\rho = 1$ .

**8. The Alternating Series Test**

Let  $a_n > 0$  for all  $n \in \mathbb{Z}^+$ . The alternating series

$$\sum_{n=1}^{\infty} (-1)^n a_n \text{ and } \sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converge by Alternating Series Test (A.S.T) when the two conditions listed below met.

1.  $a_{n+1} \leq a_n$  for all  $n \in \mathbb{Z}^+ \leftrightarrow f'(x) < 0$
2.  $\lim_{n \rightarrow \infty} a_n = 0$

## NOTE 1.5.1

A series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \dots$$

Is called a **power series in powers of  $x - c$**  or a **power series about  $c$** .

The constants  $a_0, a_1, a_2, \dots$  are called the **coefficients** of the power series.

## NOTE 1.5.2

The convergence of the series  $\sum_{n=0}^{\infty} a_n(x-c)^n$  is described by one of the following three cases:

1. There is a positive number  $R$  such that the series diverges for  $x$  with  $|x - c| > R$  but converges absolutely for  $x$  with  $|x - c| < R$ . The series may or may not converge at either of the endpoints  $x = c - R$  and  $x = c + R$ .  
Note that (Radius)  $R$  and (Center)  $c$
2. The series converges absolutely for every ( $R = \infty$ ).
3. The series converges at  $x = c$  and diverges elsewhere ( $R = 0$ ).

## NOTE 1.5.3

### How to Test a Power Series for Convergence

1. Use the Ratio Test (or Root Test) to find the interval where the series converges absolutely. Ordinarily, this is an open interval

$$|x - a| < R \quad \text{or} \quad a - R < x < a + R.$$

2. If the interval of absolute convergence is finite, test for convergence or divergence at each endpoint. Use a Comparison Test, the Integral Test, or the Alternating Series Test.
3. If the interval of absolute convergence is  $a - R < x < a + R$ , the series diverges for  $|x - a| > R$  (it does not even converge conditionally) because the  $n^{\text{th}}$  term does not approach zero for those values of  $x$ .

Determine the radius of convergence and interval of convergence for the following power series.

### Example 1

$$\sum_{n=0}^{\infty} n! x^n$$

### Answer

$$\lim_{n \rightarrow \infty} \left| \frac{(n+1)! x^{n+1}}{n! x^n} \right| = \lim_{n \rightarrow \infty} |(n+1)x| = \lim_{n \rightarrow \infty} (n+1)|x| = 0 < 1 \text{ if } x = 0$$

The series converges at  $x = 0$ . That means ( $R = 0$ )

### Example 2

### Example 3

$$\sum_{n=1}^{\infty} \frac{x^n}{n}$$

### Answer

$$\lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{n+1} \div \frac{x^n}{n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n} \right| = \lim_{n \rightarrow \infty} \left( \frac{n}{n+1} \right) |x| = |x| < 1$$

$$|x| < 1 \rightarrow (\text{Radius}) R = 1 \text{ and (Center) } c = 0$$

$$|x| < 1 \rightarrow -1 < x < 1 \text{ the endpoints are at } x = -1 \text{ and at } x = 1$$

### Endpoints:

at  $x = -1$ :  $\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$  is convergent since it is alternating harmonic series.

at  $x = 1$ :  $\sum_{n=0}^{\infty} \frac{1}{n}$  is divergent since it is harmonic series.

The interval of convergence is  $[-1, 1)$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

### Answer

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{2n+2}}{(2n+2)!} \div \frac{(-1)^n x^{2n}}{(2n)!} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{(-1)^n x^{2n}} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(-1) x^2}{(2n+2)(2n+1)} \right| = \lim_{n \rightarrow \infty} \frac{x^2}{(2n+2)(2n+1)} = 0 < 1 \text{ for all } x \end{aligned}$$

The series converges absolutely for every ( $R = \infty$ ). Then the radius of convergence is  $(-\infty, \infty)$

### Example 4

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2 \cdot 3^n}$$

### Answer

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{(x-2)^{n+1}}{(n+1)^2 \cdot 3^{n+1}} \div \frac{(x-2)^n}{n^2 \cdot 3^n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(x-2)^{n+1}}{(n+1)^2 \cdot 3^{n+1}} \cdot \frac{n^2 \cdot 3^n}{(x-2)^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(x-2) n^2}{3(n+1)^2} \right| \end{aligned}$$

$$\begin{aligned} &= \lim_{n \rightarrow \infty} \left( \frac{n}{n+1} \right)^2 \left| \frac{(x-2)}{3} \right| = \left| \frac{(x-2)}{3} \right| < 1 \\ |x-2| < 3 &\rightarrow (\text{Radius}) R = 3 \text{ and (Center) } c = 2 \end{aligned}$$

$$|x-2| < 3 \rightarrow -1 < x < 5 \text{ the endpoints are at } x = -1 \text{ and at } x = 5$$

### Endpoints:

at  $x = -1$ :  $\sum_{n=1}^{\infty} \frac{(-3)^n}{n^2 \cdot 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$  is convergent by Alternating Series Test (A.S.T).

at  $x = 5$ :  $\sum_{n=1}^{\infty} \frac{(3)^n}{n^2 \cdot 3^n} = \sum_{n=1}^{\infty} \frac{1}{n^2}$  is convergent by  $p$ -series ( $p = 2 > 1$ )

The interval of convergence is  $[-1, 5]$

**Example 5**

$$\sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^n}{\sqrt{n+1}}$$

**Answer**

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} 2^{n+1} x^{n+1}}{\sqrt{n+2}} \div \frac{(-1)^n 2^n x^n}{\sqrt{n+1}} \right| \\ = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} 2^{n+1} x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-1)^n 2^n x^n} \right| \\ = \lim_{n \rightarrow \infty} \left| \sqrt{\frac{n+1}{n+2}} \cdot 2x \right| = \lim_{n \rightarrow \infty} \left| \sqrt{\frac{n+1}{n+2}} \cdot 2x \right| = \lim_{n \rightarrow \infty} \sqrt{\frac{n+1}{n+2}} |2x| = |2x| < 1 \end{aligned}$$

$$|x| < \frac{1}{2} \rightarrow \text{(Radius) } R = \frac{1}{2} \text{ and (Center) } c = 0$$

$$|x| < \frac{1}{2} \rightarrow -\frac{1}{2} < x < \frac{1}{2} \text{ the endpoints are at } x = -\frac{1}{2} \text{ and at } x = \frac{1}{2}$$

**Endpoints:**

$$\text{at } x = -\frac{1}{2}: \sum_{n=1}^{\infty} \frac{(-1)^n 2^n \left(-\frac{1}{2}\right)^n}{\sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}} \text{ is divergent by limit comparison test.}$$

$$\text{Since } 0 < \frac{1}{\sqrt{n}} \text{ for } n \geq 1$$

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{\sqrt{n+1}}}{\frac{1}{\sqrt{n}}} = \lim_{n \rightarrow \infty} \sqrt{\frac{n}{n+1}} = 1$$

$$\text{Since } \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ is divergent } p\text{-series } \left(p = \frac{1}{2} < 1\right), \text{ then } \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}} \text{ is divergent by the limit comparison test.}$$

$$\text{at } x = \frac{1}{2}: \sum_{n=1}^{\infty} \frac{(-1)^n 2^n \left(\frac{1}{2}\right)^n}{\sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$$

$$1. \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n+1}} = 0$$

$$2. a_{n+1} = \frac{1}{\sqrt{n+2}} < \frac{1}{\sqrt{n+1}} = a_n$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} \text{ is convergent by Alternating Series Test (A.S.T).}$$

$$\text{The interval of convergence is } \left(-\frac{1}{2}, \frac{1}{2}\right]$$



## NOTE 1.6.1

Series Representation

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots + a_n(x-c)^n + \cdots$$

With a positive radius of convergence.

## NOTE 1.6.2

By repeated term by term differentiation within the interval of convergence we obtain

$$f'(x) = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots + na_n(x-c)^{n-1} + \cdots$$

$$f''(x) = 1 \cdot 2a_2 + 2 \cdot 3a_3(x-c) + 3 \cdot 4a_4(x-c)^2 + \cdots$$

$$f'''(x) = 1 \cdot 2 \cdot 3a_3 + 2 \cdot 3 \cdot 4a_4(x-c) + 3 \cdot 4 \cdot 5a_5(x-c)^2 + \cdots$$

With the  $n$ th derivative, for all  $n$ .

In general:

$$f^{(n)}(c) = n! a_n \quad \rightarrow \quad a_n = \frac{f^{(n)}(c)}{n!}$$

Thus, if  $f$  has a series representation, then the series must be

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

## NOTE 1.6.3

The **Taylor series** generated by  $f$  at  $x = c$  (i.e., center is  $c$ ) is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

The **Maclaurin series** is **Taylor series** generated by  $f$  at  $x = 0$  (i.e., center is zero) is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

**Example 1**

$$f(x) = e^x$$

Find Taylor series at  $c = 0$

**Answer**

1. Find the derivative of  $f$  and look for its pattern.

$$f(x) = e^x$$

$$f(0) = e^0 = 1$$

2. Plug in the value of  $x$  which is  $x = 0$  in this case.

$$f'(x) = e^x$$

$$f'(0) = e^0 = 1$$

3. Set up the sum form.

$$f''(x) = e^x$$

$$f''(0) = e^0 = 1$$

4. Check the interval of convergence.

$$f'''(x) = e^x$$

$$f'''(0) = e^0 = 1$$

$$f^n(x) = e^x$$

$$f^n(0) = e^0 = 1$$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

We can apply **Ratio Test**:  $\lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)!} \div \frac{x^n}{n!} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x}{n+1} \right| = \lim_{n \rightarrow \infty} \frac{1}{n+1} |2x| = 0 < 1$  for all  $x$ .

Therefore, the interval of convergence is  $(-\infty, \infty)$

## NOTE 1.6.4

**Frequently used Taylor series**

$$\frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + \dots = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \dots + (-x)^n + \dots = \sum_{n=0}^{\infty} (-1)^n x^n, \quad |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \quad |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad |x| < \infty \leftrightarrow \text{for all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad |x| < \infty \leftrightarrow \text{for all } x$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n-1} \frac{x^n}{n} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \quad -1 < x \leq 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + (-1)^n \frac{x^{2n+1}}{2n+1} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \quad |x| \leq 1$$

**Example 2**

Find the Maclaurin series of  $f(x) = x^2 \sin 2x$

**Answer**

Since  $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$  for all  $x$

$$\sin 2x = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} \quad \text{for all } x$$

$$f(x) = x^2 \sin 2x$$

$$f(x) = \sum_{n=0}^{\infty} x^2 \cdot \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!}$$

$$f(x) = \sum_{n=0}^{\infty} x^2 \cdot \frac{(-1)^n (2)^{2n+1} (x)^{2n+1}}{(2n+1)!}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2)^{2n+1} (x)^{2n+3}}{(2n+1)!} \quad \text{for all } x$$

Evaluate the following limits without using l'hopital rule:

### Example 1

$$\lim_{x \rightarrow 0} \frac{x - \sin x}{x^3}$$

### Answer

$$\lim_{x \rightarrow 0} \frac{x - \sin x}{x^3} \left[ \frac{0}{0} \right]$$

$$\lim_{x \rightarrow 0} \frac{x - \left( x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right)}{x^3} = \lim_{x \rightarrow 0} \frac{\frac{x^3}{3!} - \frac{x^5}{5!} + \dots}{x^3} = \lim_{x \rightarrow 0} \left( \frac{1}{3!} - \frac{x^2}{5!} + \dots \right) = \frac{1}{3!} = \frac{1}{6}.$$

### Example 2

$$\lim_{x \rightarrow 0} \frac{(e^{2x} - 1) \ln(1 + x^3)}{(1 - \cos 3x)^2}$$

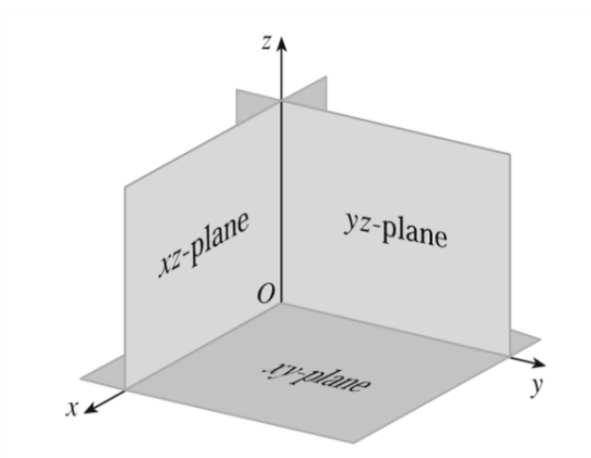
### Answer

$$\lim_{x \rightarrow 0} \frac{(e^{2x} - 1) \ln(1 + x^3)}{(1 - \cos 3x)^2} \left[ \frac{0}{0} \right] = \lim_{x \rightarrow 0} \frac{\left( 1 + (2x) + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \dots - 1 \right) \left( x^3 - \frac{x^6}{2} + \dots \right)}{\left( 1 - \left( 1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} - \dots \right) \right)^2}$$

$$= \lim_{x \rightarrow 0} \frac{2x^4 + 2x^5 + \dots}{\left( \frac{9}{2}x^2 - \frac{3^4}{4!}x^4 + \dots \right)^2} = \lim_{x \rightarrow 0} \frac{2 + 2x + \dots}{\left( \frac{9}{2} - \frac{3^4}{4!}x^2 + \dots \right)^2} = \frac{2}{\left( \frac{9}{2} \right)^2} = \frac{8}{81}.$$

## NOTE 2.1.1

In order to represent any shape (points /planes/ surfaces) in space, we need to determine the three coordinate planes as shown:



The equation  $z = 0$  is the  $xy$  – (**Horizontal**) plane contains  $x$  - and  $y$  - axes.

The equation  $x = 0$  is the  $yz$  – (**Vertical**) plane contains  $y$  - and  $z$  - axes.

The equation  $y = 0$  is the  $xz$  – (**Vertical**) plane contains  $x$  - and  $z$  - axes.

For any constant  $c$ , the equation  $z = c$  is a **horizontal** plane which is parallel to  $xy$  – plane.

For any constant  $c$ , the equation  $x = c$  is a **vertical** plane which is parallel to  $yz$  – plane.

For any constant  $c$ , the equation  $y = c$  is a **vertical** plane which is parallel to  $xz$  – plane.

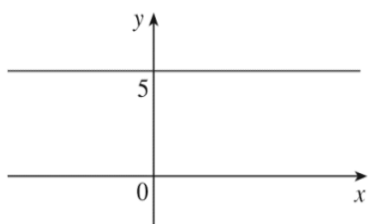
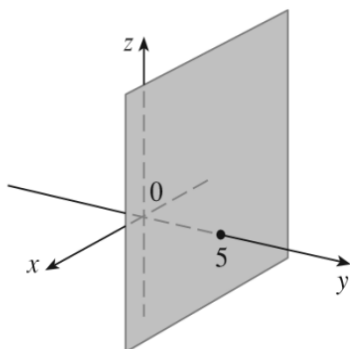
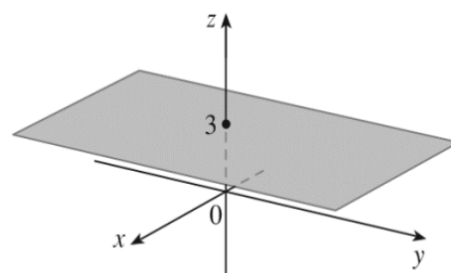
In **two-dimensional** analytic geometry, the graph of an equation involving  $x$  and  $y$  is a **curve** in  $\mathbb{R}^2$ .

In **three-dimensional** analytic geometry, the graph of an equation involving  $x$ ,  $y$  and  $z$  is a **surface** in  $\mathbb{R}^3$ .

## Example 1

What do the following equations represent?

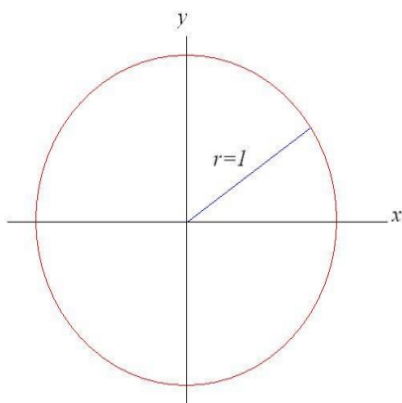
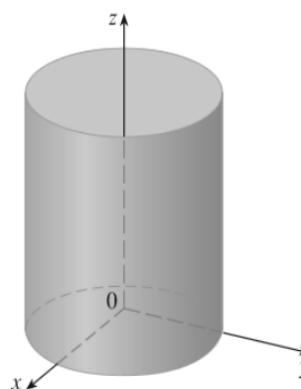
- a)  $y = 5$  in  $\mathbb{R}^2$
- b)  $y = 5$  in  $\mathbb{R}^3$
- c)  $z = 3$  in  $\mathbb{R}^3$

**Answer**a)  $y = 5$ , line in  $\mathbb{R}^2$ b)  $y = 5$ , a plane in  $\mathbb{R}^3$ c)  $z = 3$ , a plane in  $\mathbb{R}^3$ **Example 2**

What do the following equations represent?

a)  $x^2 + y^2 = 1$  in  $\mathbb{R}^2$

b)  $x^2 + y^2 = 1$  in  $\mathbb{R}^3$

**Answer**a)  $x^2 + y^2 = 1$ , circle in  $\mathbb{R}^2$ b)  $x^2 + y^2 = 1$ , cylinder in  $\mathbb{R}^3$ **Example 3**

Which points  $(x, y, z)$  satisfy the equations

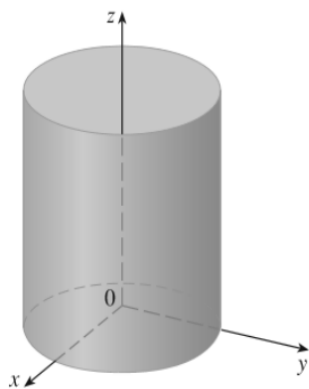
$$x^2 + y^2 = 1 \text{ and } z = 3$$

**Answer**

When the cylinder  $x^2 + y^2 = 1$  (from example 2b) passes through the plane  $z = 3$  (from example 1c), then the result of their intersection is circle with radius 1.

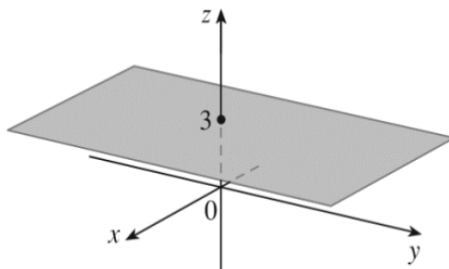
Thus, all points that satisfy the given equations lie on the circle with radius 1 with center on the z-axis.

That is  $\{(x, y, z) | x^2 + y^2 = 1 \text{ and } z = 3, x, y, z \in \mathbb{R}\}$



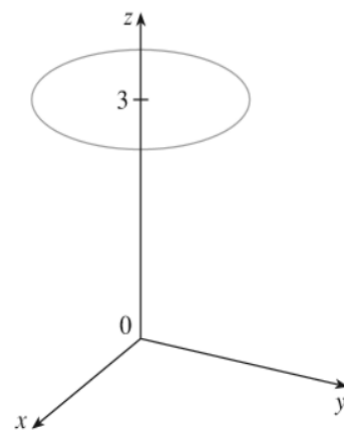
$x^2 + y^2 = 1$ , cylinder in  $\mathbb{R}^3$

+



$z = 3$ , a plane in  $\mathbb{R}^3$

=



Circle with radius 1 at  $z = 3$  in  $\mathbb{R}^3$

### NOTE 2.1.2

#### Distance formula in Three Dimensions

The distance  $|P_1P_2|$  between the points  $P_1(x_1, y_1, z_1)$  and  $P_2(x_2, y_2, z_2)$  is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

#### Example 4

The distance from the point  $P(2, -1, 7)$  to the point  $Q(1, -3, 5)$  is

#### Answer

$$|PQ| = \sqrt{(1 - 2)^2 + (-3 + 1)^2 + (5 - 7)^2} = \sqrt{1 + 4 + 4} = 3$$

### NOTE 2.1.3

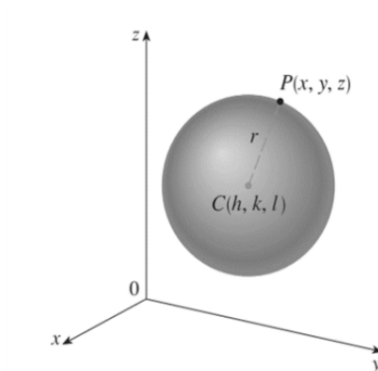
#### Equation of a Sphere

An equation of a sphere with center  $C(h, k, l)$  and radius  $r$  is

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

In particular, if the center is the origin  $O$ , then an equation of the sphere is

$$x^2 + y^2 + z^2 = r^2$$





**Example 5**

Show that  $x^2 + y^2 + z^2 + 4x - 6y + 2z + 6 = 0$  is the equation of a sphere and find its center and radius.

**Answer**

By rewriting the equation in complete squares form:

$$\begin{aligned}(x^2 + 4x + 4) + (y^2 - 6y + 9) + (z^2 + 2z + 1) &= -6 + 4 + 9 + 1 \\(x + 2)^2 + (y - 3)^2 + (z + 1)^2 &= 8\end{aligned}$$

Thus, the center of the sphere is located at  $(-2, 3, -1)$  and radius  $\sqrt{8} = 2\sqrt{2}$

**Example 6**

What region in  $\mathbb{R}^3$  is represented by the following inequalities?

$$1 \leq x^2 + y^2 + z^2 \leq 4 \quad z \leq 0$$

**Answer**

$1 \leq x^2 + y^2 + z^2$  represents the region **outside** the sphere with radius 1.

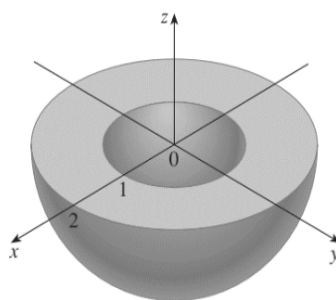
$x^2 + y^2 + z^2 \leq 4$  represents the region **inside** the sphere with radius 2.

$z \leq 0$  the points lie **on or below** the  $xy$  -plane.

Thus, the given inequalities represent the region that:

lies between (or on) the spheres  $x^2 + y^2 + z^2 = 1$  and  $x^2 + y^2 + z^2 = 4$  and beneath (or on) the  $xy$  -plane.

The region is sketched as following:



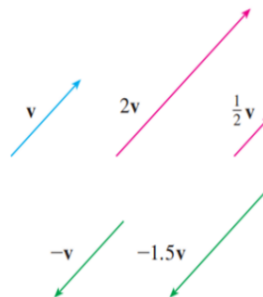
## NOTE 2.2.1

**Vector** is a quantity that has both magnitude and direction.

### Definition of Scalar Multiplication

If  $c$  is a scalar and  $\mathbf{v}$  is a vector, then the **scalar multiple**  $c\mathbf{v}$  is the vector.

The length of the vector is  $|c|$  times the length of  $\mathbf{v}$  and whose direction is the same as  $\mathbf{v}$  if  $c > 0$  and is opposite to  $\mathbf{v}$  if  $c < 0$ . If  $c = 0$  or  $\mathbf{v} = \mathbf{0}$ , then  $c\mathbf{v} = \mathbf{0}$ .



## NOTE 2.2.2

Given the initial point  $A(x_1, y_1, z_1)$  and terminal point  $B(x_2, y_2, z_2)$ , the vector  $\mathbf{a}$  with representation  $\overrightarrow{AB}$  is:

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

The **magnitude** or **length** of the vector  $\mathbf{v}$  is denoted by the symbol  $|\mathbf{v}|$  or  $\|\mathbf{v}\|$ .

The length of the two-dimensional vector  $\mathbf{a} = \langle a_1, a_2 \rangle$  is:

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$$

The length of the three-dimensional vector  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$  is:

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

## NOTE 2.2.3

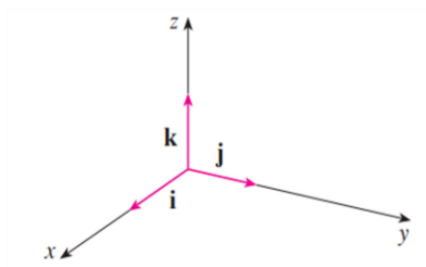
### Properties of Vectors

If  $\mathbf{A}$ ,  $\mathbf{B}$ , and  $\mathbf{C}$  are vectors in  $V_n$  and  $c$  and  $d$  are scalars, then

- |                                                                                      |                                                             |
|--------------------------------------------------------------------------------------|-------------------------------------------------------------|
| 1. $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$                               | 5. $c(\mathbf{A} + \mathbf{B}) = c\mathbf{A} + c\mathbf{B}$ |
| 2. $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$ | 6. $(c + d)\mathbf{A} = c\mathbf{A} + d\mathbf{A}$          |
| 3. $\mathbf{A} + \mathbf{0} = \mathbf{A}$                                            | 7. $(cd)\mathbf{A} = c(d\mathbf{A})$                        |
| 4. $\mathbf{A} + (-\mathbf{A}) = \mathbf{0}$                                         | 8. $1\mathbf{A} = \mathbf{A}$                               |

## NOTE 2.2.4

The vectors  $\mathbf{i}$ ,  $\mathbf{j}$ , and  $\mathbf{k}$  are called the standard basis vectors.



A **unit vector** is a vector whose length is 1. For instance,  $\mathbf{i}$ ,  $\mathbf{j}$ , and  $\mathbf{k}$  are all unit vectors.

In general, if  $\mathbf{a} \neq \mathbf{0}$ , then the unit vector that has the same direction as  $\mathbf{a}$  is

$$\mathbf{u} = \frac{\mathbf{a}}{|\mathbf{a}|} \text{ or } \mathbf{u} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$$

$|\mathbf{a}| = \|\mathbf{a}\|$  is the magnitude (scalar value) of the vector  $\mathbf{a}$ .

$\mathbf{a}$  is the direction (vector value) of the vector  $\mathbf{a}$ .

### Example 1

Find the unit vector in the direction of the vector  $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ .

### Answer

The given vector has length  $|2\mathbf{i} - \mathbf{j} - 2\mathbf{k}| = \sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{9} = 3$

So, the unit vector with the same direction is

$$\frac{(2\mathbf{i} - \mathbf{j} - 2\mathbf{k})}{3} = \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$$

**Note:**

To prove/show that  $\frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$  is a unit vector, its length has to be 1.

$$\sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(-\frac{2}{3}\right)^2} = 1$$

## NOTE 2.2.5

**Definition:**

If  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$  and  $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ , then the **dot product** of  $\mathbf{a}$  and  $\mathbf{b}$  is the number  $\mathbf{a} \cdot \mathbf{b}$  given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

**Properties of the Dot Product:**

If  $\mathbf{A}$ ,  $\mathbf{B}$ , and  $\mathbf{C}$  are vectors in  $V_3$  and  $c$  is scalar, then

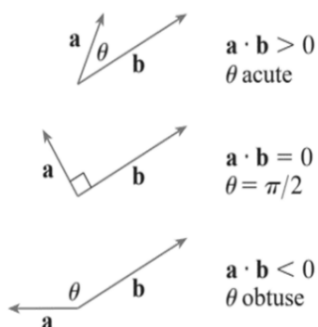
1.  $\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}|^2$
2.  $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$
3.  $\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$
4.  $(c\mathbf{A}) \cdot \mathbf{B} = c(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \cdot (c\mathbf{B})$
5.  $\mathbf{0} \cdot \mathbf{A} = 0$

**Theorem:**

If  $\theta$  is the angle between the vectors  $\mathbf{a}$  and  $\mathbf{b}$ , then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta \quad \text{that is} \quad \cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

Two vectors  $\mathbf{a}$  and  $\mathbf{b}$  are orthogonal /perpendicular if and only if  $\mathbf{a} \cdot \mathbf{b} = 0$

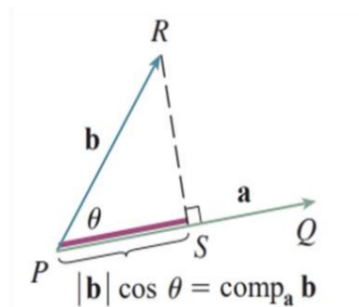
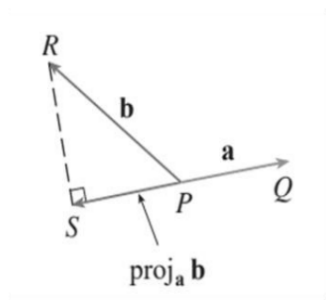
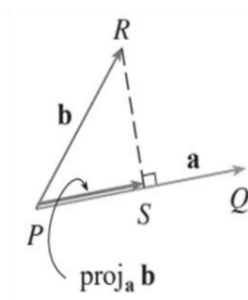


## NOTE 2.2.6

The scalar projection of  $\mathbf{b}$  onto  $\mathbf{a}$  :  $\text{comp}_a \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$

The scalar projection of  $\mathbf{b}$  onto  $\mathbf{a}$  :  $\text{proj}_a \mathbf{b} = \left( \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} \right) \frac{\mathbf{a}}{|\mathbf{a}|} = (\text{comp}_a \mathbf{b}) \frac{\mathbf{a}}{|\mathbf{a}|}$

Note:  $\frac{\mathbf{a}}{|\mathbf{a}|}$  is the unit vector of  $\mathbf{a}$



Vector projections

Scalar projection

**Example 2**

Find the scalar projection and vector projection of  $\mathbf{b} = \langle 1, 1, 2 \rangle$  onto  $\mathbf{a} = \langle -2, 3, 1 \rangle$ .

**Answer**

Since  $|\mathbf{a}| = \sqrt{(-2)^2 + 3^2 + 1^2} = \sqrt{14}$

The scalar projection of  $\mathbf{b}$  onto  $\mathbf{a}$  is

$$\text{comp}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{(-2)(1) + 3(1) + 1(2)}{\sqrt{14}} = \frac{3}{\sqrt{14}}$$

The vector projection is this scalar projection times the unit vector in the direction of  $\mathbf{a}$ :

$$\text{proj}_{\mathbf{a}} \mathbf{b} = \left( \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} \right) \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{3}{\sqrt{14}} \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{3}{14} \mathbf{a} = \left\langle \frac{-3}{7}, \frac{9}{14}, \frac{3}{14} \right\rangle$$

**Section 2.3 The Cross Product in 3-Space****NOTE 2.3.1****Definition:**

If  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$  and  $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ , then the **cross product** of  $\mathbf{a}$  and  $\mathbf{b}$  is the vector:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

$$\mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2) \mathbf{i} - (a_3 b_1 - a_1 b_3) \mathbf{j} + (a_1 b_2 - a_2 b_1) \mathbf{k}$$

$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

**Commonly Used Determinant Types:**

A determinant of order 2

$$\text{The determinant of } \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

A determinant of order 3

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_1 & b_2 \\ c_1 & c_2 & c_3 & c_1 & c_2 \end{bmatrix}$$

The determinant is  $(a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2) - (c_1 b_2 a_3 + c_2 b_3 a_1 + c_3 b_1 a_2)$

**Example 1**

Find  $\mathbf{a} \times \mathbf{b}$  if  $\mathbf{a} = \langle 1, 3, 4 \rangle$  and  $\mathbf{b} = \langle 2, 7, -5 \rangle$ .

**Answer**

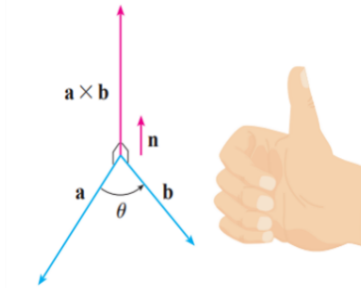
$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} = \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} \mathbf{k}$$

$$\mathbf{a} \times \mathbf{b} = (-15 - 28)\mathbf{i} - (-5 - 8)\mathbf{j} + (7 - 6)\mathbf{k}$$

$$\mathbf{a} \times \mathbf{b} = -43\mathbf{i} + 13\mathbf{j} + \mathbf{k} = \langle -43, 13, 1 \rangle$$

**NOTE 2.3.2**

The vector  $\mathbf{a} \times \mathbf{b}$  is orthogonal /perpendicular to both  $\mathbf{a}$  and  $\mathbf{b}$ .



If  $\theta$  is the angle between  $\mathbf{a}$  and  $\mathbf{b}$  (so  $0 \leq \theta \leq \pi$ ), then

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$$

Two nonzero  $\mathbf{a}$  and  $\mathbf{b}$  are parallel if and only if  $\mathbf{a} \times \mathbf{b} = \mathbf{0}$

The length of the cross product  $\mathbf{a} \times \mathbf{b}$  is equal to the area of the parallelogram determined by  $\mathbf{a}$  and  $\mathbf{b}$ .

**Recall**

$$\mathbf{i} \times \mathbf{i} = \mathbf{0}$$

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}$$

$$\mathbf{i} \cdot \mathbf{j} = 0$$

$$\mathbf{j} \times \mathbf{j} = \mathbf{0}$$

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k}$$

$$\mathbf{j} \cdot \mathbf{i} = 0$$

$$\mathbf{k} \times \mathbf{k} = \mathbf{0}$$

$$\mathbf{j} \times \mathbf{k} = \mathbf{i}$$

$$\mathbf{j} \cdot \mathbf{k} = 0$$

$$\mathbf{i} \cdot \mathbf{i} = 1$$

$$\mathbf{k} \times \mathbf{j} = -\mathbf{i}$$

$$\mathbf{k} \cdot \mathbf{j} = 0$$

$$\mathbf{j} \cdot \mathbf{j} = 1$$

$$\mathbf{k} \times \mathbf{i} = \mathbf{j}$$

$$\mathbf{k} \cdot \mathbf{i} = 0$$

$$\mathbf{k} \cdot \mathbf{k} = 1$$

$$\mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

$$\mathbf{i} \cdot \mathbf{k} = 0$$

Note: the result of the **dot product** is a **scalar** quantity, whereas the result of the **cross product** is a **vector** quantity. The result is a **scalar** quantity, so it has only **magnitude** but **no direction**.

**Example 2**

Find a vector perpendicular to the plane that passes through the points  $P(1,4,6)$ ,  $Q(-2,5,-1)$ , and  $R(1,-1,1)$

**Answer**

The vector  $\overrightarrow{PQ} \times \overrightarrow{PR}$  is perpendicular to both  $\overrightarrow{PQ}$  and  $\overrightarrow{PR}$  and is therefore perpendicular to the plane through  $P$ ,  $Q$ , and  $R$ . We have that

$$\overrightarrow{PQ} = (-2 - 1)\mathbf{i} + (5 - 4)\mathbf{j} + (-1 - 6)\mathbf{k} = -3\mathbf{i} + \mathbf{j} - 7\mathbf{k}$$

$$\overrightarrow{PR} = (1 - 1)\mathbf{i} + (-1 - 4)\mathbf{j} + (1 - 6)\mathbf{k} = -3\mathbf{i} + \mathbf{j} - 7\mathbf{k}$$

We compute the cross product of these vectors:

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix} = (-5 - 35)\mathbf{i} - (15 - 0)\mathbf{j} + (15 - 0)\mathbf{k} = -40\mathbf{i} - 15\mathbf{j} + 15\mathbf{k}$$

So, the vector  $\langle -40, -15, 15 \rangle$  is perpendicular to the given plane. Any nonzero scalar multiple of this vector, such as  $\langle -8, -3, 3 \rangle$ , is also perpendicular to the plane.

**Example 3**

Find the area of the triangle with vertices  $P(1,4,6)$ ,  $Q(-2,5,-1)$ , and  $R(1,-1,1)$

**Answer**

In Example 2, we computed that

$$\overrightarrow{PQ} \times \overrightarrow{PR} = -40\mathbf{i} - 15\mathbf{j} + 15\mathbf{k}$$

The area of the parallelogram with adjacent sides  $PQ$  and  $PR$  is the length of this cross product:

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = \sqrt{(-40)^2 + (-15)^2 + 15^2} = 5\sqrt{82}$$

The area  $A$  of the triangle  $PQR$  is half the area of this parallelogram, that is  $\frac{5\sqrt{82}}{2}$



## NOTE 2.3.3

**Properties of the Dot Product:**

If  $A$ ,  $B$ , and  $C$  are vectors in  $V_3$  and  $c$  is scalar, then

$$A \times B = -B \times A$$

$$(cA) \times B = c(A \times B) = A \times (cB)$$

$$A \times (B + C) = A \times B + A \times C$$

$$(A + B) \times C = A \times C + B \times C$$

$$A \cdot (B \times C) = (A \times B) \cdot C$$

$$A \times (B \times C) = (A \cdot C)B - (A \cdot B)C$$

**Triple Products**

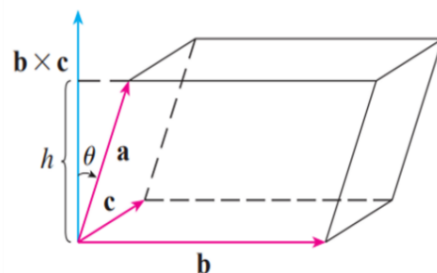
$a \cdot (b \times c)$  is called the scalar triple product of the vectors  $a$ ,  $b$ , and  $c$ .

$$a \cdot (b \times c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

If the value of the scalar triple product of the vectors  $a$ ,  $b$ , and  $c$  is zero, then the vectors must lie in the same plane; that is, they are **coplanar**.

The volume of the parallelepiped determined by the vectors  $a$ ,  $b$ , and  $c$  is the magnitude of their scalar triple product:

$$V = |a \cdot (b \times c)|$$

**Example 4**

Use the scalar triple product to show that the vectors  $a = \langle 1, 4, -7 \rangle$ ,  $b = \langle 2, -1, 4 \rangle$ , and  $c = \langle 0, -9, 18 \rangle$  are coplanar.

**Answer**

$$a \cdot (b \times c) = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix} = 1 \begin{vmatrix} -1 & 4 \\ -9 & 18 \end{vmatrix} - 4 \begin{vmatrix} 2 & 4 \\ 0 & 18 \end{vmatrix} - 7 \begin{vmatrix} 2 & -1 \\ 0 & -9 \end{vmatrix} = 1(18) - 4(36) - 7(-18) = 0$$

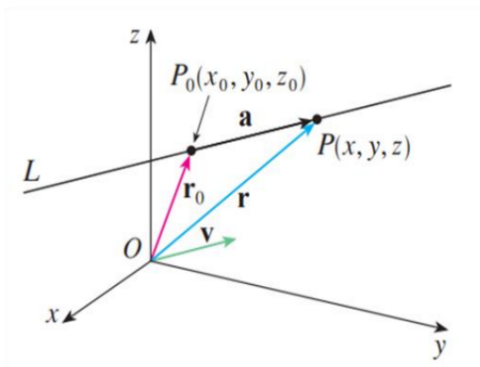
Therefore, the volume of the parallelepiped determined by  $a$ ,  $b$ , and  $c$  is 0. This means that  $a$ ,  $b$ , and  $c$  are coplanar.

### NOTE 2.4.1

#### Lines

If  $\mathbf{a}$  is the vector with representation  $\overrightarrow{P_0P}$ , as in the figure below, then the Triangle Law for vector addition gives (position vector)  $\mathbf{r} = \mathbf{r}_0 + \mathbf{a}$ . But, since  $\mathbf{a}$  and  $\mathbf{v}$  are parallel vectors, there is a scalar  $t$  such that  $\mathbf{a} = t\mathbf{v}$ . Thus

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v} \quad \text{which is a vector equation of } L$$



$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases} \text{ where } t \in \mathbb{R}. \text{ These equations are called } \mathbf{parametric equations} \text{ of the line } L.$$

$$\langle x, y, z \rangle = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle \text{ This equation is called } \mathbf{vector equation} \text{ of the line } L.$$

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c} \text{ These equations are called } \mathbf{symmetric equations} \text{ of the line } L.$$

The line  $L$  passes through the point  $P_0(x_0, y_0, z_0)$  and parallel to the vector  $\mathbf{v} = \langle a, b, c \rangle$

The **skew lines** do not intersect and are not parallel (and therefore do not lie in the same plane).

#### Example 1

- Find a vector, parametric, and symmetric equations for the line that passes through the point  $(5, 1, 3)$  and is parallel to the vector  $\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ .
- Find two other points on the line obtained from part (a).
- Find a vector, parametric, and symmetric equations for the line that passes through the points  $A(2, 4, -3)$  and  $B(3, -1, 1)$ .
- At what point does this line obtained from part (c) intersect the  $xy$ -plane?

## Answer

(a) the point  $(5,1,3)$  and is parallel to the vector  $\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ .

Here  $\mathbf{r}_0 = \langle 5,1,3 \rangle = 5\mathbf{i} + \mathbf{j} + 3\mathbf{k}$  and  $\mathbf{v} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$  so the vector equation becomes

$$\mathbf{r} = (5\mathbf{i} + \mathbf{j} + 3\mathbf{k}) + t(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$$

$$\mathbf{r} = (5+t)\mathbf{i} + (1+4t)\mathbf{j} + (3-2t)\mathbf{k}$$

$$\langle x, y, z \rangle = \langle 5+t, 1+4t, 3-2t \rangle$$

Parametric equations are

$$x = 5 + t$$

$$y = 1 + 4t$$

$$z = 3 - 2t$$

Symmetric equations are

$$\frac{x-5}{1} = \frac{y-1}{4} = \frac{z-3}{-2}$$

(b) Choosing the parameter value  $t = 1$  gives  $x = 6$ ,  $y = 5$ , and  $z = 1$  so  $(6,5,1)$  is a point on the line. Similarly,  $t = -1$  gives the point  $(4, -3, 5)$ .

(c) the points  $A(2,4,-3)$  and  $B(3,-1,1)$ .

Here  $\mathbf{r}_0 = \langle 2,4,-3 \rangle = 2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$  and  $\mathbf{v} = 1\mathbf{i} - 5\mathbf{j} + 4\mathbf{k}$  so the vector equation becomes

$$\mathbf{r} = (2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}) + t(1\mathbf{i} - 5\mathbf{j} + 4\mathbf{k})$$

$$\mathbf{r} = (2+t)\mathbf{i} + (4-5t)\mathbf{j} + (-3+4t)\mathbf{k}$$

$$\langle x, y, z \rangle = \langle 2+t, 4-5t, -3+4t \rangle$$

Parametric equations are

$$x = 2 + t$$

$$y = 4 - 5t$$

$$z = -3 + 4t$$

Symmetric equations are

$$\frac{x-2}{1} = \frac{y-4}{-5} = \frac{z+3}{4}$$

(d) The line intersects the  $xy$ -plane when  $z = 0$ , so we put  $z = 0$  in the symmetric equations and obtain

$$\frac{x-2}{1} = \frac{y-4}{-5} = \frac{3}{4}$$

This gives  $x = \frac{11}{4}$  and  $y = \frac{1}{4}$ , so the line intersects the  $xy$ -plane at the point  $\left(\frac{11}{4}, \frac{1}{4}, 0\right)$

### Example 2

Show that the lines  $L_1$  and  $L_2$  with parametric equations

$$\begin{array}{lll} x = 1 + t & y = -2 + 3t & z = 4 - t \\ x = 2s & y = 3 + s & z = -3 + 4s \end{array}$$

are **skew lines**; that is, they do not intersect and are not parallel (and therefore do not lie in the same plane).

### Answer

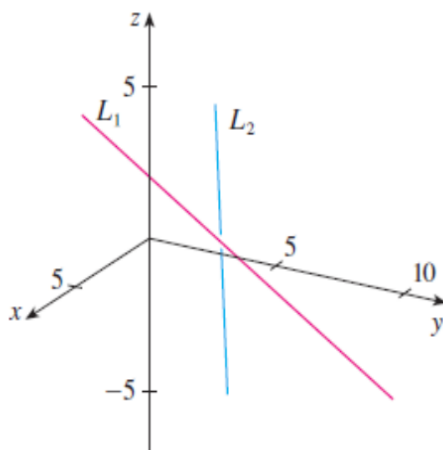
The lines are not parallel because the corresponding vectors  $\langle 1, 3, -1 \rangle$  and  $\langle 2, 1, 4 \rangle$  are not parallel.

If  $L_1$  and  $L_2$  had a point of intersection, there would be values of  $t$  and  $s$  such that

$$\begin{aligned} 1 + t &= 2s \\ -2 + 3t &= 3 + s \\ 4 - t &= -3 + 4s \end{aligned}$$

But if we solve the first two equations, we get  $t = \frac{11}{5}$  and  $s = \frac{8}{5}$ , and these values don't satisfy the third equation.

Therefore, there are no values of  $t$  and  $s$  that satisfy the three equations, so  $L_1$  and  $L_2$  do not intersect. Thus,  $L_1$  and  $L_2$  are skew lines.



## NOTE 2.4.2

### ➤ Planes

A line in space is determined by a **point** and a **direction**.

The line  $L$  passes through the point  $P_0(x_0, y_0, z_0)$  and parallel to the vector  $\mathbf{v} = \langle a, b, c \rangle$ .

A plane in space is determined by a **point** in the plane and a **vector** that is orthogonal/ perpendicular to the plane.

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

is called a **vector equation of the plane**.

To obtain a scalar equation for the plane, we write

$$\mathbf{n} = \langle a, b, c \rangle$$

$$\mathbf{r} = \langle x, y, z \rangle$$

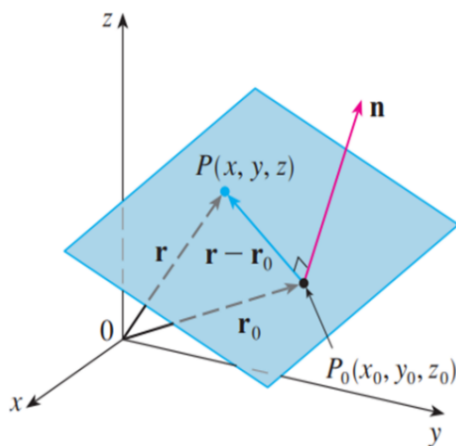
$$\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$$

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

is called the **scalar equation of the plane**.

When we represent a plane, we use the **scalar equation form**. The scalar equation of the plane passes through  $P_0(x_0, y_0, z_0)$  with **normal vector**  $\mathbf{n} = \langle a, b, c \rangle$



**Example 3**

Find an equation of the plane that passes through the points  $P(1,3,2)$ ,  $Q(3,-1,6)$ , and  $R(5,2,0)$ .

**Answer**

The vectors  $\mathbf{a}$  and  $\mathbf{b}$  corresponding to  $\overrightarrow{PQ}$  and  $\overrightarrow{PR}$  are

$$\mathbf{a} = \langle 2, -4, 4 \rangle \quad \mathbf{b} = \langle 4, -1, -2 \rangle$$

Since both  $\mathbf{a}$  and  $\mathbf{b}$  lie in the plane, their cross product  $\mathbf{a} \times \mathbf{b}$  is orthogonal to the plane and can be taken as the normal vector. Thus

$$\mathbf{n} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = 12\mathbf{i} + 20\mathbf{j} + 14\mathbf{k}$$

With the point  $P(1,3,2)$  and the normal vector  $\mathbf{n}$ , an equation of the plane is

$$12(x - 1) + 20(y - 3) + 14(z - 2) = 0$$

$$6x + 10y + 7z = 50$$

**Example 4**

Find the point at which the line with parametric equations

$$x = 2 + 3t \quad y = -4t \quad z = 5 + t$$

intersects the plane  $4x + 5y - 2z = 18$

**Answer**

We substitute the expressions for  $x$ ,  $y$ , and  $z$  from the parametric equations into the equation of the plane:

$$4(2 + 3t) + 5(-4t) - 2(5 + t) = 18$$

This simplifies to  $-10t = 20$ , so  $t = -2$ . Therefore, the point of intersection occurs when the parameter value is  $t = -2$ . Then,

$$x = 2 + 3(-2) = -4 \quad y = -4(-2) = 8 \quad z = 5 + (-2) = 3$$

and so, the point of intersection is  $(-4, 8, 3)$

**Example 5**

Find symmetric equations for the line of intersection  $L$  of these two planes  $x + y + z = 1$  and  $x - 2y + 3z = 1$

**Answer**

We first need to find a point on  $L$ . For instance, we can find the point where the line intersects the  $xy$ -plane by setting  $z = 0$  in the equations of both planes. This gives the equations  $x + y = 1$  and  $x - 2y = 1$ , whose solution is  $x = 1$ ,  $y = 0$ . So, the point  $(1, 0, 0)$  lies on  $L$ .

Now we observe that, since  $L$  lies in both planes, it is perpendicular to both of the normal vectors. Thus, a vector  $\mathbf{v}$  parallel to  $L$  is given by the cross product

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 5\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$$

and so, the symmetric equations of  $L$  can be written as

$$\frac{x-1}{5} = \frac{y}{-2} = \frac{z}{-3}$$

### Example 6

Find the distance between the parallel planes  $10x + 2y - 2z = 5$  and  $5x + y - z = 1$ .

### Answer

First, we note that the planes are parallel because their normal vectors  $\langle 10, 2, -2 \rangle$  and  $\langle 5, 1, -1 \rangle$  are parallel. To find the distance  $D$  between the planes, we choose any point on one plane and calculate its distance to the other plane. In particular, if we put  $y = z = 0$  in the equation of the first plane, we get  $10x = 5$  and so  $(\frac{1}{2}, 0, 0)$  is a point in this plane. The distance between  $(\frac{1}{2}, 0, 0)$  and the plane  $5x + y - z = 1$  is

$$D = \frac{\left| 5\left(\frac{1}{2}\right) + 1(0) + 1(0) - 1 \right|}{\sqrt{5^2 + 1^2 + (-1)^2}} = \frac{3/2}{3\sqrt{3}} = \frac{\sqrt{3}}{6}$$

So, the distance between the planes is  $\frac{\sqrt{3}}{6}$ .

### Example 7

The lines

$$\begin{array}{lll} L_1: & x = 1 + t & y = -2 + 3t & z = 4 - t \\ L_2: & x = 2s & y = 3 + s & z = -3 + 4s \end{array}$$

are skew (check Example 2 to see how such lines are skew). Find the **distance** between them.

### Answer

The distance between  $L_1$  and  $L_2$  is the same as the distance between  $P_1$  and  $P_2$ , which can be computed as in Example 6.

The common normal vector to both planes must be orthogonal to both  $\mathbf{v}_1 = \langle 1, 3, -1 \rangle$  (the direction of  $L_1$ ) and  $\mathbf{v}_2 = \langle 2, 1, 4 \rangle$  (the direction of  $L_2$ ). So, a normal vector is

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -1 \\ 2 & 1 & 4 \end{vmatrix} = 13\mathbf{i} - 6\mathbf{j} - 5\mathbf{k}$$

If we put  $s = 0$  in the equations of  $L_2$ , we get the point  $(0, 3, -3)$  on  $L_2$  and so an equation for  $P_2$  is

$$13(x - 0) - 6(y - 3) - 5(z + 3) = 0 \quad \text{or} \quad 13x - 6y - 5z + 3 = 0$$

If we put  $t = 0$  in the equations of  $L_1$ , we get the point  $(1, -2, 4)$  on  $L_1$  and so

The **distance** between  $L_1$  and  $L_2$  is the same as the distance from  $(1, -2, 4)$  to  $13x - 6y - 5z + 3 = 0$ .

This distance is

$$D = \frac{|13(1) - 6(-2) - 5(4) + 3|}{\sqrt{13^2 + (-6)^2 + (-5)^2}} = \frac{8}{\sqrt{230}} \approx 0.53$$

## Section 2.5 Quadric Surfaces

### NOTE 2.5.1

In order to **sketch** the graph of a **surface**, it is useful to determine the curves of intersection of the surface with planes parallel to the coordinate planes. These curves are called **traces** (or cross-sections) of the surface.

#### Cylinders

A **cylinder** is a surface that consists of all lines (called **rulings**) that are parallel to a given line and pass-through a given plane curve.

If one of the variables  $x$ ,  $y$ , or  $z$  is missing from the equation of a surface, then the surface is a **cylinder**.

#### Example 1

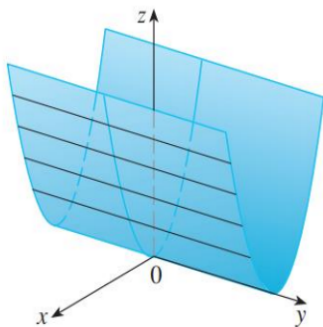
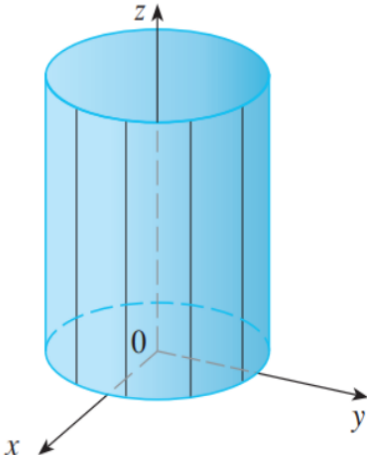
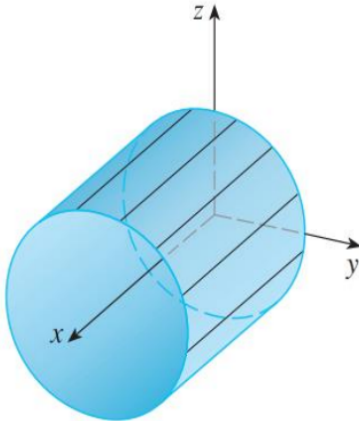
Identify and sketch the surfaces.

(a)  $z = x^2$

(b)  $x^2 + y^2 = 1$

(c)  $y^2 + z^2 = 1$

#### Answer

| (a) $z = x^2$                                                                       | (b) $x^2 + y^2 = 1$                                                                 | (c) $y^2 + z^2 = 1$                                                                   |
|-------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------|
|  |  |  |
| $y$ is missing                                                                      | $z$ is missing                                                                      | $x$ is missing                                                                        |
| The rulings of the cylinder are parallel to the $y$ -axis.                          | The rulings are vertical lines, which are parallel to the $z$ -axis                 | The rulings are horizontal lines, which are parallel to the $x$ -axis                 |
| The trace of the cylinder in the $xz$ -plane is parabola.                           | The trace of the cylinder in the $xy$ -plane is circle.                             | The trace of the cylinder in the $yz$ -plane is circle.                               |
| <b>Parabolic Cylinder</b>                                                           | <b>Circular Cylinder</b>                                                            | <b>Circular Cylinder</b>                                                              |



## NOTE 2.5.2

### Quadric Surfaces

A **quadric surface** is the graph of a second-degree equation in three variables  $x$ ,  $y$ , and  $z$ .

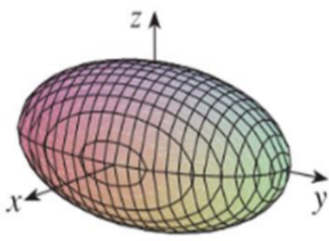
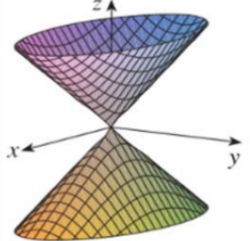
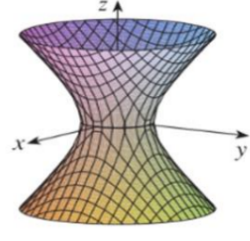
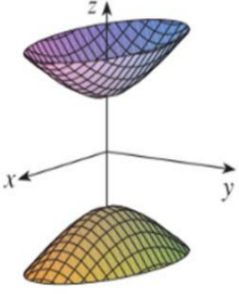
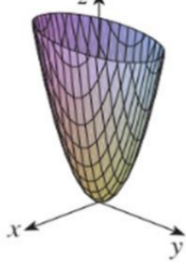
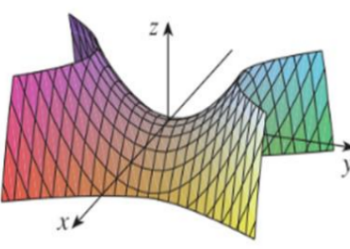
$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

where  $A, B, C, D, E, F, I, J$  are constants, but by translation and rotation it can be brought into one of the two standard forms

$$Ax^2 + By^2 + Cz^2 + J = 0 \quad \text{or} \quad Ax^2 + By^2 + Iz = 0$$

## NOTE 2.5.3

### Quadric Surfaces

| Ellipsoid                                                                                                                                                                                                                                                   | Cone                                                                                                                                                                                                                                                           | Hyperboloid of One Sheet                                                                                                                                                                                             |
|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
|                                                                                                                                                                           |                                                                                                                                                                              |                                                                                                                                  |
| $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ <p>All traces are ellipses.<br/>If <math>a = b = c</math>, the ellipsoid is a sphere.</p>                                                                                                         | $\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ <p>Horizontal traces are ellipses.<br/>Vertical traces in the planes <math>x = k</math> and <math>y = k</math> are hyperbolas if <math>k \neq 0</math> but are pairs of lines if <math>k = 0</math>.</p> | $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ <p>Horizontal traces are ellipses.<br/>Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.</p> |
| Hyperboloid of Two Sheets                                                                                                                                                                                                                                   | Elliptic Paraboloid                                                                                                                                                                                                                                            | Hyperbolic Paraboloid                                                                                                                                                                                                |
|                                                                                                                                                                          |                                                                                                                                                                             |                                                                                                                                 |
| $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ <p>Horizontal traces in <math>z = k</math> are ellipses if <math>k &gt; c</math> or <math>k &lt; -c</math>.<br/>Vertical traces are hyperbolas.<br/>The two minus signs indicate two sheets.</p> | $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ <p>Horizontal traces are ellipses.<br/>Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.</p>                                                       | $\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ <p>Horizontal traces are hyperbolas.<br/>Vertical traces are parabolas. The case <math>c &lt; 0</math> is illustrated.</p>                                         |

**Example 2**

Use traces to sketch the quadric surface with equation

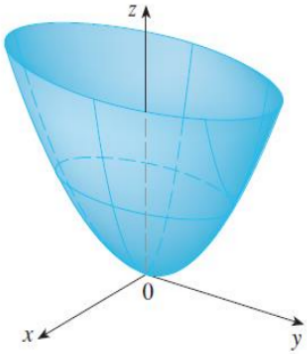
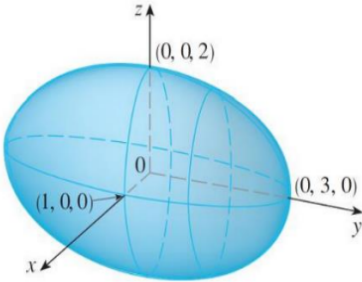
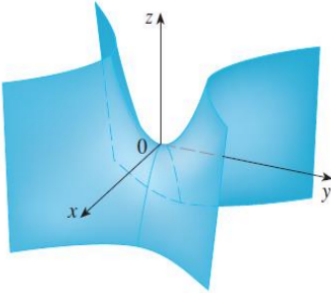
(a)  $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$

(b)  $z = 4x^2 + y^2$

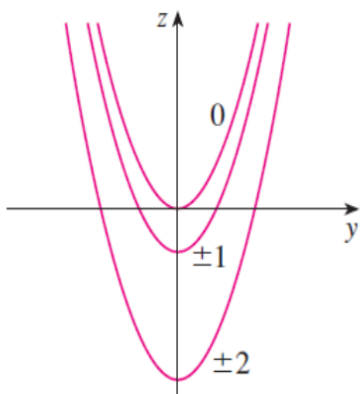
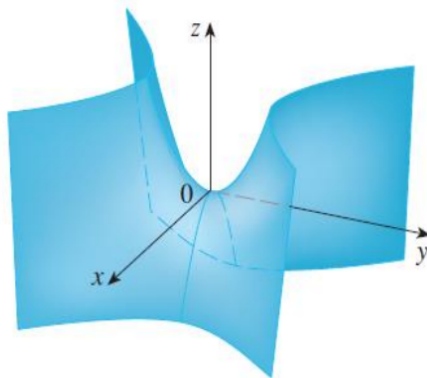
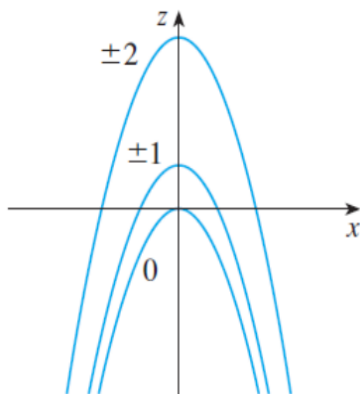
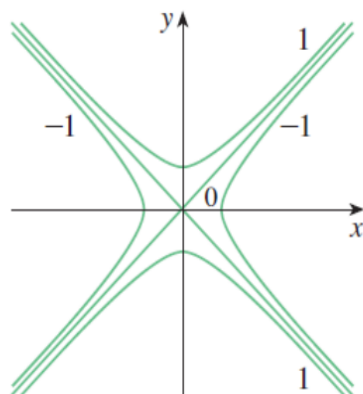
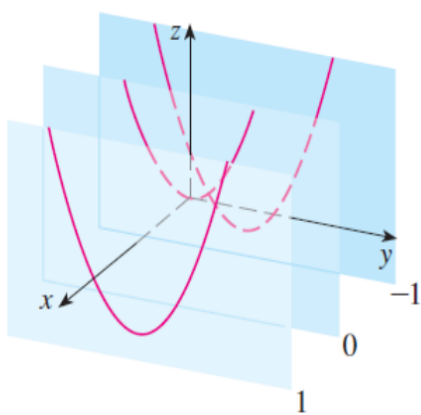
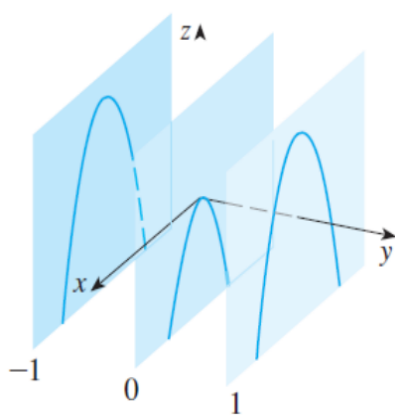
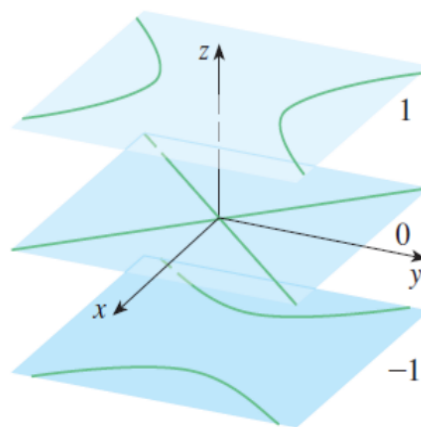
(c)  $z = y^2 - x^2$

**Answer**

| Trace Type                                                         | (a) $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$                   | (b) $z = 4x^2 + y^2$      | (c) $z = y^2 - x^2$         |
|--------------------------------------------------------------------|-----------------------------------------------------------------|---------------------------|-----------------------------|
| Horizontal Trace<br>( $z = k$ )<br>parallel to the<br>$xy$ - plane | $x^2 + \frac{y^2}{9} = 1 - \frac{k^2}{4}$<br>(If $-2 < k < 2$ ) | $4x^2 + y^2 = k$          | $y^2 - x^2 = k$             |
|                                                                    | Ellipse                                                         | Ellipse                   | Family of Hyperbolas        |
| Vertical Trace<br>( $y = k$ )<br>parallel to the<br>$xz$ - plane   | $x^2 + \frac{z^2}{4} = 1 - \frac{k^2}{9}$<br>(If $-3 < k < 3$ ) | $z = 4x^2 + k^2$          | $z = -x^2 + k^2$            |
|                                                                    | Ellipse                                                         | Parabola<br>(Open Upward) | Parabola<br>(Open Downward) |
| Vertical Trace<br>( $x = k$ )<br>parallel to the<br>$yz$ - plane   | $\frac{y^2}{9} + \frac{z^2}{4} = 1 - k^2$<br>(If $-1 < k < 1$ ) | $z = y^2 + 4k^2$          | $z = y^2 - k^2$             |
|                                                                    | Ellipse                                                         | Parabola<br>(Open Upward) | Parabola<br>(Open Upward)   |
| Surface Name                                                       | Ellipsoid                                                       | Elliptic Paraboloid       | Hyperbolic Paraboloid       |

|                                                                                     |                                                                                     |                                                                                       |
|-------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------|
|  |  |  |
| $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$                                           | $z = 4x^2 + y^2$                                                                    | $z = y^2 - x^2$                                                                       |
| Ellipsoid                                                                           | Elliptic Paraboloid                                                                 | Hyperbolic Paraboloid                                                                 |

# Hyperbolic Paraboloid

$$z = y^2 - x^2$$
Traces in  $x = k$  are  $z = y^2 - k^2$ Traces in  $y = k$  are  $z = -x^2 + k^2$ Traces in  $z = k$  are  $y^2 - x^2 = k$ Traces in  $x = k$ Traces in  $y = k$ Traces in  $z = k$

### NOTE 3.1.1

#### One Dimension

$$x \rightarrow \text{Function } f(x) \rightarrow y$$

Domain (Input):  $\forall x$  variables

Range (Output):  $\forall y$  variables

#### Two Dimensions

$$(x, y) \rightarrow \text{Function } f(x, y) \rightarrow z$$

Domain (Input):  $\forall (x, y)$  pairs

Range (Output):  $\forall z$  variables

In the function  $z = f(x, y)$  the  $x$  and  $y$  variables are called the **independent** variables and the  $z$  variable is the **dependent** variable.

#### Example 1

For each of the following functions, evaluate  $f(3,2)$  and find and sketch the domain.

$$(a) f(x, y) = \frac{\sqrt{x+y+1}}{x-1}$$

$$(b) f(x, y) = x \ln(y^2 - x)$$

#### Answer

$$(a) f(2,3) = \frac{\sqrt{3+2+1}}{3-1} = \frac{\sqrt{6}}{2}$$

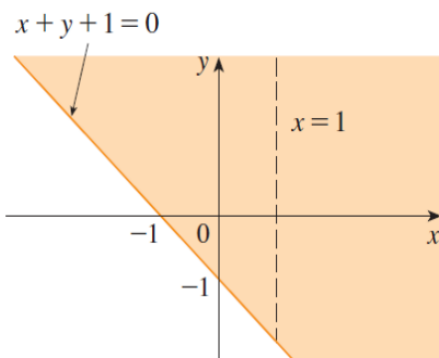
The expression for  $f(x, y)$  makes sense if the denominator is not 0 and the quantity under the square root sign is nonnegative. So, the domain of  $f(x, y)$  is

$$D = \{(x, y) | x + y + 1 \geq 0, x \neq 1\}$$

The inequality  $x + y + 1 \geq 0$ , or  $y \geq -x - 1$ , describes the points that lie on or above the line

$$y = -x - 1$$

While  $x \neq 1$  means that the points on the line  $x = 1$  must be excluded from the domain.



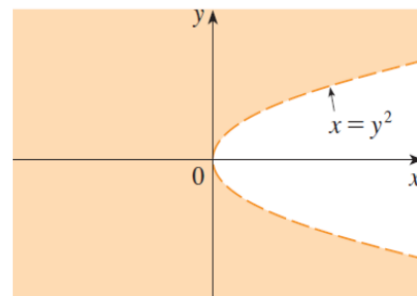
$$(b) f(x, y) = x \ln(y^2 - x)$$

since  $\ln(y^2 - x)$  is defined only when

$x^2 - x > 0$ , that is,  $x < y^2$ , the domain of  $f(x, y)$  is

$$D = \{(x, y) | x < y^2\}$$

This is the set of points to left of the parabola  $x = y^2$

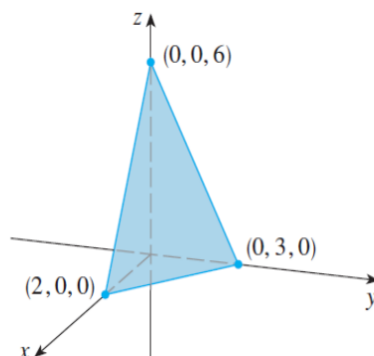


**Example 2**

Sketch the graph of the function  $f(x, y) = 6 - 3x - 2y$

**Answer**

To graph the plane, we first find the intercepts. Putting  $y = z = 0$  in the equation, we get  $x = 2$  as the  $x$ -intercept. Similarly, the  $y$ -intercept is 3 and the  $z$ -intercept is 6.



The function in this example is a special case of the function

$$f(x, y) = ax + by + c$$

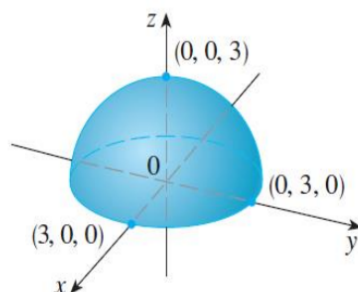
which is called a **linear function**.

**Example 3**

Sketch the graph of the function  $g(x, y) = \sqrt{9 - x^2 - y^2}$ .

**Answer**

The graph has equation  $z = \sqrt{9 - x^2 - y^2}$ . We square both sides of this equation to obtain  $z^2 = 9 - x^2 - y^2$ , or  $x^2 + y^2 + z^2 = 9$ , which we recognize as an equation of the sphere with center the origin and radius 3. But, since  $z \geq 0$ , the graph of  $g(x, y)$  is just the top half of this sphere.

**NOTE 3.1.2**

An entire sphere can't be represented by a single function of  $x$  and  $y$ . The upper hemisphere of the sphere  $x^2 + y^2 + z^2 = r^2$  is represented by the function  $g(x, y) = \sqrt{r^2 - x^2 - y^2}$ . The lower hemisphere is represented by the function  $g(x, y) = -\sqrt{r^2 - x^2 - y^2}$ .

**NOTE 3.1.3**

The **level curves** (contour lines are formed from points of constant elevation that are joined together) of a function  $f(x, y)$  of two variables are the curves with equations  $f(x, y) = k$ , where  $k$  is a constant (in the range of  $f(x, y)$ ).

**Example 4**

Sketch the level curves of the function  $f(x, y) = 6 - 3x - 2y$  for the values  $k = -6, 0, 6, 12$ .

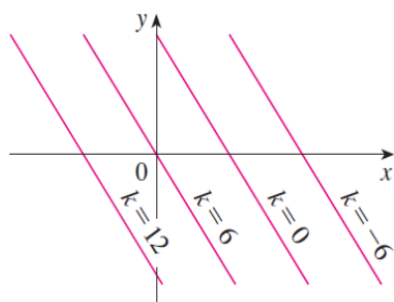
**Answer**

The level curves are

$$6 - 3x - 2y = k \quad \text{or} \quad 3x + 2y + (k - 6) = 0$$

$$\text{or} \quad y = \frac{-3x}{2} - \frac{(k-6)}{2}$$

This is a family of lines with slope  $-\frac{3}{2}$ .



**Example 5**

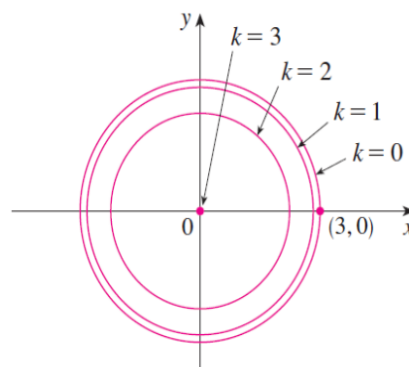
Sketch the level curves of the function  $g(x, y) = \sqrt{9 - x^2 - y^2}$  for the values  $k = 0, 1, 2, 3$ .

**Answer**

The level curves are

$$\sqrt{9 - x^2 - y^2} = k \quad \text{or} \quad x^2 + y^2 = 9 - k^2$$

This is a family of concentric circles with center  $(0, 0)$  and radius  $\sqrt{9 - k^2}$ .





## NOTE 3.2.1

Let  $f(x, y)$  be a function of two variables whose domain  $D$  includes points arbitrarily close to  $(a, b)$ . Then we say that the **limit of  $f(x, y)$  as  $(x, y)$  approach  $(a, b)$  is  $L$**  and we write

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$$

The definition refers only to the distance between  $(x, y)$  and  $(a, b)$ . It does not refer to the direction of approach.

Therefore, if the limit exists, then  $f(x, y)$  must approach the same limit no matter how  $(x, y)$  approaches  $(a, b)$ . Thus, if we can find two different paths of approach along which the function  $f(x, y)$  has different limits, then it follows that does not exist.

**how to show that a limit does exist or not**

1. Use any possible path method (valid ONLY to confirm that the limit does not exist)
2. Use the Squeeze Theorem (valid ONLY to confirm that the limit exists)
3. Use the polar coordinate trick (valid ONLY to confirm that the limit does not exist)

[Recall:  $x = r \cos \theta$ ,  $y = r \sin \theta$ ,  $x^2 + y^2 = r^2$ , and  $(x, y) \rightarrow (0, 0)$  convert to  $r \rightarrow 0^+$ ]

**Example 1**

Is  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2}$  exist, or not?

**Answer**

**Method 1:** Use any possible path

Path 1: Along  $x$  -axis  $\rightarrow y = 0 \rightarrow \lim_{(x,0) \rightarrow (0,0)} \frac{x^2}{x^2} = 1$

Path 2: Along  $y$  -axis  $\rightarrow x = 0 \rightarrow \lim_{(0,y) \rightarrow (0,0)} \frac{-y^2}{y^2} = -1$

Since these limits along two paths do not match, we say that the limit does not exist.

**Method 2:** The Squeeze Theorem does not hold for this limit

**Method 3:** Use the polar coordinate trick

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{r \rightarrow 0^+} \frac{r^2 \cos^2 \theta - r^2 \sin^2 \theta}{r^2} = \lim_{r \rightarrow 0^+} \cos^2 \theta - \sin^2 \theta = \cos^2 \theta - \sin^2 \theta$$

The resulting limit depends on the choice of  $\theta$  along the path, and so the limit does not exist.

**Example 2**

Is  $\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2+y^2}$  exist, or not?

**Answer**

**Method 1:** Use any possible path

Path 1: Along  $x$  –axis  $\rightarrow y = 0 \rightarrow \lim_{(x,0) \rightarrow (0,0)} \frac{0}{x^2} = 0$

Path 2: Along  $y$  –axis  $\rightarrow x = 0 \rightarrow \lim_{(0,y) \rightarrow (0,0)} \frac{0}{y^2} = 0$

Path 3: Along  $y = mx \rightarrow \lim_{(x,mx) \rightarrow (0,0)} \frac{x(mx)}{x^2+(mx)^2} = \lim_{(x,mx) \rightarrow (0,0)} \frac{mx^2}{x^2+mx^2} = \lim_{(x,mx) \rightarrow (0,0)} \frac{m}{1+m} = \frac{m}{1+m}$

Since we have obtained different limits along different paths, the given limit does not exist.

The resulting limit depends on the choice of  $m$  along the path, and so the limit does not exist.

**Method 2:** The Squeeze Theorem does not hold for this limit

**Method 3:** Use the polar coordinate trick

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2+y^2} = \lim_{r \rightarrow 0^+} \frac{(r \cos \theta)(r \sin \theta)}{r^2} = \lim_{r \rightarrow 0^+} \cos \theta \sin \theta = \cos \theta \sin \theta$$

The resulting limit depends on the choice of  $\theta$  along the path, and so the limit does not exist.

**Example 3**

Is  $\lim_{(x,y) \rightarrow (0,0)} \frac{3x^2y}{x^2+y^2}$  exist, or not?

**Answer**

**Method 1:** Use any possible path

Path 1: Along  $x$  –axis  $\rightarrow y = 0 \rightarrow \lim_{(x,0) \rightarrow (0,0)} \frac{0}{x^2} = 0$

Path 2: Along  $y$  –axis  $\rightarrow x = 0 \rightarrow \lim_{(0,y) \rightarrow (0,0)} \frac{0}{y^2} = 0$

Path 3: Along  $y = mx \rightarrow \lim_{(x,mx) \rightarrow (0,0)} \frac{3x^2(mx)}{x^2+(mx)^2} = \lim_{(x,mx) \rightarrow (0,0)} \frac{3mx^3}{x^2+mx^2} = \lim_{(x,mx) \rightarrow (0,0)} \frac{3mx}{1+m} = 0$

This doesn't prove that the given limit is 0 but let us try the limits along the following parabolas.

Path 4: Along  $y = x^2 \rightarrow \lim_{(x,x^2) \rightarrow (0,0)} \frac{3x^2(x^2)}{x^2+(x^2)^2} = \lim_{(x,x^2) \rightarrow (0,0)} \frac{3x^4}{x^2+x^4} = \lim_{(x,x^2) \rightarrow (0,0)} \frac{3x^2}{1+x^2} = 0$

Path 5: Along  $x = y^2 \rightarrow \lim_{(y^2,y) \rightarrow (0,0)} \frac{3(y^2)^2y}{(y^2)^2+y^2} = \lim_{(y^2,y) \rightarrow (0,0)} \frac{3y^5}{y^4+y^2} = \lim_{(y^2,y) \rightarrow (0,0)} \frac{3y^3}{y^2+1} = 0$



The resulting limit also turn out to be 0, so we begin to suspect (means method 1 does not confirm the existence of the limit, then method 2 is required) that the limit does exist and is equal to 0.

### Method 2 (Required):

Not that  $\frac{x^2}{x^2+y^2} \leq 1$  because  $x^2 \leq x^2 + y^2$

$$\lim_{(x,y) \rightarrow (0,0)} -3|y| \leq \lim_{(x,y) \rightarrow (0,0)} \frac{3x^2y}{x^2+y^2} \leq \lim_{(x,y) \rightarrow (0,0)} 3|y|$$

$$\text{since } \lim_{(x,y) \rightarrow (0,0)} 3|y| = \lim_{(x,y) \rightarrow (0,0)} -3|y| = 0$$

So, we conclude by the Squeeze Theorem that

$$\lim_{(x,y) \rightarrow (0,0)} \frac{3x^2y}{x^2+y^2} = 0$$

**Method 3:** Use the polar coordinate trick

$$\lim_{(x,y) \rightarrow (0,0)} \frac{3x^2y}{x^2+y^2} = \lim_{r \rightarrow 0^+} \frac{3(r \cos \theta)^2(r \sin \theta)}{r^2} = \lim_{r \rightarrow 0^+} \frac{3r^3 \cos^2 \theta \sin \theta}{r^2} = \lim_{r \rightarrow 0^+} 3r \cos^2 \theta \sin \theta = 0$$

The resulting limit depends NOT on the choice of  $\theta$  along the path, and so the limit exists.



### NOTE 3.2.2

A function  $f(x, y)$  of two variables is called continuous at  $(a, b)$  if

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = f(a, b)$$

We say  $f(x, y)$  is continuous on  $D$  if  $f(x, y)$  is continuous at every point  $(a, b)$  in  $D$ .

Some Special Cases

1. A polynomial in two variables is continuous everywhere.
2. A rational function is continuous everywhere on its domain.

Recall that if you know that a function is continuous at a point that you're taking a limit to, you can use direct substitution to figure out the limit.

### Example 4

Evaluate  $\lim_{(x,y) \rightarrow (1,2)} (x^2y^3 - x^3y^2 + 3x + 2y)$

### Answer

Since  $f(x, y) = x^2y^3 - x^3y^2 + 3x + 2y$  is a polynomial, it is continuous everywhere, so we can find the limit by direct substitution:

$$\lim_{(x,y) \rightarrow (1,2)} (x^2y^3 - x^3y^2 + 3x + 2y) = 11$$

**Example 5**

Suppose that  $g(x, y) = \begin{cases} \frac{x^2 - y^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$ ,

Is  $\lim_{(x,y) \rightarrow (0,0)} g(x, y)$  exist, or not?

**Answer**

$g(x, y)$  is defined at  $(0, 0)$  but  $g(x, y)$  is still discontinuous there because does not exist (see Example 1 in this section).

**Example 6**

Suppose that  $f(x, y) = \begin{cases} \frac{3x^2y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$ ,

Is  $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$  exist, or not?

**Answer**

We know is continuous for  $(x, y) \neq (0, 0)$  since it is equal to a rational function there. Also, from Example 3 in this section, we have

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = \lim_{(x,y) \rightarrow (0,0)} \frac{3x^2y}{x^2 + y^2} = 0 = f(0, 0)$$

Therefore  $f(x, y)$  is continuous at  $(0, 0)$ , and so it is continuous on  $\mathbb{R}^2$ .

**Section 3.3 Partial Derivatives****NOTE 3.3.1**

➤ **The Definition of a Derivative:**

- $f_x(a, b)$  is the first partial derivative of with respect to  $x$  at  $(a, b)$ .

$$f_x(a, b) = \lim_{h \rightarrow 0} \frac{f(a + h, b) - f(a, b)}{h}$$

- $f_y(a, b)$  is the first partial derivative of with respect to  $y$  at  $(a, b)$ .

$$f_y(a, b) = \lim_{h \rightarrow 0} \frac{f(a, b + h) - f(a, b)}{h}$$

➤ **Notations for Partial Derivatives:** If  $z = f(x, y)$ , we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

➤ **Rule for Finding Partial Derivatives of  $z = f(x, y)$**

To find  $f_x$ , regard  $y$  as a constant and differentiate  $f(x, y)$  with respect to  $x$ .

To find  $f_y$ , regard  $x$  as a constant and differentiate  $f(x, y)$  with respect to  $y$ .

**Example 1**

If  $f(x, y) = x^3 + x^2y^3 - 2y^2$ , find  $f_x(2, 1)$  and  $f_y(2, 1)$

**Answer**

$$f_x(x, y) = 3x^2 + 2xy^3$$

$$f_x(2, 1) = 3(2)^2 + 2(2)(1)^3 = 16$$

$$f_y(x, y) = 3x^2y^2 - 4y$$

$$f_y(2, 1) = 3(2)^2(1)^2 - 4(1) = 8$$

**Example 2**

If  $f(x, y) = \sin\left(\frac{x}{1+y}\right)$ , find  $\frac{\partial f}{\partial x}$  and  $\frac{\partial f}{\partial y}$

**Answer**

Using the Chain Rule for functions of one variable, we have

$$\frac{\partial f}{\partial x} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial x}\left(\frac{x}{1+y}\right) = \cos\left(\frac{x}{1+y}\right) \cdot \frac{1}{1+y}$$

$$\begin{aligned} \frac{\partial f}{\partial y} &= \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial y}\left(\frac{x}{1+y}\right) \\ &= -\cos\left(\frac{x}{1+y}\right) \cdot \frac{x}{(1+y)^2} \end{aligned}$$

**NOTE 3.3.2**

- If the graph  $z = f(x, y)$  is a smooth surface near the point  $P$  with coordinates  $(a, b, f(a, b))$ , then that graph will have a **tangent plane** and a **normal line** at  $P$ .
- The **normal vector** to  $z = f(x, y)$  at  $(a, b, f(a, b))$  is

$$n = f_1(a, b)\mathbf{i} + f_2(a, b)\mathbf{j} - \mathbf{k}$$

- The **equation of tangent plane** to  $z = f(x, y)$  at  $(a, b, f(a, b))$  with  $n = f_1(a, b)\mathbf{i} + f_2(a, b)\mathbf{j} - \mathbf{k}$  is

$$z = f(a, b) + f_1(a, b)(x - a) + f_2(a, b)(y - b).$$

- The **normal line** to  $z = f(x, y)$  at  $(a, b, f(a, b))$  with  $n = f_1(a, b)\mathbf{i} + f_2(a, b)\mathbf{j} - \mathbf{k}$  is

- $\frac{x-a}{f_1(a, b)} = \frac{y-b}{f_2(a, b)} = \frac{z-f(a, b)}{-1}$  is a **symmetric equation form**.

- or  $\begin{cases} x = a + f_1(a, b)t \\ y = b + f_2(a, b)t \\ z = f(a, b) - t \end{cases}$  where  $t \in \mathbb{R}$  is a **parametric equation form**.

- or  $\langle x, y, z \rangle = \langle a + f_1(a, b)t, b + f_2(a, b)t, z = f(a, b) - t \rangle$  is a **vector equation form**.

- The horizontal plane equation which is tangent to surface  $z = f(x, y)$  is

$$z = f(a, b), \text{ it is independent of } x \text{ and } y.$$

- The point of tangency of horizontal plane where  $f_1(a, b) = f_2(a, b) = 0$  is  $(a, b, f(a, b))$ .

**Example 3**

Find the **tangent plane** and **normal line** to the elliptic paraboloid  $z = 2x^2 + y^2$  at the point  $(1,1,3)$ .

**Answer**

Let  $f(x, y) = 2x^2 + y^2$ . Then

$$f_x(x, y) = 4x \quad f_x(1,1) = 4$$

$$f_y(x, y) = 2y \quad f_y(1,1) = 2$$

$$z = f(1,1) + f_1(1,1)(x-1) + f_2(1,1)(y-1)$$

$$z = 3 + 4(x-1) + 2(y-1)$$

$z = 4x + 2y - 3$  is the **equation of the tangent plane** at the point  $(1,1,3)$ .

$$\frac{x-1}{4} = \frac{y-1}{2} = \frac{z-3}{-1} \text{ is the } \mathbf{normal\ line} \text{ to the elliptic}$$

paraboloid  $z = 2x^2 + y^2$  in **symmetric equation form**.

**Example 4**

Find the **horizontal plane** which tangent to the elliptic paraboloid  $z = 2x^2 + y^2 + 2022$  and what is the **point of tangency**?

**Answer**

We have  $f_1(a, b) = f_2(a, b) = 0$  at the point of tangency.

$$f_x(x, y) = 4x \quad \text{as } f_x(x, y) = 0 \quad \therefore x = 0$$

$$f_y(x, y) = 2y \quad \text{as } f_y(x, y) = 0 \quad \therefore y = 0$$

For these values we have  $z = 2022$

The required tangent plane has equation  $z = 2022$  and the point of tangency is  $(0,0,2022)$ .

## NOTE 3.4.1

- If  $z = f(x, y)$ , we use the following notations to represent the **second partial derivatives**:

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

means that we first differentiate with respect  $x$  to and then with respect to  $x$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

means that we first differentiate with respect  $x$  to and then with respect to  $y$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

means that we first differentiate with respect  $y$  to and then with respect to  $x$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

means that we first differentiate with respect  $y$  to and then with respect to  $y$

- Clairaut's Theorem: If the functions  $f_{xy}$  and  $f_{yx}$  are both continuous on  $D$ , that contains the point  $(a, b)$  then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

## Example 1

Find the second partial derivatives of  $f(x, y) = x^3 + x^2y^3 - 2y^2$

## Answer

$$f_x(x, y) = 3x^2 + 2xy^3 \qquad f_y(x, y) = 3x^2y^2 - 4y$$

$$f_{xy}(x, y) = 6xy^2 \qquad f_{yx}(x, y) = 6xy^2$$

$$f_{xx}(x, y) = 6x + 2y^3 \qquad f_{yy}(x, y) = 6x^2y - 4$$

### NOTE 3.4.2

- Partial derivatives of order 3 or higher can also be defined. For instance,

$$f_{xyy} = (f_{xy})_y = \frac{\partial}{\partial y} \left( \frac{\partial^2 f}{\partial y \partial x} \right) = \frac{\partial^3 f}{\partial y^2 \partial x}$$

means that we first differentiate with respect to  $x$  and then differentiate two times with respect to  $y$

- Clairaut's Theorem:  $f_{xyy} = f_{yxy} = f_{yyx}$  if these functions are continuous.

#### Example 2

Calculate  $f_{xxyz}$  if  $f(x, y) = \sin(3x + yz)$ .

#### Answer

$$f_x = 3 \cos(3x + yz)$$

$$f_{xx} = -9 \sin(3x + yz)$$

$$f_{xxy} = -9 \cos(3x + yz)$$

$$f_{xxyz} = -9 \cos(3x + yz) + 9yz \sin(3x + yz)$$

### Section 3.5 The Chain Rule

#### NOTE 3.5.1

**The Chain Rule (Case 1):** Suppose that  $z = f(x, y)$  is a differentiable function of  $x$  and  $y$ , where  $x = g(t)$  and  $y = h(t)$  are both differentiable functions of  $t$ . Then  $z$  is a differentiable function of  $t$  and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

#### Example 1

If  $z = x^2y + 3xy^4$ , where  $x = \sin 2t$  and  $y = \cos t$ , find  $\frac{dz}{dt}$  when  $t = 0$

#### Answer

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = (2xy + 3y^4)(2 \cos 2t) + (x^2 + 12xy^3)(-\sin t)$$

As  $t = 0$ , then  $x = \sin 2(0) = 0$  and,  $y = \cos 0 = 1$ .

$$\left. \frac{dz}{dt} \right|_{t=0} = (2(0)(1) + 3(1)^4)(2 \cos 2(0)) + ((0)^2 + 12(0)(1)^3)(-\sin 0) = 6$$

## NOTE 3.5.2

**The Chain Rule (Case 2):** Suppose that  $z = f(x, y)$  is a differentiable function of  $x$  and  $y$ , where  $x = g(s, t)$  and  $y = h(s, t)$  are both differentiable functions of  $s$  and  $t$ . Then

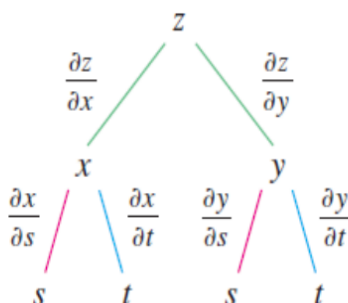
$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t}$$

## Example 2

If  $z = e^x \sin y$ , where  $x = st^2$  and  $y = s^2t$ , find  $\frac{\partial z}{\partial s}$  and  $\frac{\partial z}{\partial t}$ .

## Answer

To remember the Chain Rule, it's helpful to draw the tree diagram in Figure below.



$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} = (e^x \sin y)(t^2) + (e^x \cos y)(2st) = t^2 e^{st^2} \sin(s^2t) + 2ste^{st^2} \cos(s^2t)$$

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} = (e^x \sin y)(2st) + (e^x \cos y)(s^2) = 2ste^{st^2} \sin(s^2t) + s^2 e^{st^2} \cos(s^2t)$$

## Example 3

If  $g(s, t) = f(s^2 - t^2, t^2 - s^2)$  and  $f$  is differentiable, show that  $g$  satisfies the equation

$$t \frac{\partial g}{\partial s} + s \frac{\partial g}{\partial t} = 0$$

## Answer

Let  $x = s^2 - t^2$  and  $y = t^2 - s^2$ . Then  $(s, t) = f(x, y)$  and Chain Rule gives

$$\frac{\partial g}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} = \frac{\partial f}{\partial x} (2s) + \frac{\partial f}{\partial y} (-2s)$$

$$\frac{\partial g}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} = \frac{\partial f}{\partial x} (-2t) + \frac{\partial f}{\partial y} (2t)$$

$$\text{Therefore, } t \frac{\partial g}{\partial s} + s \frac{\partial g}{\partial t} = t \left( \frac{\partial f}{\partial x} (2s) + \frac{\partial f}{\partial y} (-2s) \right) + s \left( \frac{\partial f}{\partial x} (-2t) + \frac{\partial f}{\partial y} (2t) \right) = 0$$





## NOTE 3.6.1

An equation of the **tangent plane** to the graph of a function  $f$  of two variables at the point  $(a, b, f(a, b))$  is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

The linear function whose graph is this tangent plane, namely

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linearization** of  $f$  at and the approximation

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linear approximation** or the **tangent plane approximation** of  $f$  at  $(a, b)$

## Example 1

Show that  $f(x, y) = xe^{xy}$  is differentiable at  $(1, 0)$  and find its linearization there. Then use it to approximate  $f(1.1, -0.1)$

## Answer

The partial derivatives are

$$f_x(x, y) = e^{xy} + xye^{xy} \quad f_x(1, 0) = 1$$

$$f_y(x, y) = x^2e^{xy} \quad f_y(1, 0) = 1$$

Both  $f_x$  and  $f_y$  are continuous functions, so  $f(x, y)$  is differentiable. The linearization is

$$\begin{aligned} L(x, y) &= f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0) \\ &= 1 + 1(x - 1) + 1 \cdot y = x + y \end{aligned}$$

The corresponding linear approximation is  $xe^{xy} \approx x + y$

$$\text{So } f(1.1, -0.1) = 1.1 - 0.1 = 1$$

(Optional) Using calculator, compare this with the actual value of  $f(1.1, -0.1) = 1.1e^{-0.11} \approx 0.98542$ .

## NOTE 3.7.1

- To begin, it is useful to combine the first partial derivatives of a function into single *vector function* called a **gradient**.
  - We define the **gradient vector**  $\nabla f(x, y) = \mathbf{grad} f(x, y) = f_1(x, y)\mathbf{i} + f_2(x, y)\mathbf{j}$ .
  - If  $f(x, y)$  is differentiable at  $(a, b)$  and  $\nabla f(a, b) \neq \mathbf{0}$ , then  $\nabla f(a, b)$  is a **normal vector** to the level curve of  $f$  that passes through  $(a, b)$ .
- 
- Let  $\mathbf{u} = u\mathbf{i} + v\mathbf{j}$  be a unit vector, so that  $u^2 + v^2 = 1$ . The **directional derivative** of  $f(x, y)$  at  $(a, b)$  in the direction of  $\mathbf{u}$  is

$$D_{\mathbf{u}}f(a, b) = \lim_{h \rightarrow 0^+} \frac{f(a + hu, b + hv) - f(a, b)}{h}$$

- **Using the gradient to find directional derivatives**

If  $f$  is differentiable at  $(a, b)$  and  $\mathbf{u} = u\mathbf{i} + v\mathbf{j}$  is a unit vector, then the directional derivative of  $f$  at  $(a, b)$  in the direction of  $\mathbf{u}$  is given by

$$D_{\mathbf{u}}f(a, b) = \mathbf{u} \cdot \nabla f(a, b)$$

- Recall: In general, if  $\mathbf{v} \neq \mathbf{0}$ , then the **unit vector** that has the same direction as  $\mathbf{v}$  is

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} \text{ or } u = \frac{v}{\|\mathbf{v}\|}$$

## Example 1

Find the directional derivative of the function  $f(x, y) = x^2y^3 - 4y$  at the point  $(2, -1)$  in the direction of the vector  $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$ .

## Answer

We first compute the gradient vector at  $(2, -1)$ :

$$\nabla f(x, y) = 2xy^3\mathbf{i} + (3x^2y^2 - 4)\mathbf{j}$$

$$\nabla f(2, -1) = -4\mathbf{i} + 8\mathbf{j}$$

Note that  $\mathbf{v}$  is not a unit vector, but since  $\|\mathbf{v}\| = \sqrt{29}$ , the unit vector in the direction of  $\mathbf{v}$  is

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{2}{\sqrt{29}}\mathbf{i} + \frac{5}{\sqrt{29}}\mathbf{j}$$

$$D_{\mathbf{u}}f(2, -1) = \mathbf{u} \cdot \nabla f(2, -1) = \left( \frac{2}{\sqrt{29}}\mathbf{i} + \frac{5}{\sqrt{29}}\mathbf{j} \right) \cdot (-4\mathbf{i} + 8\mathbf{j}) = \frac{-4 \cdot 2 + 8 \cdot 5}{\sqrt{29}} = \frac{32}{\sqrt{29}}$$

## NOTE 3.7.2

**Geometric properties of the gradient vector**

- At  $(a, b)$ ,  $f(x, y)$  **increases** most rapidly in the direction of the gradient vector  $\nabla f(a, b)$ . The maximum rate of **increase** is  $|\nabla f(a, b)|$ .
- At  $(a, b)$ ,  $f(x, y)$  **decreases** most rapidly in the direction of the gradient vector  $-\nabla f(a, b)$ . The maximum rate of **decrease** is  $|\nabla f(a, b)|$ .
- The **rate of change** of  $f(x, y)$  at  $(a, b)$  is zero in directions tangent to the level curve of  $f$  that passes through  $(a, b)$ .

**Example 2**

- (a) If  $f(x, y) = xe^y$ , find the rate of change of  $f$  at the point  $P(2, 0)$  in the direction from  $P(2, 0)$  to  $Q\left(\frac{1}{2}, 2\right)$ .
- (b) In what direction does  $f$  have the maximum rate of change? What is this maximum rate of change?

**Answer**

- (a) We first compute the gradient vector:  $\nabla f(x, y) = \langle e^y, xe^y \rangle$        $\nabla f(2, 0) = \langle 1, 2 \rangle$

The unit vector in the direction of  $\overrightarrow{PQ} = \langle -1.5, 2 \rangle$  is  $\mathbf{u} = \left\langle -\frac{3}{5}, \frac{4}{5} \right\rangle$

so, the rate of change of in the direction from  $P$  to  $Q$  is

$$D_{\mathbf{u}}f(2, 0) = \mathbf{u} \cdot \nabla f(2, 0) = \left\langle -\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j} \right\rangle \cdot \langle 1, 2 \rangle = 1\left(-\frac{3}{5}\right) + 2\left(\frac{4}{5}\right) = 1$$

- (b)  $f$  increases fastest in the direction of the gradient vector  $\nabla f(2, 0) = \langle 1, 2 \rangle$ . The maximum rate of change is

$$|\nabla f(2, 0)| = |\langle 1, 2 \rangle| = \sqrt{5}$$

**Example 3**

Let  $f(x, y, z) = x^2 + y^2 + z^2$ .

- (a) Find  $\nabla f(x, y, z)$  and  $\nabla f(1, -1, 2)$ .
- (b) Find an equation of the tangent plane to the sphere  $x^2 + y^2 + z^2 = 6$  at the point  $(1, -1, 2)$ .
- (c) What is the maximum rate of increase of  $f$  at  $(1, -1, 2)$ ?
- (d) What is the rate of change with respect to distance of  $f$  at  $(1, -1, 2)$  measured in the direction from that point toward the point  $c$ ?

## Answer

(a) Find  $\nabla f(x, y, z) = 2x \mathbf{i} + 2y \mathbf{j} + 2z \mathbf{k} = \langle 2x, 2y, 2z \rangle$  so  $\nabla f(1, -1, 2) = 2 \mathbf{i} - 2 \mathbf{j} + 4 \mathbf{k} = \langle 2, -2, 4 \rangle$ .

(b) The normal to tangent plane at the point  $(1, -1, 2)$  is  $\nabla f(1, -1, 2) = 2 \mathbf{i} - 2 \mathbf{j} + 4 \mathbf{k}$

Therefore, its equation is given by  $2(x - 1) - 2(y + 1) + 4(z - 2) = 0$  or, more simply,  $x - y + 2z = 6$ .

(c) The maximum rate of increase of  $f$  at the point  $(1, -1, 2)$  is

$$|\nabla f(1, -1, 2)| = |\langle 2, -2, 4 \rangle| = 2\sqrt{6}$$

(d) The direction from  $(1, -1, 2)$  toward  $(3, 1, 1)$  is specified by  $2 \mathbf{i} + 2 \mathbf{j} - \mathbf{k}$

The rate of change of  $f$  with respect to distance in this direction is

$$D_u f(1, -1, 2) = u \cdot \nabla f(1, -1, 2) = \left( \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{\sqrt{4 + 4 + 1}} \right) \cdot (2 \mathbf{i} - 2 \mathbf{j} + 4 \mathbf{k}) = \frac{4 - 4 - 4}{3} = -\frac{4}{3}$$

That is,  $f$  decreases at rate  $\frac{4}{3}$  of a unit per horizontal unit moved.

## Section 3.8

## Implicit Functions



## NOTE 3.8.1

- We suppose that  $y$  is given implicitly as a function  $y = f(x)$  by an equation of the form  $F(x, y) = 0$

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

- We suppose that  $z$  is given implicitly as a function  $z = f(x, y)$  by an equation of the form  $F(x, y, z) = 0$

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{F_x}{F_z} \quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{F_y}{F_z}$$

**Example 1**

Find  $y' = \frac{dy}{dx}$  if  $x^3 + y^3 = 6xy$

**Answer**

The given equation can be written as  $F(x, y) = x^3 + y^3 - 6xy = 0$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2 - 6y}{3y^2 - 6x} = -\frac{x^2 - 2y}{y^2 - 2x}$$

**Example 2**

Find  $\frac{\partial z}{\partial x}$  and  $\frac{\partial z}{\partial y}$  if  $x^3 + y^3 + z^3 + 6xyz = 1$

**Answer**

Let  $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + 6yz}{3z^2 + 6xy} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{3y^2 + 6xz}{3z^2 + 6xy} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$

**Section 4.1****Extreme Values****NOTE 4.1.1**

**Method I: Second Derivatives Test:** Suppose the second partial derivatives of  $f$  are continuous on a disk with center  $(a, b)$ , and suppose that  $f_x(a, b) = 0$  and  $f_y(a, b) = 0$  [that is,  $(a, b)$  is a **critical point** of  $f$ ]. Let

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

If  $D > 0$  and  $f_{xx}(a, b) > 0$ , then  $f(a, b)$  has a local **minimum** at  $(a, b)$ .

If  $D > 0$  and  $f_{xx}(a, b) < 0$ , then  $f(a, b)$  has a local **maximum** at  $(a, b)$ .

If  $D < 0$ , then  $f(a, b)$  has a **saddle point** at  $(a, b)$ .

If  $D = 0$ , then this test provides **no information**.

To remember the formula for  $D$ , it's helpful to write it as a determinant:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2$$

### NOTE 4.1.2

**Method II: Second Derivatives Test:** Suppose the second partial derivatives of  $f$  are continuous on a disk with center  $(a, b)$ , and suppose that  $f_x(a, b) = 0$  and  $f_y(a, b) = 0$  [that is,  $(a, b)$  is a **critical point** of  $f$ ]. Let

$$A = f_{11}(a, b), \quad B = f_{12}(a, b) = f_{21}(a, b), \text{ and } C = f_{22}(a, b)$$

If  $B^2 - AC < 0$  and  $A > 0$ , then  $f(a, b)$  has a local **minimum** at  $(a, b)$ .

If  $B^2 - AC < 0$  and  $A < 0$ , then  $f(a, b)$  has a local **maximum** at  $(a, b)$ .

If  $B^2 - AC > 0$ , then  $f(a, b)$  has a **saddle point** at  $(a, b)$ .

If  $B^2 - AC = 0$ , then this test provides **no information**.

#### Example 1

Find the local maximum and minimum values and saddle points of  $f(x, y) = x^4 + y^4 - 4xy + 1$ .

#### Answer

$$f_x = 4x^3 - 4y \quad \text{and} \quad f_y = 4y^3 - 4x$$

Setting these partial derivatives equal to 0, we obtain the equations

$$x^3 - y = 0 \quad \text{and} \quad y^3 - x = 0$$

To solve these equations, we substitute  $y = x^3$  from the first equation into the second one. This gives

$$x^9 - x = 0 \quad x(x^8 - 1) = x(x^4 - 1)(x^4 + 1) = x(x^2 - 1)(x^2 + 1)(x^4 + 1) = 0$$

so, there are three real roots:  $x = 0, 1, -1$ . The three critical points are  $(0, 0)$ ,  $(1, 1)$ , and  $(-1, -1)$

For critical points classification, you can use method I (see note: 4.1.1) or method II (see note: 4.1.2), in this question, method I is used.

$$f_{xx} = 12x^2 \quad f_{xy} = f_{yx} = -4 \quad f_{yy} = 12y^2$$

$$D = f_{xx}f_{yy} - f_{xy}^2 = 144x^2y^2 - 16$$

For  $(0, 0)$ , we have  $D(0, 0) = -16 < 0 \therefore (0, 0)$  is a saddle point by the Second Derivatives Test.

For  $(1, 1)$ , we have  $D(1, 1) = 128 > 0$  and  $f_{xx}(1, 1) = 12 > 0$

$\therefore (1, 1)$  is a minimum local point and its value is  $f(1, 1) = -1$  by the Second Derivatives Test.

For  $(-1, -1)$ , we have  $D(-1, -1) = 128 > 0$  and  $f_{xx}(-1, -1) = 12 > 0$

$\therefore (-1, -1)$  is a minimum local point and its value is  $f(-1, -1) = -1$  by the Second Derivatives Test.

**Example 2**

Find the shortest distance from the point  $(1, 0, -2)$  to the plane  $x + 2y + z = 4$

**Answer**

The distance from any point  $(x, y, z)$  to the point  $(1, 0, -2)$  is

$$d = \sqrt{(x-1)^2 + y^2 + (z+2)^2}$$

but if  $(x, y, z)$  lies on the plane  $x + 2y + z = 4$ , then  $z = 4 - x - 2y$  and so we have

$$d = \sqrt{(x-1)^2 + y^2 + (6-x-2y)^2}$$

We can minimize by minimizing the simpler expression

$$d^2 = f(x, y) = (x-1)^2 + y^2 + (6-x-2y)^2$$

$$f_x = 2(x-1) - 2(6-x-2y) = 4x + 4y - 14 \quad \text{and} \quad f_y = 2y - 4(6-x-2y) = 4x + 10y - 24$$

Setting these partial derivatives equal to 0, we obtain the equations

$$4x + 4y - 14 = 0 \quad \text{and} \quad 4x + 10y - 24 = 0$$

We find that the only critical point is  $\left(\frac{11}{6}, \frac{5}{3}\right)$

$$f_{xx} = 4 > 0 \quad f_{xy} = f_{yx} = 4 \quad f_{yy} = 10$$

$$D = f_{xx}f_{yy} - f_{xy}^2 = 24 > 0$$

For  $\left(\frac{11}{6}, \frac{5}{3}\right)$ , we have  $D\left(\frac{11}{6}, \frac{5}{3}\right) = 24 > 0$  and  $f_{xx}\left(\frac{11}{6}, \frac{5}{3}\right) = 4 > 0$

$\therefore \left(\frac{11}{6}, \frac{5}{3}\right)$  is a minimum local point by the Second Derivatives Test.

Intuitively, we can see that this local minimum is an absolute minimum because there must be a point on the given plane that is closest to  $(1, 0, -2)$ .

$$d = \sqrt{(x-1)^2 + y^2 + (6-x-2y)^2} = \sqrt{\left(\frac{5}{6}\right)^2 + \left(\frac{5}{3}\right)^2 + \left(\frac{5}{6}\right)^2} = \frac{5}{6}\sqrt{6}$$

Therefore, the shortest distance from the point  $(1, 0, -2)$  to the plane  $x + 2y + z = 4$  is  $\frac{5}{6}\sqrt{6}$



## NOTE 4.2.1

## ➤ Necessary conditions for extreme values

A function  $f(x, y)$  can have a local or absolute extreme value at a point  $(a, b)$  in its domain only if  $(a, b)$  is one of the following:

- (a) a **critical point** of  $f$ , that is, a point satisfying  $\nabla f(a, b) = 0$  i.e.,  $f_x(a, b) = f_y(a, b) = 0$
- (b) a **singular point** of  $f$ , that is, a point where  $\nabla f(a, b)$  does not exist
- (c) a **boundary point** of the domain of  $f$ .

## ➤ Absolute Maximum and Minimum Values

Extreme Value Theorem for Functions of Two Variables: If  $f$  is continuous on a closed, bounded set  $D$  in  $\mathbb{R}^2$ , then  $f$  attains an absolute maximum value  $f(x_1, y_1)$  and an absolute minimum value  $f(x_2, y_2)$  at some points  $(x_1, y_1)$  and  $(x_2, y_2)$  in  $D$ .

## NOTE 4.2.2

To find the absolute maximum and minimum values of a continuous function  $f$  on a closed, bounded set  $D$ :

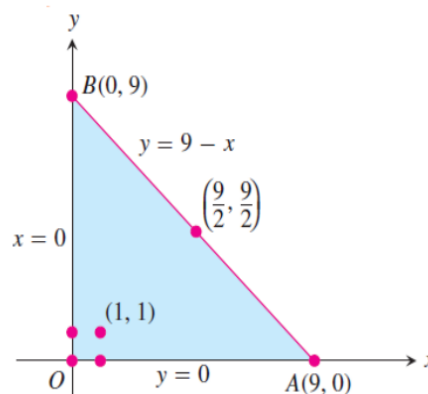
1. Find the values of  $f$  at the critical points of  $f$  in  $D$ .
2. Find the extreme values of  $f$  on the boundary of  $D$ .
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

## Example 1

Find the absolute maximum and minimum values of

$$f(x, y) = 2 + 2x + 2y - x^2 - y^2$$

on the triangular region in the first quadrant bounded by the lines  $x = 0$ ,  $y = 0$ ,  $y = 9 - x$ .



Answer

(a) **Interior points.** For these we have

$$f_x = 2 - 2x = 0, \quad f_y = 2 - 2y = 0$$

yielding the single point  $(x, y) = (1, 1)$ . The value of  $f$  there is  $f(1, 1) = 4$

(b) **Boundary points.** For these we have

(i) On the segment OA,  $y = 0$

$$\text{The function } f(x, y) = f(x, 0) = 2 + 2x - x^2$$

$$f'(x, 0) = 2 - 2x = 0 \quad \text{Then, } x = 1 \therefore f(1, 0) = 3$$

**End points** for the segment OA are  $f(0, 0) = 2$  and  $f(9, 0) = -61$

(ii) On the segment OB,  $x = 0$

$$\text{The function } f(x, y) = f(0, y) = 2 + 2y - y^2$$

$$f'(0, y) = 2 - 2y = 0 \quad \text{Then, } y = 1 \therefore f(0, 1) = 3$$

**End points** for the segment OB are  $f(0, 0) = 2$  and  $f(0, 9) = -61$

(iii) On the segment AB,  $y = 9 - x$

$$\text{The function } f(x, y) = f(x, 9 - x) = 2 + 2x + 2(9 - x) - x^2 - (9 - x)^2 = -61 + 18x - 2x^2$$

$$f'(x, 9 - x) = 18 - 4x = 0 \quad \text{Then, } x = \frac{9}{2} \therefore f\left(\frac{9}{2}, \frac{9}{2}\right) = -\frac{41}{2}$$

**End points** for the segment AB are  $f(9, 0) = -61$  and  $f(0, 9) = -61$

Therefore, the maximum is 4, which  $f$  assumes at  $(1, 1)$ . The minimum is  $-61$ , which assumes at  $(0, 9)$  and  $(9, 0)$ .

## NOTE 4.3.1

**Method I of Lagrange Multipliers ( $\lambda$ )**

To find the maximum and minimum values of  $f(x, y, z)$  subject to the constraint  $g(x, y, z) = k$  [assuming that these extreme values exist and  $\nabla g \neq 0$  on the surface  $g(x, y, z) = k$ ]:

(a) Find all values of  $x, y, z$  and  $\lambda$  such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \quad \text{i.e., 3 equations}$$

and  $g(x, y, z) = k$  i.e., 1 constraint equation

(b) Evaluate  $f$  at all the points  $(x, y, z)$  that result from step (a). The largest of these values is the maximum value of  $f$ ; the smallest is the minimum value of  $f$ .

- 
- Recall:  $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$  is  $f_x = \lambda g_x$        $f_y = \lambda g_y$        $f_z = \lambda g_z$
  - BEWARE: This is a system of four equations in the four unknowns  $x, y, z$  and  $\lambda$ , but it is not necessary to find explicit values for  $\lambda$ .

## NOTE 4.3.2

**Method II of Lagrange Multipliers ( $\lambda$ )**

To find the maximum and minimum values of  $f(x, y, z)$  subject to the constraint  $g(x, y, z) = k$  [assuming that these extreme values exist and  $\nabla g \neq 0$  on the surface  $g(x, y, z) = k$ ]:

(a) Find all values of  $x, y, z$  and  $\lambda$  such that

At any critical point of  $L$  we must have

$$L(x, y, z, \lambda) = f(x, y, z) + \lambda g(x, y, z)$$

$$\text{That is, } \nabla f \text{ is parallel } \nabla g, \quad \begin{cases} 0 = \frac{\partial L}{\partial x} = f_1(x, y, z) + \lambda g_1(x, y, z) \\ 0 = \frac{\partial L}{\partial y} = f_2(x, y, z) + \lambda g_2(x, y, z) \\ 0 = \frac{\partial L}{\partial z} = f_3(x, y, z) + \lambda g_3(x, y, z) \end{cases}$$

$$0 = \frac{\partial L}{\partial \lambda} = g(x, y, z)$$

(b) Evaluate  $f$  at all the points  $(x, y, z)$  that result from step (a). The largest of these values is the maximum value of  $f$ ; the smallest is the minimum value of  $f$ .

**Example 1**

Find the shortest distance from the origin to the curve  $x^2y = 16$

**Answer**

We want to solve the problem

$$\text{minimize } f(x, y) = x^2 + y^2 \quad \text{subject to} \quad g(x, y) = x^2y - 16 = 0$$

**Method I of Lagrange Multipliers**

As  $\nabla f(x, y) = \lambda \nabla g(x, y)$ , then we have

$$2x = 2\lambda xy \quad (A)$$

$$2y = \lambda x^2 \quad (B)$$

$$x^2y = 16 \quad (C)$$

Then dividing the (A) by (B), we have  $\frac{x}{y} = \frac{2y}{x}$  that is  $x^2 = 2y^2$

Plug  $x^2 = 2y^2$  in equation (C). Then, we have  $(2y^2)y - 16 = 0$   $y^3 = 8$  so,  $y = 2$

Thus,  $x^2 = 8$   $x = \pm 2\sqrt{2}$ .

Therefore, there are two candidates for points on  $x^2y = 16$  closest to the origin,  $(\pm 2\sqrt{2}, 2)$ .

Both of these points are at distance  $\sqrt{8 + 4} = 2\sqrt{3}$  units from the origin, so this must be the minimum distance from the origin to the curve.

**Method II of Lagrange Multipliers**

Let  $L(x, y, \lambda) = x^2 + y^2 + \lambda(x^2y - 16)$

$$0 = \frac{\partial L}{\partial x} = 2x + 2\lambda xy \quad (A)$$

$$0 = \frac{\partial L}{\partial y} = 2y + \lambda x^2 \quad (B)$$

$$0 = \frac{\partial L}{\partial \lambda} = x^2y - 16 \quad (C)$$

By rewriting equation (A), we have  $2x = -2\lambda xy$  (D)

By rewriting equation (B), we have  $2y = -\lambda x^2$  (E)

Then dividing the (D) by (E), we have  $\frac{x}{y} = \frac{2y}{x}$  that is  $x^2 = 2y^2$

**Example 2**

Find the maximum and minimum values of  $f(x, y, z) = xyz$  on the sphere  $x^2 + y^2 + z^2 = 12$ .

**Answer**

We want to solve the problem

$$\text{minimize } f(x, y, z) = xyz \quad \text{subject to} \quad g(x, y, z) = x^2 + y^2 + z^2 - 12 = 0$$

**Method I of Lagrange Multipliers**

As  $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ , then we have

$$yz = 2\lambda x \quad (A)$$

$$xz = 2\lambda y \quad (B)$$

$$xy = 2\lambda z \quad (C)$$

$$x^2 + y^2 + z^2 = 12 \quad (D)$$

Then dividing the (A) by (B), we have  $\frac{y}{x} = \frac{x}{y}$  that is  $x^2 = y^2$

Then dividing the (B) by (C), we have  $\frac{z}{y} = \frac{y}{z}$  that is  $z^2 = y^2$

Thus,  $x^2 = y^2 = z^2$

Plug  $x^2 = y^2 = z^2$  in equation (D). Then, we have  $3x^2 = 12$   $x^2 = 4$  so,  $x = \pm 2$

Thus,  $x = \pm 2$  or  $y = \pm 2$  or  $z = \pm 2$

We obtain eight distinct points, and they are classified as following:

At four of these points,  $(2, 2, 2)$ ,  $(-2, -2, 2)$ ,  $(-2, 2, -2)$ , and  $(2, -2, -2)$ , we have  $xyz = 8$

At four of these points,  $(-2, -2, -2)$ ,  $(-2, 2, 2)$ ,  $(2, -2, 2)$ , and  $(2, 2, -2)$ , we have  $xyz = -8$

**Method II of Lagrange Multipliers**

Let  $L(x, y, z, \lambda) = xyz + \lambda(x^2 + y^2 + z^2 - 12)$

$$0 = \frac{\partial L}{\partial x} = yz + 2\lambda x \quad (A)$$

$$0 = \frac{\partial L}{\partial y} = xz + 2\lambda y \quad (B)$$

$$0 = \frac{\partial L}{\partial z} = xy + 2\lambda z \quad (C)$$

$$0 = \frac{\partial L}{\partial \lambda} = x^2 + y^2 + z^2 - 12 \quad (D)$$

By rewriting equation (A), we have  $yz = -2\lambda x$  (D)

By rewriting equation (B), we have  $xz = -2\lambda y$  (E)

By rewriting equation (C), we have  $xy = -2\lambda z$  (F)

Then dividing the (D) by (E), we have  $\frac{y}{x} = \frac{x}{y}$  that is  $x^2 = y^2$

Then dividing the (E) by (F), we have  $\frac{z}{y} = \frac{y}{z}$  that is  $z^2 = y^2$

Thus,  $x^2 = y^2 = z^2$

**Example 3**

Find the minimum value of  $f(x, y, z) = y$  subject to the constraint equation  $g(x, y) = y^3 - x^2 = 0$ .

**Answer**

We want to solve the problem

**minimize**  $f(x, y) = y$  **subject to**  $g(x, y) = y^3 - x^2 = 0$

**Method I of Lagrange Multipliers**

As  $\nabla f(x, y) = \lambda \nabla g(x, y)$ , then we have

$$0 = -2\lambda x \quad (A)$$

$$1 = 3\lambda y^2 \quad (B)$$

$$y^3 - x^2 = 0 \quad (C)$$

**Method II of Lagrange Multipliers**

Let  $L(x, y, \lambda) = y + \lambda(y^3 - x^2)$

$$0 = \frac{\partial L}{\partial x} = -2\lambda x \quad (A)$$

$$0 = \frac{\partial L}{\partial y} = 1 + 3\lambda y^2 \quad (B)$$

$$0 = \frac{\partial L}{\partial \lambda} = y^3 - x^2 \quad (C)$$

From equation (A), we obtain that either  $\lambda = 0$  or  $x = 0$ . Although, as  $x = 0$  from equation (A), we obtained from equation (C) that  $y = 0$ . Such solution of  $x$  and  $y$  cannot satisfy the equation (B)

Therefore, the three equations have no solution  $(x, y, \lambda)$

The method of Lagrange multipliers breaks down because  $\nabla g = 0$  at the solution point  $(0, 0)$  where the curve  $g(x, y)$  is not smooth at the origin.

**Section 5.1 Double Integrals****NOTE 5.1.1**

If  $f(x, y) \geq 0$ , then the volume  $V$  of the solid that lies above the rectangle  $R$  and below the surface

$Z = f(x, y)$  is

$$V = \iint_R f(x, y) dA = \int_c^d \int_a^b f(x, y) dx dy = \int_a^b \int_c^d f(x, y) dy dx$$

Where  $R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 | a \leq x \leq b, c \leq y \leq d\}$

Where  $dA = dx dy$  or  $dA = dy dx$

## NOTE 5.1.2

## Main Properties of Double Integrals

- $\iint_R [f(x, y) + g(x, y)] dA = \iint_R f(x, y) dA + \iint_R g(x, y) dA$
- $\iint_R c f(x, y) dA = c \iint_R f(x, y) dA$  where  $c$  is a constant
- If  $f(x, y) \geq g(x, y)$  for all  $(x, y)$  in  $R$ , then

$$\iint_R f(x, y) dA \geq \iint_R g(x, y) dA$$

$\iint_R dA$  is the **area** of a closed, bounded plane region  $R$ , where  $f(x, y) = 1$

## Example 1

Evaluate  $\int_0^{\pi/2} \int_0^1 y \sin x \, dy \, dx$

## Answer

$$\begin{aligned} & \int_0^{\pi/2} \int_0^1 y \sin x \, dy \, dx \\ &= \int_0^{\pi/2} \left( \underbrace{\int_0^1 y \sin x \, dy}_{\text{work out treating } x \text{ as constant}} \right) dx \\ &= \int_0^{\pi/2} \left[ \frac{y^2}{2} \sin x \right]_{y=0}^{y=1} dx = \int_0^{\pi/2} \frac{1}{2} \sin x \, dx = \left[ -\frac{1}{2} \cos x \right]_0^{\pi/2} \\ &= \frac{1}{2} \end{aligned}$$

## Example 2

Find the volume of the solid bounded above by the plane  $z = 4 - x - y$  and below by the rectangle

$$R = \{(x, y): 0 \leq x \leq 1, 0 \leq y \leq 2\}$$

## Answer

The volume under any surface  $z = f(x, y)$  and above a region  $R$  is given by

$$V = \iint_R f(x, y) \, dx \, dy$$

In our case

$$\begin{aligned} V &= \int_0^2 \int_0^1 (4 - x - y) \, dx \, dy \\ &= \int_0^2 \left[ 4x - \frac{1}{2}x^2 - yx \right]_{x=0}^{x=1} dy = \int_0^2 \left( \frac{7}{2} - y \right) dy \\ &= \left[ \frac{7}{2}y - \frac{y^2}{2} \right]_0^2 = (7 - 2) - (0) = 5 \end{aligned}$$



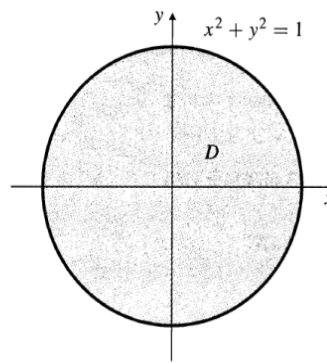
**Example 3**

Evaluate  $\iint_{x^2+y^2 \leq 1} 2022 \, dA$

**Answer**

The domain of the integration is a circular disk of radius 1 centered at the origin.

$$\iint_{x^2+y^2 \leq 1} 2022 \, dA = 2022 \times \text{area of } D = 2022\pi$$

**Example 3**

Evaluate  $\iint_{x^2+y^2 \leq 1} \sqrt{1-x^2-y^2} \, dA$

**Answer**

This integral represents the volume of a hemisphere of radius 1 and so has the value  $2\pi/3$ .

## Section 5.2 Iteration of Double Integrals in Cartesian Coordinates

### NOTE 5.2.1

**Iteration** is the process of reducing the problem of evaluating a double (or multiple) integral to one of evaluating two (or more) successive single definite integral.

➤ Let  $f(x, y)$  be continuous on a region  $R$ .

1. If  $R$  is defined by  $a \leq x \leq b$ ,  $g_1(x) \leq y \leq g_2(x)$  with  $g_1$  and  $g_2$  continuous on  $[a, b]$ , then

$$\iint_R f(x, y) \, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \, dx$$

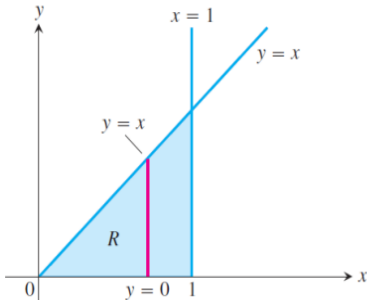
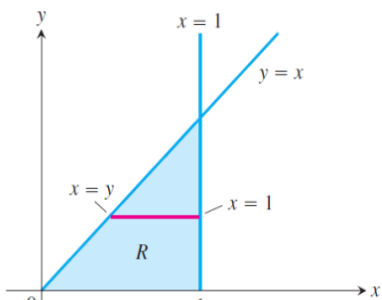
2. If  $R$  is defined by  $c \leq y \leq d$ ,  $h_1(y) \leq x \leq h_2(y)$  with  $h_1$  and  $h_2$  continuous on  $[c, d]$ , then

$$\iint_R f(x, y) \, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) \, dx \, dy$$

**Example 1**

Find the volume of the prism whose base is the triangle in the  $xy$  –plane bounded by the  $x$  –axis and the lines  $y = x$  and  $x = 1$  and whose top lies in the plane  $z = f(x, y) = 3 - x - y$

**Answer**

| Using Vertical Cross-section ( $dydx$ )                                                                                                                                                                                         | Using Horizontal Cross-section ( $dx dy$ )                                                                                                                                                                                                                                                                                       |
|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
|                                                                                                                                                |                                                                                                                                                                                                                                                |
| <p>The <math>y</math> –limits of integration<br/> <math>0 \leq y \leq x</math><br/> The <math>x</math> –limits of integration<br/> <math>0 \leq x \leq 1</math></p>                                                             | <p>The <math>x</math> –limits of integration<br/> <math>y \leq x \leq 1</math><br/> The <math>y</math> –limits of integration<br/> <math>0 \leq y \leq 1</math></p>                                                                                                                                                              |
| $V = \int_0^1 \int_0^x (3 - x - y) dy dx$ $= \int_0^1 \left[ 3y - xy - \frac{y^2}{2} \right]_{y=0}^{y=x} dx = \int_0^1 \left( 3x - \frac{3x^2}{2} \right) dx$ $= \left[ \frac{3x^2}{2} - \frac{x^3}{2} \right]_{x=0}^{x=1} = 1$ | $V = \int_0^1 \int_y^1 (3 - x - y) dx dy$ $= \int_0^1 \left[ 3x - \frac{x^2}{2} - xy \right]_{x=y}^{x=1} dy$ $= \int_0^1 \left( 3 - \frac{1}{2} - y - 3y + \frac{y^2}{2} + y^2 \right) dy$ $= \int_0^1 \left( \frac{5}{2} - 4y + \frac{3}{2}y^2 \right) dy = \left[ \frac{5}{2}y - 2y^2 - \frac{y^3}{2} \right]_{y=0}^{y=1} = 1$ |

**Example 2**

Find the volume of the solid in the first octant bounded by the region above line  $y = 1 - x$  and inside the circle with radius 1 and whose top lies in the plane  $z = f(x, y)$

**Answer**

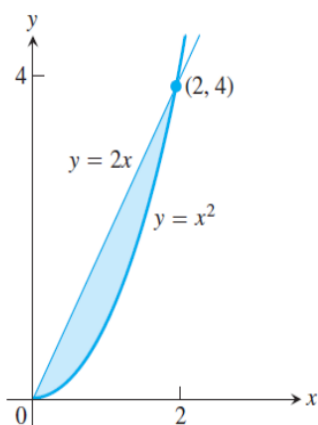
| Using Vertical Cross-section ( $dydx$ )                                                                                                                                              | Using Horizontal Cross-section ( $dx dy$ )                                                                                                                                           |
|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <p>Leaves at <math>y = \sqrt{1-x^2}</math><br/>Enters at <math>y = 1-x</math><br/>Smallest <math>x</math> is <math>x = 0</math><br/>Largest <math>x</math> is <math>x = 1</math></p> | <p>Largest <math>y</math> is <math>y = 1</math><br/>Enters at <math>x = 1-y</math><br/>Smallest <math>y</math> is <math>y = 0</math><br/>Leaves at <math>x = \sqrt{1-y^2}</math></p> |
| <p>The <math>y</math> –limits of integration<br/><math>1-x \leq y \leq \sqrt{1-x^2}</math><br/>The <math>x</math> –limits of integration<br/><math>0 \leq x \leq 1</math></p>        | <p>The <math>x</math> –limits of integration<br/><math>1-y \leq x \leq \sqrt{1-y^2}</math><br/>The <math>y</math> –limits of integration<br/><math>0 \leq y \leq 1</math></p>        |
| $V = \int_0^1 \int_{1-x}^{\sqrt{1-x^2}} f(x, y) dy dx$                                                                                                                               | $V = \int_0^1 \int_{1-y}^{\sqrt{1-y^2}} f(x, y) dx dy$                                                                                                                               |

**Example 3**

Sketch the region of integration for the integral

$$\int_0^2 \int_{x^2}^{2x} (4x + 2) dy dx$$

and write an equivalent integral with the order of integration reversed [DO NOT EVALUTE THE INTEGRAL].

**Answer**

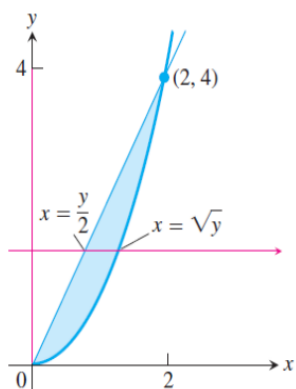
The  $x$  –limits of integration

$$\frac{y}{2} \leq x \leq \sqrt{y}$$

The  $y$  –limits of integration

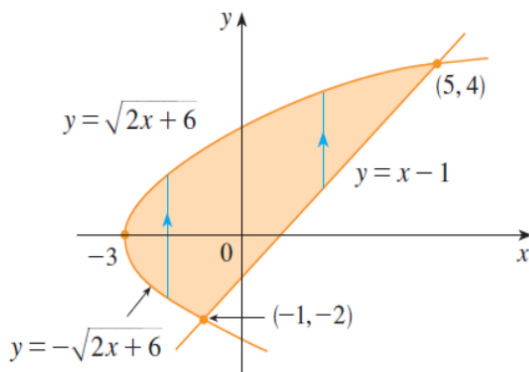
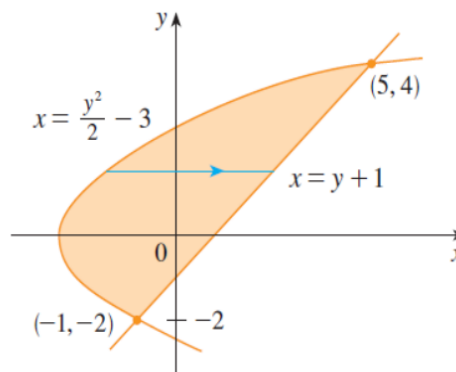
$$0 \leq y \leq 4$$

$$V = \int_0^4 \int_{\frac{y}{2}}^{\sqrt{y}} (4x + 2) dx dy$$



**Example 4**

Evaluate  $\iint_D xy \, dA$ , where  $D$  is the region bounded by the line  $y = x - 1$  and the parabola  $y^2 = 2x + 6$ .

**Answer**(a)  $D$  as a type I region(b)  $D$  as a type II region

$D$  is both type I and type II, but the description of  $D$  as a type I region is more complicated because the lower boundary consists of two parts. Therefore, we prefer to express  $D$  as a type II region:

**Type II region:**

The  $x$  –limits of integration

$$\frac{y^2}{2} - 3 \leq x \leq y + 1$$

The  $y$  –limits of integration

$$-2 \leq y \leq 4$$

$$\begin{aligned} \text{Therefore, } \iint_D xy \, dA &= \int_{-2}^4 \int_{\frac{y^2}{2}-3}^{y+1} xy \, dx \, dy = \int_{-2}^4 \left[ \frac{x^2}{2} y \right]_{x=\frac{y^2}{2}-3}^{x=y+1} dy = \frac{1}{2} \int_{-2}^4 y \left[ (y+1)^2 - \left( \frac{y^2}{2} - 3 \right)^2 \right] dy \\ &= \frac{1}{2} \int_{-2}^4 \left( -\frac{y^5}{4} + 4y^3 + 2y^2 - 8y \right) dy = \frac{1}{2} \left[ -\frac{y^6}{24} + y^4 + 2\frac{y^3}{3} - 4y^2 \right]_{-2}^4 = 36 \end{aligned}$$

**Type I region:**

Part I: the  $y$  –limits of integration  $-\sqrt{2x+6} \leq y \leq \sqrt{2x+6}$  and the  $x$  –limits of integration  $-3 \leq x \leq -1$

Part II: the  $y$  –limits of integration  $x-1 \leq y \leq \sqrt{2x+6}$  and the  $x$  –limits of integration  $-1 \leq x \leq 5$

If we had expressed  $D$  as a type I region using Figure (a), then we would have obtained

$$\text{Therefore, } \iint_D xy \, dA = \int_{-3}^{-1} \int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} xy \, dy \, dx + \int_{-1}^5 \int_{x-1}^{\sqrt{2x+6}} xy \, dy \, dx = 36$$

but this would have involved more work than the other method.

## NOTE 5.3.1

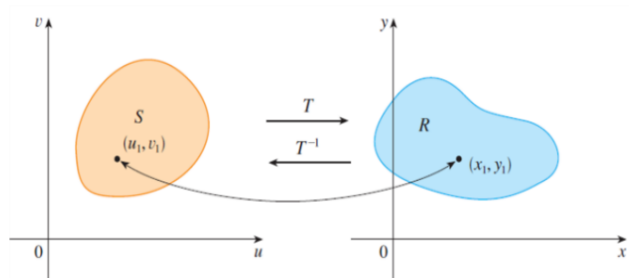
**Change of variables formula for double integrals**

Let  $x = x(u, v)$ ,  $y = y(u, v)$  be a one-to-one transformation from a domain  $S$  in the  $uv$  -plane onto a domain  $D$  in the  $xy$  -plane.

Suppose that the functions  $x$  and  $y$ , and their first partial derivatives with respect to  $u$  and  $v$ , are continuous in  $S$ . If  $f(x, y)$  is integrable on  $D$ , if  $g(u, v) = f(x(u, v), y(u, v))$ , then  $g$  is integrable on  $S$  and

$$\iint_D f(x, y) dx dy = \iint_S g(u, v) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$$

$$dA = dx dy = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$$



## NOTE 5.3.2

The Jacobian ( $J$ ) of the transformation  $T$  given by  $x = g(u, v)$  and  $y = g(u, v)$  is

$$J = T = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v}$$

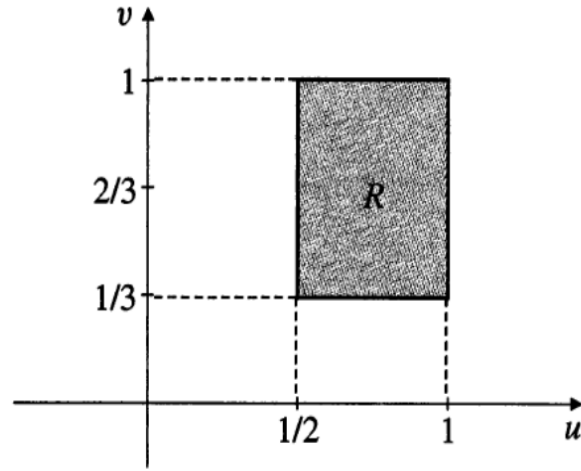
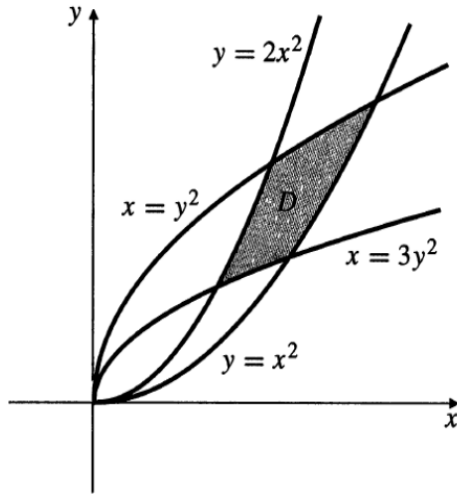
The Jacobian ( $J$ ) of the inverse transformation  $T$  given by  $u = f(x, y)$  and  $y = f(x, y)$  is

$$J^{-1} = T^{-1} = \left| \frac{\partial(u, v)}{\partial(x, y)} \right| = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial v}{\partial x} \frac{\partial u}{\partial y}$$

Note: the Jacobian ( $J$ ) is a **positive** value.

**Example 1**

Find the area of the finite plane region bounded by the four parabolas  $y = x^2$ ,  $y = 2x^2$ ,  $x = y^2$ , and  $x = 3y^2$ .

**Answer**

The integrand suggests the transformation as following

Let  $\begin{cases} u = x^2/y \rightarrow (1) \\ v = y^2/x \rightarrow (2) \end{cases}$  Routine algebra produces  $x$  and  $y$  as functions of  $u$  and  $v$ :  $\begin{cases} x = u^{2/3}v^{1/3} \rightarrow (3) \\ y = v^{2/3}u^{1/3} \rightarrow (4) \end{cases}$

$$y = x^2 \rightarrow \frac{x^2}{y} = 1 \quad \therefore u = 1 \text{ by using equation (1)}$$

$$y = 2x^2 \rightarrow \frac{x^2}{y} = \frac{1}{2} \quad \therefore u = \frac{1}{2} \text{ by using equation (1)}$$

$$x = y^2 \rightarrow \frac{y^2}{x} = 1 \quad \therefore v = 1 \text{ by using equation (2)}$$

$$x = 3y^2 \rightarrow \frac{y^2}{x} = \frac{1}{3} \quad \therefore v = \frac{1}{3} \text{ by using equation (2)}$$

Thus,

$$\triangleright \frac{1}{3} \leq v \leq 1 \text{ and } \frac{1}{2} \leq u \leq 1$$

$$\triangleright J = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{2}{3}u^{-1/3}v^{1/3} & \frac{1}{3}u^{2/3}v^{-2/3} \\ \frac{1}{3}v^{2/3}u^{-2/3} & \frac{2}{3}v^{-1/3}u^{1/3} \end{vmatrix} = \frac{4}{9} - \frac{1}{9} = \frac{3}{9} = \frac{1}{3}$$

$$\triangleright \text{or } J^{-1} = \left| \frac{\partial(u,v)}{\partial(x,y)} \right| = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{2x}{y} & \frac{-x^2}{y^2} \\ -\frac{2y^2}{x^2} & \frac{2y}{x} \end{vmatrix} = 3 \quad \therefore J = \frac{1}{3}$$

$$\triangleright f(x,y) = 1 \text{ converted to } f(u,v) = 1$$

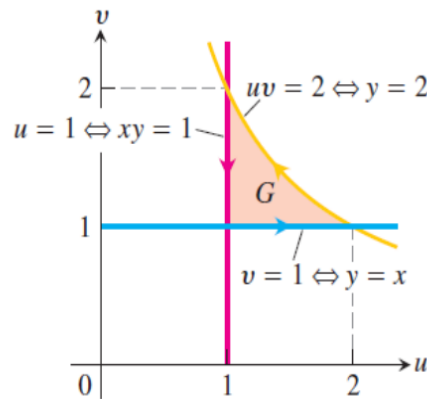
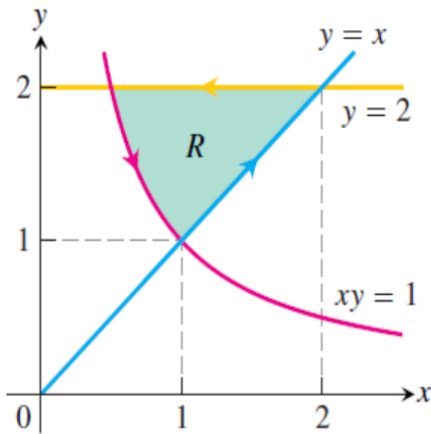
$$\iint_D f(x,y) dx dy = \int_{\frac{1}{3}}^1 \int_{\frac{1}{2}}^1 1 |J(u,v)| du dv = \int_{\frac{1}{3}}^1 \int_{\frac{1}{2}}^1 \frac{1}{3} du dv = \frac{1}{3} \times \left(1 - \frac{1}{2}\right) \times \left(1 - \frac{1}{3}\right) = \frac{1}{3} \times \frac{1}{2} \times \frac{2}{3} = \frac{1}{9}$$



### Example 2

Evaluate  $\int_1^2 \int_{\frac{1}{y}}^y \sqrt{\frac{y}{x}} e^{\sqrt{xy}} dx dy$

### Answer



The integrand suggests the transformation as following

Let  $\begin{cases} u = \sqrt{xy} \\ v = \sqrt{\frac{y}{x}} \end{cases}$

Then  $\begin{cases} u^2 = xy \rightarrow (1) \\ v^2 = y/x \rightarrow (2) \end{cases}$  Routine algebra produces  $x$  and  $y$  as functions of  $u$  and  $v$ :  $\begin{cases} x = \frac{u}{v} \rightarrow (3) \\ y = uv \rightarrow (4) \end{cases}$

$x = y \rightarrow \frac{y}{x} = 1 \quad \therefore \quad v = 1 \quad \text{by using equation (2)}$

$x = \frac{1}{y} \rightarrow xy = 1 \quad \therefore \quad u = 1 \quad \text{by using equation (1)}$

$y = 2 \rightarrow 2 = uv \quad \therefore \quad v = \frac{2}{u} \quad \text{by using equation (4)}$

Thus,

$\triangleright \quad 1 \leq v \leq \frac{2}{u} \text{ and } 1 \leq u \leq 2$

$\triangleright \quad J = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & \frac{-u}{v^2} \\ v & u \end{vmatrix} = \frac{2u}{v} \quad \text{or}$

$\triangleright \quad J^{-1} = \left| \frac{\partial(u,v)}{\partial(x,y)} \right| = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{y}{2\sqrt{xy}} & \frac{x}{2\sqrt{xy}} \\ \frac{-y/x^2}{2\sqrt{y/x}} & \frac{1/x}{2\sqrt{y/x}} \end{vmatrix} = \frac{1}{4x} + \frac{1}{4x} = \frac{1}{2x} = \frac{v}{2u} \quad \therefore J = \frac{2u}{v}$

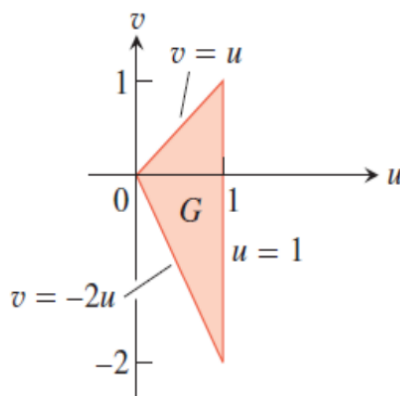
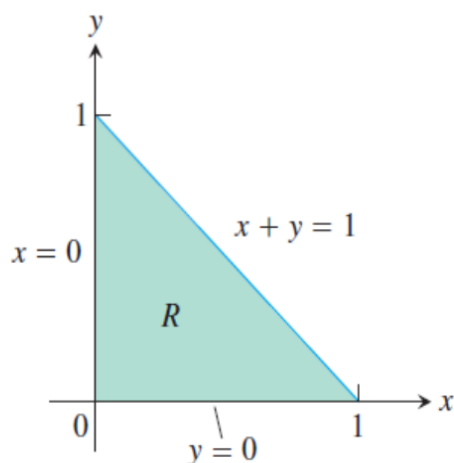
$\triangleright \quad f(x,y) = \sqrt{\frac{y}{x}} e^{\sqrt{xy}} \text{ converted to } f(u,v) = v e^u$

$$\begin{aligned} \int_1^2 \int_{\frac{1}{y}}^y \sqrt{\frac{y}{x}} e^{\sqrt{xy}} dx dy &= \int_1^2 \int_1^{\frac{2}{u}} v e^u |J(u, v)| dv du = \int_1^2 \int_1^{\frac{2}{u}} v e^u \left(\frac{2u}{v}\right) dv du = \int_1^2 \int_1^{\frac{2}{u}} 2u e^u dv du \\ &= 2 \int_1^2 [v u e^u]_{v=1}^{v=\frac{2}{u}} du = 2 \int_1^2 (2e^u - u e^u) du = 2 \int_1^2 (2 - u) e^u du = 2 \left[ (2 - u) e^u + e^u \right]_{u=1}^{u=2} = 2(e^2 - (e + e)) = 2e(e - 2) \end{aligned}$$

Integrate by parts

**Example 3**

Evaluate  $\int_0^1 \int_0^{1-x} \sqrt{x+y} (y-2x)^2 dy dx$

**Answer**

The integrand suggests the transformation as following

Let  $\begin{cases} u = x + y \rightarrow (1) \\ v = y - 2x \rightarrow (2) \end{cases}$  Routine algebra produces  $x$  and  $y$  as functions of  $u$  and  $v$ :  $\begin{cases} x = \frac{u}{3} - \frac{v}{3} \rightarrow (3) \\ y = \frac{2u}{3} + \frac{v}{3} \rightarrow (4) \end{cases}$

$y = 1 - x \rightarrow x + y = 1 \quad \therefore \quad u = 1$  by using equation (1)

$y = 0 \quad \therefore \quad v = -2u$  by using equation (4)

$x = 0 \quad \therefore \quad v = u$  by using equation (3)

Thus,

$\rightarrow -2u \leq v \leq u$  and  $0 \leq u \leq 1$

$\rightarrow J = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{3} \quad \text{or} \quad J^{-1} = \left| \frac{\partial(u,v)}{\partial(x,y)} \right| = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = 3 \quad \therefore J = \frac{1}{3}$

$\rightarrow f(x, y) = \sqrt{x+y} (y-2x)^2$  converted to  $f(u, v) = u^{1/2} v^2$

$$\begin{aligned}\int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 dy dx &= \int_0^1 \int_{-2u}^u u^{1/2} v^2 |J(u,v)| dv du = \int_0^1 \int_{-2u}^u u^{1/2} v^2 \left(\frac{1}{3}\right) dv du \\ &= \frac{1}{3} \int_0^1 u^{\frac{1}{2}} \left[ \frac{1}{3} v^3 \right]_{v=-2u}^{v=u} du = \frac{1}{9} \int_0^1 u^{\frac{1}{2}} (u^3 + 8u^3) du = \int_0^1 u^{\frac{7}{2}} du = \frac{2}{9} u^{\frac{9}{2}} \Big|_0^1 = \frac{2}{9}\end{aligned}$$

### NOTE 5.3.3

The Jacobian ( $J$ ) of the **polar coordinate transformation**  $T$  given by  $x = g(r, \theta) = r \cos \theta$  and  $y = g(r, \theta) = r \sin \theta$  is

$$J = T = \left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r \cos^2 \theta + r \sin^2 \theta = r > 0$$

$$\iint_D f(x,y) dx dy = \iint_S g(r,\theta) \left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| dr d\theta = \iint_S g(r,\theta) \mathbf{r} dr d\theta$$

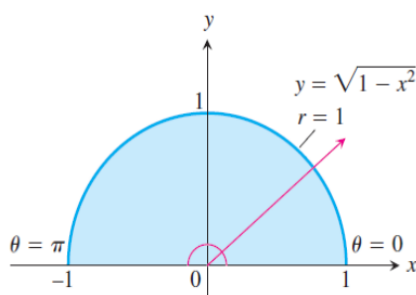
where  $0 \leq \theta \leq 2\pi$ ,  $r$  is positive and,  $dA = \left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| dr d\theta = \mathbf{r} dr d\theta$

### Example 4

Evaluate  $\iint_R e^{x^2+y^2} dy dx$

where  $R$  is the semicircular region bounded by the  $x$ -axis and the curve  $y = \sqrt{1-x^2}$

### Answer



$$\iint_R e^{x^2+y^2} dy dx = \int_0^\pi \int_0^1 e^{r^2} r dr d\theta = \int_0^\pi \left[ \frac{1}{2} e^{r^2} \right]_0^1 d\theta = \int_0^\pi \frac{1}{2} (e - 1) d\theta = \frac{\pi}{2} (e - 1).$$

## NOTE 5.3.3

## Area in Polar Coordinates

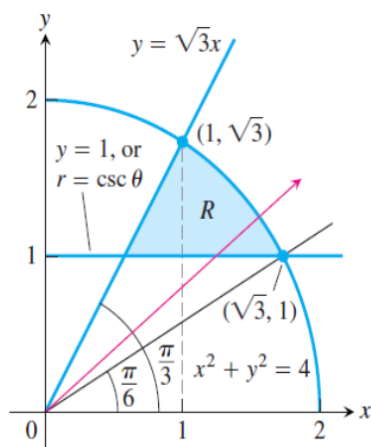
The area of a closed and bounded region  $R$  in the polar coordinate plane is

$$A = \iint_R r \, dr \, d\theta$$

## Example 5

Using polar integration, find the **area** of the region  $R$  in the  $xy$  -plane enclosed by the circle  $x^2 + y^2 = 4$  above the line  $y = 1$ , and below the line  $y = \sqrt{3}x$ .

## Answer



$$\begin{aligned} \iint_R dy \, dx &= \int_{\pi/6}^{\pi/3} \int_{\csc \theta}^2 r \, dr \, d\theta = \int_{\pi/6}^{\pi/3} \left[ \frac{1}{2} r^2 \right]_{r=\csc \theta}^{r=2} d\theta = \int_{\pi/6}^{\pi/3} \frac{1}{2} [4 - \csc^2 \theta] d\theta = \frac{1}{2} [4\theta + \cot \theta]_{\pi/6}^{\pi/3} = \\ &= \frac{1}{2} \left( \frac{4\pi}{3} + \frac{1}{\sqrt{3}} \right) - \frac{1}{2} \left( \frac{4\pi}{6} + \sqrt{3} \right) = \frac{\pi - \sqrt{3}}{3} \end{aligned}$$

## NOTE 5.4.1

## Triple integrals over rectangular regions

Rectangular Regions → The limit boundaries of the integral are all constants.

## Definition:

If  $f(x, y, z)$  is continuous on the rectangular box  $B$

$$B = [a, b] \times [c, d] \times [r, s] \quad \text{or} \quad B = \{(x, y, z) \mid a \leq x \leq b, c \leq y \leq d, r \leq z \leq s\}$$

Then

$$\iiint_B f(x, y, z) \, dV = \int_r^s \int_c^d \int_a^b f(x, y, z) \, dx \, dy \, dz$$

## NOTE 5.4.2

$$\iiint_B f(x, y, z) \, dV$$

6 possible orders of integration

$$\begin{array}{lll} 1) \int_r^s \int_c^d \int_a^b f(x, y, z) \, dx \, dy \, dz & 3) \int_r^c \int_a^b \int_c^d f(x, y, z) \, dy \, dx \, dz & 5) \int_c^d \int_a^b \int_r^s f(x, y, z) \, dz \, dx \, dy \\ 2) \int_c^d \int_r^s \int_a^b f(x, y, z) \, dx \, dz \, dy & 4) \int_a^b \int_r^s \int_c^d f(x, y, z) \, dy \, dz \, dx & 6) \int_a^b \int_c^d \int_r^s f(x, y, z) \, dz \, dy \, dx \end{array}$$

## NOTE 5.4.3

The **volume** of a closed, bounded region  $D$  in space is

$$V = \iiint_D dV$$

where  $f(x, y, z) = 1$

**Example 1**

Evaluate the triple integral:

$$\iiint_B xyz^2 \, dV$$

where  $B$  is the rectangular box given by

$$B = \{(x, y, z) \mid 0 \leq x \leq 1, -1 \leq y \leq 2, 0 \leq z \leq 3\}$$

**Answer**

$$\begin{aligned} \iiint_B xyz^2 \, dV &= \int_0^3 \int_{-1}^2 \int_0^1 xyz^2 \, dx \, dy \, dz = \int_0^3 \int_{-1}^2 \left[ \frac{x^2 y z^2}{2} \right]_{x=0}^{x=1} dy \, dz = \int_0^3 \int_{-1}^2 \frac{y z^2}{2} dy \, dz = \int_0^3 \left[ \frac{y^2 z^2}{2} \right]_{y=-1}^{y=2} dz \\ &= \int_0^3 \frac{3z^2}{2} dz = \frac{z^3}{2} \Big|_{z=0}^{z=3} = \frac{27}{2} \end{aligned}$$

**NOTE 5.4.4**

Triple integrals over non-rectangular (general) regions

Non-rectangular (General) Regions → The limit boundaries of the integral are NOT all constants.

Non-rectangular (General) Regions = Solid

Definition:

If  $f(x, y, z)$  is continuous on the solid (general) region  $E$

$$E = \{(x, y, z) \mid a \leq x \leq b, y_1(x) \leq y \leq y_2(x), z_1(x, y) \leq z \leq z_2(x, y)\}$$

Then

$$\iiint_E f(x, y, z) \, dV = \int_a^b \int_{y_1(x)}^{y_2(x)} \int_{z_1(x, y)}^{z_2(x, y)} f(x, y, z) \, dz \, dy \, dx$$

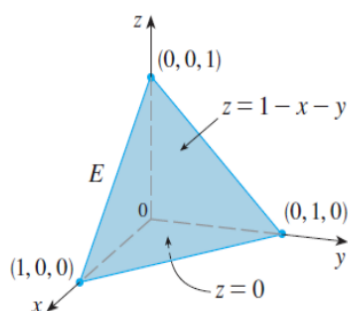
**Example 2**

Evaluate the triple integral:

$$\iiint_E z \, dV$$

where  $E$  is the solid tetrahedron bounded by the four planes

$$x = 0, \, y = 0, \, z = 0 \text{ and } x + y + z = 1$$

**Answer**

**1<sup>st</sup> step: Find Z limit boundary**

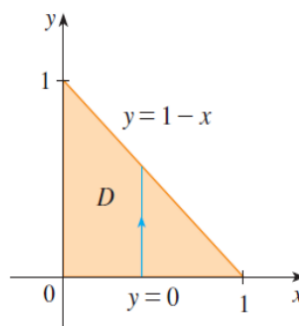
$$0 \leq z \leq 1 - x - y$$

**2<sup>nd</sup> step: Find both x and y limit boundaries**

$dy \, dx$

$$0 \leq y \leq 1 - x$$

$$0 \leq x \leq 1$$



**3<sup>rd</sup> step: Set up triple integral**

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} z \, dz \, dy \, dx$$

$$\begin{aligned} & \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z \, dz \, dy \, dx = \\ & \int_0^1 \int_0^{1-x} \left[ \frac{z^2}{2} \right]_{z=0}^{z=1-x-y} dy \, dx = \frac{1}{2} \\ & \int_0^1 \int_0^{1-x} (1-x-y)^2 \, dy \, dx \end{aligned}$$

**4<sup>th</sup> step: Evaluate the triple integral**

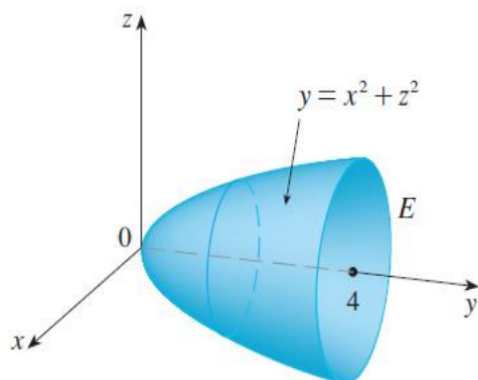
$$= \frac{1}{2} \int_0^1 \left[ \frac{(1-x-y)^3}{3} \right]_{y=0}^{y=1-x} dx = \frac{1}{6} \int_0^1 (1-x)^3 \, dx = \frac{1}{6} \left[ -\frac{(1-x)^4}{4} \right]_0^1 = \frac{1}{24}$$

**Example 3**

Evaluate the triple integral:

$$\iiint_E \sqrt{x^2 + z^2} \, dV$$

where  $E$  is the region bounded by the paraboloid  $y = x^2 + z^2$  and the plane  $y = 4$ .

**Answer****1<sup>st</sup> step: Find Z limit boundary**

$$y = x^2 + z^2 \quad z^2 = y - x^2$$

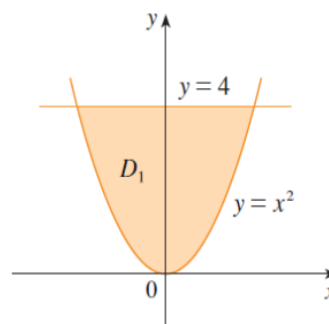
$$z = \pm \sqrt{y - x^2} = z^2$$

$$-\sqrt{y - x^2} \leq z \leq \sqrt{y - x^2}$$

**2<sup>nd</sup> step: Find both x and y limit boundaries**

As  $z = 0$ , then the region in  $xy$  plane is bounded by  $y = x^2$  and  $y = 4$

$$\begin{aligned} dy \, dx \\ x^2 \leq y \leq 4 \\ -2 \leq x \leq 2 \end{aligned}$$

**3<sup>rd</sup> step: Set up triple integral**

$$\int_{-2}^2 \int_{x^2}^4 \int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2 + z^2} \, dz \, dy \, dx \quad \text{It is extremely difficult to evaluate}$$

**Try another method****1<sup>st</sup> step: Find y limit boundary**

$$x^2 + z^2 \leq y \leq 4$$

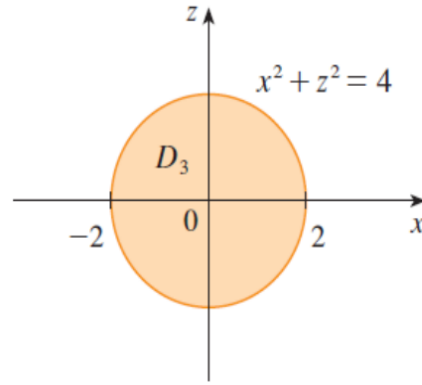
**2<sup>nd</sup> step: Find both x and z limit boundaries**

Cross section Intersection between the paraboloid  $y = x^2 + z^2$  and the plane  $y = 4$ .

$$x^2 + z^2 = 4$$



$$\begin{aligned}
 x^2 + z^2 &= r^2 \\
 0 &\leq r \leq 2 \\
 0 &\leq \theta \leq 2\pi \\
 dA &= r \, dr \, d\theta
 \end{aligned}$$



**3<sup>rd</sup> step: Set up and evaluate triple integral**

$$\begin{aligned}
 \iint_D \left( \int_{x^2+z^2}^4 \sqrt{x^2+z^2} \, dy \right) dA &= \int_0^{2\pi} \int_0^2 \left( \int_{r^2}^4 r \, dy \right) r \, dr \, d\theta = \int_0^{2\pi} \int_0^2 \int_{r^2}^4 r^2 \, dy \, dr \, d\theta = \int_0^{2\pi} \int_0^2 [r^2 y]_{y=r^2}^{y=4} \, dr \, d\theta \\
 &= \int_0^{2\pi} \int_0^2 r^2 (4 - r^2) \, dr \, d\theta = \int_0^{2\pi} \int_0^2 (4r^2 - r^4) \, dr \, d\theta = \int_0^{2\pi} \left[ \frac{4r^3}{3} - \frac{r^5}{5} \right]_{r=0}^{r=2} d\theta = \int_0^{2\pi} \frac{64}{15} \, d\theta = 2\pi \frac{64}{15} = \frac{128}{15} \pi
 \end{aligned}$$

## Section 5.5 Change of Variables in Triple Integrals

### NOTE 5.5.1

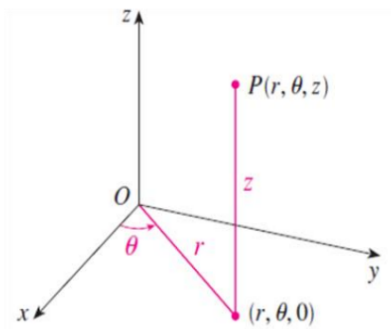
#### Triple integrals in cylindrical coordinates

To convert between (cartesian  $\Leftrightarrow$  cylindrical) coordinates, you need to know the following:

$$x = r \cos \theta \quad y = r \sin \theta \quad z = z$$

$$dV = dz \, dy \, dx \rightarrow dV = r \, dz \, dr \, d\theta \quad f(x, y, z) \rightarrow f(r \cos \theta, r \sin \theta, z)$$

$$\iiint_E f(x, y, z) \, dV = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} \int_{z_1}^{z_2} f(r \cos \theta, r \sin \theta, z) r \, dz \, dr \, d\theta$$



**Example 1**

Evaluate  $\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^2 (x^2 + y^2) dz dy dx$

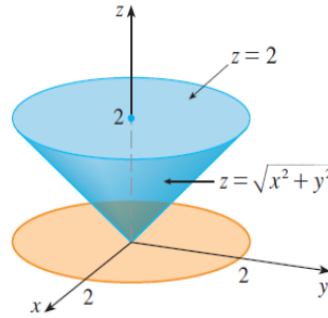
**Answer**

**1<sup>st</sup> step: Find Z limit boundary in polar form**

$$\sqrt{x^2 + y^2} \leq z \leq 2 \quad \rightarrow \quad r \leq z \leq 2$$

**2<sup>nd</sup> step: Find r and  $\theta$  limit boundaries**

$$0 \leq r \leq 2 \quad 0 \leq \theta \leq 2\pi$$



$$\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^2 (x^2 + y^2) dz dy dx = \int_0^{2\pi} \int_0^2 \int_r^2 r^2 r dz dr d\theta = \int_0^{2\pi} \int_0^2 \int_r^2 r^3 dz dr d\theta$$

**3<sup>rd</sup> step: Evaluate the triple integral**

$$= \int_0^{2\pi} \int_0^2 \int_r^2 r^3 dz dr d\theta = \int_0^{2\pi} \int_0^2 r^3 (2 - r) dr d\theta = \int_0^{2\pi} \int_0^2 (2r^3 - r^4) dr d\theta = \int_0^{2\pi} \left[ \frac{r^4}{2} - \frac{r^5}{5} \right]_{r=0}^{r=2} d\theta = \int_0^{2\pi} \frac{8}{5} d\theta = \frac{16}{5} \pi$$

## NOTE 5.5.2

## Triple integrals in Spherical coordinates

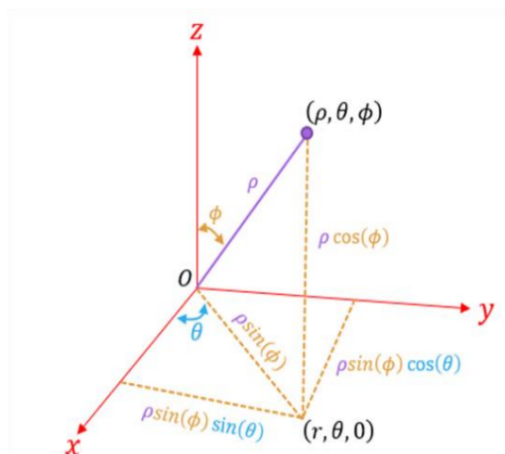
To convert between (cartesian  $\Leftrightarrow$  Spherical ) coordinates, you need to know the following:

$$x = \rho \sin \phi \cos \theta \quad y = \rho \sin \phi \sin \theta \quad z = \rho \cos \phi \quad x^2 + y^2 + z^2 = \rho^2$$

$$dV = dz dy dx \rightarrow dV = \rho^2 \sin \phi d\rho d\phi d\theta \quad f(x, y, z) \rightarrow f(\rho, \theta, \phi)$$

- $\rho$  = distance to origin,  $\rho \geq 0$
- $\theta$  = angle of projection to  $xy$  -plane with  $x$  -axis,  $0 \leq \theta \leq 2\pi$
- $\phi$  = angle to  $z$  -axis,  $0 \leq \phi \leq \pi$

$$\iiint_E f(x, y, z) dV = \int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} \int_{\rho_1}^{\rho_2} f(\rho, \theta, \phi) \rho^2 \sin \phi d\rho d\phi d\theta$$



## Example 2

Evaluate  $\iiint_B e^{(x^2+y^2+z^2)^{3/2}} dV$ , where B is the unit ball  $x^2 + y^2 + z^2 \leq 1$ .

## Answer

**1<sup>st</sup> step: Find  $\rho$  limit boundary**

$$0 \leq x^2 + y^2 + z^2 \leq 1 \quad \rightarrow \quad 0 \leq \rho \leq 1$$

**2<sup>nd</sup> step: Find  $\theta$  and  $\phi$  limit boundaries**

$$0 \leq \theta \leq 2\pi \quad 0 \leq \phi \leq \pi$$

3<sup>rd</sup> step: Set up the triple integral in spherical form

$$\iiint_E f(x, y, z) dV = \int_0^{2\pi} \int_0^{\pi} \int_0^1 e^{\rho^3} \rho^2 \sin \phi d\rho d\phi d\theta$$

4<sup>th</sup> step: Evaluate the triple integral

$$\begin{aligned} \int_0^{2\pi} \int_0^{\pi} \int_0^1 e^{\rho^3} \rho^2 \sin \phi d\rho d\phi d\theta &= \int_0^{2\pi} \int_0^{\pi} \left[ \frac{e^{\rho^3}}{3} \sin \phi \right]_{\rho=0}^{\rho=1} d\phi d\theta = \int_0^{2\pi} \int_0^{\pi} \left( \frac{e-1}{3} \sin \phi \right) d\phi d\theta \\ &= \frac{e-1}{3} \int_0^{2\pi} [-\cos \phi]_{\phi=0}^{\phi=\pi} d\theta = \frac{e-1}{3} \int_0^{2\pi} 2 d\theta = \frac{e-1}{3} \cdot 2 \cdot 2\pi = \frac{4}{3}\pi(e-1) \end{aligned}$$

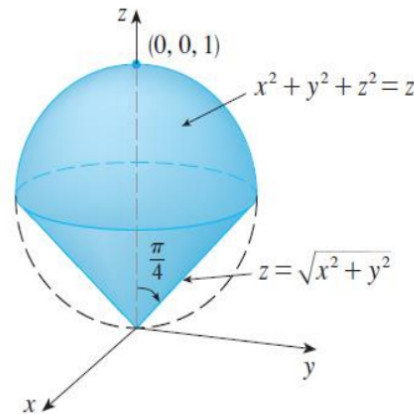
### Example 3

Use spherical coordinates to find the volume of the solid that lies above the cone  $z = \sqrt{x^2 + y^2}$  and below the sphere  $x^2 + y^2 + z^2 = z$

### Answer

1<sup>st</sup> step: Find  $\rho$  limit boundary

$$\begin{aligned} x^2 + y^2 + z^2 &= z \\ \rho^2 &= \rho \cos \phi \\ \rho &= \cos \phi \\ 0 \leq \rho &\leq \cos \phi \end{aligned}$$



2<sup>nd</sup> step: Find  $\theta$  and  $\phi$  limit boundaries

$$\begin{aligned} 0 \leq \theta &\leq 2\pi \\ 0 \leq \phi &\leq \frac{\pi}{4} \end{aligned}$$

3<sup>rd</sup> step: Set up the triple integral in spherical form

$$\begin{aligned} \text{volume} &= \iiint_E f(x, y, z) dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\cos \phi} \rho^2 \sin \phi d\rho d\phi d\theta = \int_0^{2\pi} \int_0^{\pi/4} \left[ \frac{\rho^3}{3} \sin \phi \right]_{\rho=0}^{\rho=\cos \phi} d\phi d\theta \\ &= \frac{1}{3} \int_0^{2\pi} \int_0^{\pi/4} \cos^3 \phi \sin \phi d\phi d\theta = \frac{1}{3} \int_0^{2\pi} \frac{3}{16} d\theta = \frac{1}{3} \cdot \frac{3}{16} \cdot 2\pi = \frac{\pi}{8} \end{aligned}$$

$$\text{Where } \int_0^{\pi/4} \cos^3 \phi \sin \phi d\phi \stackrel{(*)}{=} - \int_1^{\sqrt{2}/2} u^3 du = \frac{u^4}{4} \Big|_1^{\sqrt{2}/2} = - \left( \frac{1}{16} - \frac{1}{4} \right) = \frac{3}{16}$$

$$(*) \text{ Where } u = \cos \phi \quad du = -\sin \phi d\phi \quad \text{As } \phi = 0 \rightarrow u = 1 \quad \text{As } \phi = \frac{\pi}{4} \rightarrow u = \frac{\sqrt{2}}{2}$$

## NOTE 6.1.1

When a particle moves through space during a time interval  $I$ , we think of the particle's coordinates as functions defined on  $I$ :

$$x = f(t), \quad y = g(t), \quad z = h(t), \quad t \in I \quad (1)$$

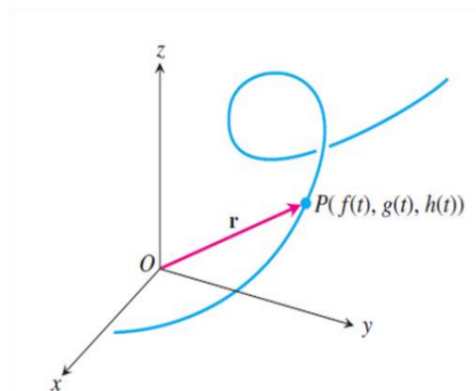
The points  $(x, y, z) = (f(t), g(t), h(t))$ ,  $t \in I$  make up the curve in space that we call the particle's path.

$$\mathbf{r}(t) = \overrightarrow{OP} = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \quad (2)$$

The vector from the origin to the particle's position  $P(f(t), g(t), h(t))$  at time  $t$  is the particle's position vector.

Equation (2) defines  $\mathbf{r}$  as a vector function (vector-valued function) of the real variable  $t$  on the interval  $I$ .

Real-valued functions are called **scalar functions** to distinguish them from vector functions. The components of  $\mathbf{r}$  in Equation (2) are scalar functions of  $t$ .



## NOTE 6.1.2

If  $\mathbf{r}$  is the position vector of a particle moving along a smooth curve in space, then

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} \quad \text{is the particle's velocity vector, tangent to the curve.}$$

1. Velocity is the derivative of position:  $\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} \quad \leftrightarrow \quad \text{velocity} = |\mathbf{v}| \left( \frac{\mathbf{v}}{|\mathbf{v}|} \right) = (\text{speed})(\text{direction}).$
2. Speed is the magnitude of velocity:  $\text{Speed} = |\mathbf{v}|$
3. Acceleration is the derivative of velocity:  $\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$
4. The unit vector  $\mathbf{v}/|\mathbf{v}|$  is the direction of motion at time  $t$ .

**Example 1**

Find the velocity, speed, and acceleration of a particle whose motion in space is given by the position vector

$$\mathbf{r}(t) = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j} + 5 \cos^2 t \mathbf{k}$$

**Answer**

The velocity and acceleration vectors at time  $t$  are

$$\begin{aligned}\mathbf{v}(t) = \mathbf{r}'(t) &= -2 \sin t \mathbf{i} + 2 \cos t \mathbf{j} - 10 \cos t \sin t \mathbf{k} \\ &= -2 \sin t \mathbf{i} + 2 \cos t \mathbf{j} - 5 \sin 2t \mathbf{k}\end{aligned}$$

$$\mathbf{a}(t) = \mathbf{r}''(t) = -2 \cos t \mathbf{i} - 2 \sin t \mathbf{j} - 10 \cos 2t \mathbf{k}$$

and the speed is  $|\mathbf{v}| =$

$$\begin{aligned}&\sqrt{(-2 \sin t)^2 + (2 \cos t)^2 + (-5 \sin 2t)^2} = \\ &\sqrt{4 + 25 \sin^2 2t}\end{aligned}$$

**NOTE 6.1.3****Differentiation Rules for Vector Functions**

Let  $\mathbf{u}$  and  $\mathbf{v}$  be differentiable vector functions of  $t$ ,  $\mathbf{C}$  a constant vector,  $c$  any scalar, and  $f$  any differentiable scalar function.

1. Constant Function Rule:  $\frac{d}{dt} \mathbf{C} = \mathbf{0}$

2. Scalar Multiple Rules:  $\frac{d}{dt} [c \mathbf{u}(t)] = c \mathbf{u}'(t)$

$$\frac{d}{dt} [f(t) \mathbf{u}(t)] = f'(t) \mathbf{u}(t) + f(t) \mathbf{u}'(t)$$

3. Sum/ Difference Rule:  $\frac{d}{dt} [\mathbf{u}(t) \pm \mathbf{v}(t)] = \mathbf{u}'(t) \pm \mathbf{v}'(t)$

4. Dot Product Rule:  $\frac{d}{dt} [\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$

5. Cross Product Rule:  $\frac{d}{dt} [\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$

6. Chain Rule:  $\frac{d}{dt} [\mathbf{u}(f(t))] = f'(t) \mathbf{u}'(f(t))$

6. At any point where  $\mathbf{u}(t) \neq \mathbf{0}$   $\frac{d}{dt} |\mathbf{u}(t)| = \frac{\mathbf{u}(t) \cdot \mathbf{u}'(t)}{|\mathbf{u}(t)|}$

### NOTE 6.1.4

A vector function  $\mathbf{r}$  is **differentiable** if it is differentiable at every point of its domain.

The curve traced by  $\mathbf{r}$  is **smooth** if  $d\mathbf{r}/dt$  is continuous and never 0, that is, if  $f$ ,  $g$ , and  $h$  have continuous first derivatives that are not simultaneously 0.

A curve that is made up of a finite number of smooth curves pieced together in a continuous fashion is called **piecewise smooth**.

## Section 6.2 Curves and Parametrizations

### NOTE 6.2.1

The set of points whose positions are given by position vector function

$$\mathbf{r} = \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \quad a \leq t \leq b$$

The variable  $t$ , which need no longer represent time or any other specific physical quantity, is called a parameter for the curve.

The point  $(f(a), g(a))$  is the **initial point** of the curve and  $(f(b), g(b))$  is the **terminal point**.

#### Example 1

The plane  $x + y = 1$  intersects the paraboloid  $z = x^2 + y^2$  in a parabola. Parametrize the whole parabola using  $t = x$  as parameter. Could  $t = y$  has been used as parameter? what about  $t = z$ ?

#### Answer

➤ Parametrizing the whole parabola using  $t = x$  as parameter.

$$x + y = 1 \quad y = 1 - x = 1 - t$$

$$z = x^2 + y^2 \quad z = t^2 + (1 - t)^2 = 1 - 2t + 2t^2$$

Thus, the required parametrization is  $\mathbf{r}(t) = t\mathbf{i} + (1 - t)\mathbf{j} + (1 - 2t + 2t^2)\mathbf{k}$ ,  $-\infty < t < \infty$

➤ Parametrizing the whole parabola using  $t = y$  as parameter.

$$x + y = 1 \quad x = 1 - y = 1 - t$$

$$z = x^2 + y^2 \quad z = (1 - t)^2 + t^2 = 1 - 2t + 2t^2$$

Thus, the required parametrization is  $\mathbf{r}(t) = (1 - t)\mathbf{i} + t\mathbf{j} + (1 - 2t + 2t^2)\mathbf{k}$ ,  $-\infty < t < \infty$

➤ Parametrizing the whole parabola using  $t = z$  as parameter.

The equation  $x + y = 1$  is independent of  $z$ .

To solve the system of equations  $x + y = 1$ ,  $x^2 + y^2 = t$  for  $x$  and  $y$ , we have two possible solutions, each corresponding to a different half of the parabola starting at the lowest point  $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$  because there are two points on the parabola at each height  $z > \frac{1}{2}$ .

Thus, the whole parabola cannot be parametrized using  $z$  as the parameter.

### Example 2

Show that each of the vector functions

$$\mathbf{r}_1(t) = \sin t \mathbf{i} + \cos t \mathbf{j} \quad \left(-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}\right)$$

$$\mathbf{r}_2(t) = (t-1) \mathbf{i} + \sqrt{2t-t^2} \mathbf{j} \quad (0 \leq t \leq 2)$$

$$\mathbf{r}_3(t) = t\sqrt{2-t^2} \mathbf{i} + (1-t^2) \mathbf{j} \quad (-1 \leq t \leq 1)$$

all represent the same curve. Describe the curve.

### Answer

$$\mathbf{r}_1\left(-\frac{\pi}{2}\right) = -\mathbf{i} \quad \mathbf{r}_1\left(\frac{\pi}{2}\right) = \mathbf{i}$$

$$|\mathbf{r}_1(t)| = \sqrt{(\sin t)^2 + (\cos t)^2} = 1$$

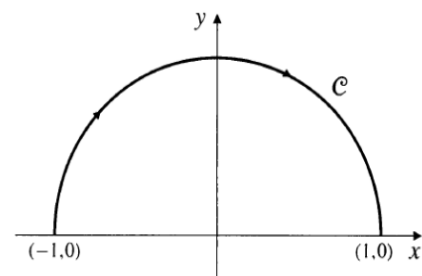
$$\mathbf{r}_2(0) = -\mathbf{i} \quad \mathbf{r}_2(2) = \mathbf{i}$$

$$|\mathbf{r}_2(t)| = \sqrt{(t-1)^2 + 2t-t^2} = 1$$

$$\mathbf{r}_3(-1) = -\mathbf{i} \quad \mathbf{r}_3(1) = \mathbf{i}$$

$$|\mathbf{r}_3(t)| = \sqrt{t^2(2-t^2) + (1-t^2)^2} = 1$$

Thus, all three functions represent the same semicircle although the three parametrizations trace out the curve with different velocities.



### Example 3

Parametrize the curve of intersection of the plane  $x + 2y + 4z = 4$  and the elliptic cylinder  $x^2 + 4y^2 = 4$ .

### Answer

It is easier to start parametrize the equation (with two variables/unknowns)  $x^2 + 4y^2 = 4$ , which is independent of  $z$ .

Such equation can be parametrized in many ways; one convenient way is

$$x = 2 \cos t \quad y = 2 \sin t \quad 0 \leq t \leq 2\pi$$

$$\text{Then, } z = \frac{1}{4}(4 - x - 2y) = 1 - \frac{1}{2}(\cos t + \sin t)$$

Thus, the given surfaces intersect in the curve

$$\mathbf{r}(t) = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j} + \left(1 - \frac{\cos t + \sin t}{2}\right) \mathbf{k}, \quad 0 \leq t \leq 2\pi$$



**Example 4**

Find a parametric representation of the curve of the intersection of the two surfaces

$$x^2 + y + z = 2 \quad \text{and} \quad xy + z = 1$$

**Answer**

1) Since we have 3 variables/unknowns in both equations, we should eliminate at least one variable. We might eliminate the variable  $z$  and obtain the following equation by subtracting both given equations.

$$x^2 + y - xy = 1$$

2) To parametrize this equation by let  $x = t$ , then we have

$$t^2 + y - ty = 1 \quad y = \frac{1-t^2}{1-t} = 1+t$$

3) Either of the given equations can then be used to express  $z$  in terms of  $t$ :

$$xy + z = 1$$

$$z = 1 - xy = 1 - t(1+t) = 1 - t - t^2$$

4) Thus, a possible parametrization of the curve is

$$\mathbf{r}(t) = t\mathbf{i} + (1+t)\mathbf{j} + (1-t-t^2)\mathbf{k},$$

$$-\infty < t < \infty$$

**BEWARE:** this answer is NOT unique. Many other parameterizations can be found for the curve, providing orientations in either direction.

**NOTE 6.2.2**

The length of a space curve  $\mathcal{C}$  is the limit of lengths of inscribed polygons as shown below

Suppose that the curve has the vector equation

$$\mathbf{r} = \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \quad a \leq t \leq b$$

The arc length of that curve  $\mathcal{C}$  is

$$L = \int_a^b ds = \int_a^b |\mathbf{r}'(t)| dt = \int_a^b |\mathbf{v}(t)| dt = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} dt$$

**Example 5**

Find the length  $s$  of that part of the circular helix

$$\mathbf{r} = a \cos t \mathbf{i} + a \sin t \mathbf{j} + bt \mathbf{k}, \quad \text{between the points } (a, 0, 0) \text{ and } (a, 0, 2\pi b)$$

**Answer**

$$\text{We have } \mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = -a \sin t \mathbf{i} + a \cos t \mathbf{j} + b \mathbf{k} \quad |\mathbf{v}(t)| = \sqrt{a^2 + b^2}$$

The required length  $s$  corresponds to parameter interval  $[0, 2\pi]$  is described by

$$s = \int_0^{2\pi} |\mathbf{v}(t)| dt = \int_0^{2\pi} \sqrt{a^2 + b^2} dt = 2\pi\sqrt{a^2 + b^2}$$

## NOTE 6.2.3

| Curve                                                       | Parametric Equations                                                                                    |                                                                                 |
|-------------------------------------------------------------|---------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------|
| $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$<br>(Ellipse)        | <u>Counter-Clockwise</u><br>$x = a \cos(t)$<br>$y = b \sin(t)$<br>$0 \leq t \leq 2\pi$                  | <u>Clockwise</u><br>$x = a \cos(t)$<br>$y = -b \sin(t)$<br>$0 \leq t \leq 2\pi$ |
| $x^2 + y^2 = r^2$<br>(Circle)                               | <u>Counter-Clockwise</u><br>$x = r \cos(t)$<br>$y = r \sin(t)$<br>$0 \leq t \leq 2\pi$                  | <u>Clockwise</u><br>$x = r \cos(t)$<br>$y = -r \sin(t)$<br>$0 \leq t \leq 2\pi$ |
| $y = f(x)$                                                  | $x = t$<br>$y = f(t)$                                                                                   |                                                                                 |
| $x = g(y)$                                                  | $x = g(t)$<br>$y = t$                                                                                   |                                                                                 |
| Line Segment From<br>$(x_0, y_0, z_0)$ to $(x_1, y_1, z_1)$ | $x = x_0 + (x_1 - x_0)t$<br>$y = y_0 + (y_1 - y_0)t, \quad 0 \leq t \leq 1$<br>$z = z_0 + (z_1 - z_0)t$ |                                                                                 |

## Section 7.1 Line Integrals

## NOTE 7.1.1

- To calculate the total mass of a wire lying along a curve in space, or to find the work done by a variable force acting along such a curve, we need a more general notion of integral than was defined in Calculus I. We need to integrate over a curve  $\mathcal{C}$  rather than over an interval  $[a, b]$ . These more general integrals are called **line integrals**.
- Suppose that  $f(x, y, z)$  is a real-valued (scalar) function we wish to integrate over the curve  $\mathcal{C}$  lying within the domain of  $f$  and parametrized by

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \quad a \leq t \leq b$$

- If  $f$  is defined on a curve  $\mathcal{C}$  given parametrically by  $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \quad a \leq t \leq b$

then the line integral of  $f$  over  $\mathcal{C}$  is  $\int_{\mathcal{C}} f(x, y, z) ds = \int_a^b f(x, y, z) \cdot |\mathbf{r}'(t)| dt = \int_a^b f(x, y, z) \cdot |\mathbf{v}(t)| dt$

where  $|\mathbf{r}'(t)| = |\mathbf{v}(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}$

**Example 1**

Evaluate  $\int_C (2 + x^2 y) ds$ , where  $C$  is the upper half of the unit circle  $x^2 + y^2 = 1$ .

**Answer**

1<sup>st</sup> step: Find a smooth parametrization of  $C$

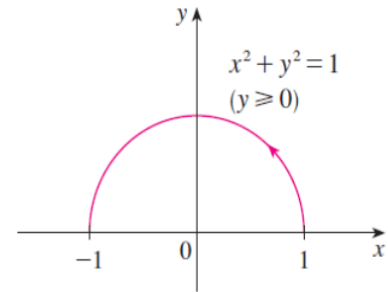
Recall that the unit circle can be parametrized by means of the equations

$$x = \cos t \quad y = \sin t \quad 0 \leq t \leq \pi \text{ (Upper half of the circle)}$$

Thus, the simplest parametrization we can think of:

$$\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} \quad 0 \leq t \leq \pi$$

$$\mathbf{v}(t) = -\sin t \mathbf{i} + \cos t \mathbf{j} \quad 0 \leq t \leq \pi$$



The components have continuous first derivatives and  $|\mathbf{v}(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1$  is never 0, so the parametrization is smooth.

2<sup>nd</sup> step: Evaluate the integral as

$$\int_C f(x, y, z) ds = \int_a^b f(x, y, z) \cdot |\mathbf{v}(t)| dt = \int_0^\pi (2 + (\cos t)^2 (\sin t)) \cdot (1) dt = \left[ 2t - \frac{\cos^3 t}{3} \right]_0^\pi = 2\pi + \frac{2}{3}$$

**Example 2**

Evaluate  $\int_C y^2 dx + x dy$ , where

- (a)  $C = C_1$  is the line segment from  $(-5, -3)$  to  $(0, 2)$  and  
 (b)  $C = C_2$  is the arc of the parabola  $x = 4 - y^2$  from  $(-5, -3)$  to  $(0, 2)$ .

**Answer**

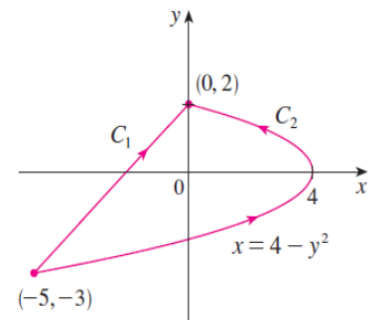
(a) 1<sup>st</sup> step: Find a smooth parametrization of  $C_1$

Recall that the line segment can be parametrized by means of the equations

$$x = -5 + 5t \quad y = -3 + 5t \quad 0 \leq t \leq 1$$

Thus, the simplest parametrization we can think of:

$$\mathbf{r}(t) = (-5 + 5t) \mathbf{i} + (-3 + 5t) \mathbf{j} \quad 0 \leq t \leq 1$$



Since  $x = -5 + 5t$ , and  $y = -3 + 5t$ , then we have  $dx = 5 dt$  and  $dy = 5 dt$

2<sup>nd</sup> step: Evaluate the integral as

$$\int_{c_1} y^2 dx + x dy = \int_0^1 (5t-3)^2 \cdot 5 dt + (5t-5) \cdot 5 dt = 5 \int_0^1 (25t^2 - 25t + 4) dt = \left[ \frac{25t^3}{3} - \frac{25t^2}{2} + 4t \right]_0^1 = \frac{-5}{6}$$

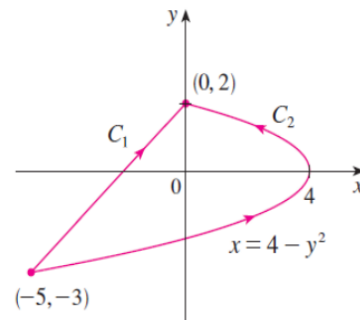
(b) 1<sup>st</sup> step: Find a smooth parametrization of  $C_2$

The parabola can be parametrized by means of the equations

$$x = 4 - y^2 \quad \text{and} \quad y = y \quad -3 \leq t \leq 2$$

Since  $x = 4 - y^2$  and  $y = y$ , then we have  $dx = -2y dy$

2<sup>nd</sup> step: Evaluate the integral as



$$\int_{c_2} y^2 dx + x dy = \int_{-3}^2 y^2 \cdot (-2y) dy + (4 - y^2) dy = \int_{-3}^2 (-2y^3 - y^2 + 4) dy = \left[ \frac{-y^4}{2} - \frac{y^3}{3} + 4y \right]_{-3}^2 = 40 \frac{5}{6}$$



### NOTE 7.1.2

- Notice that we got different answers in parts (a) and (b) of Example 2 even though the two curves had the same endpoints. Thus, in general, the value of a **line integral depends not just on the endpoints of the curve but also on the path.** (But see Section 7.5 for conditions under which the integral is independent of the path.)
- Notice also that the answers in Example 2 **depend on the direction, or orientation, of the curve.** If  $-C_1$  denotes the line segment from  $(0, 2)$  to  $(-5, -3)$ , you can verify, using the parametrization

$$x = -5t \quad y = 2 - 5t \quad 0 \leq t \leq 1$$

$$\int_{-C_1} y^2 dx + x dy = \frac{5}{6}$$

## NOTE 7.1.3

- If  $-C$  denotes the curve consisting of the same points as  $C$  but with the opposite orientation (from  $B$  initial point to  $A$  terminal point in Figure on the right), then we have

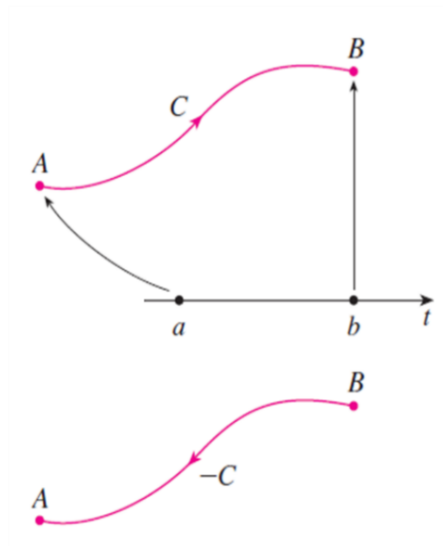
$$\int_{-c} f(x, y) dx = - \int_c f(x, y) dx$$

$$\int_{-c} f(x, y) dy = - \int_c f(x, y) dy$$

- But if we integrate with respect to **arc length**, the value of the line integral does **not change** when we reverse the orientation of the curve:

$$\int_{-c} f(x, y) ds = \int_c f(x, y) ds$$

This is because  $\Delta s_i$  (the length of the subarcs) is always positive, whereas  $\Delta x_i$  and  $\Delta y_i$  direction change sign when we reverse the orientation of  $C$ .



## Example 3

Evaluate  $\int_C y \sin z ds$ , where  $C$  is the circular helix given by the equations  $x = \cos t$ ,  $y = \sin t$ ,  $z = t$   $0 \leq t \leq 2\pi$

## Answer

1<sup>st</sup> step: Find a smooth parametrization of  $C$

$$\text{where } x = \cos t, y = \sin t, z = t \quad 0 \leq t \leq 2\pi$$

Thus, the simplest parametrization we can think of:

$$\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k} \quad 0 \leq t \leq 2\pi$$

$$\mathbf{v}(t) = -\sin t \mathbf{i} + \cos t \mathbf{j} + \mathbf{k} \quad 0 \leq t \leq 2\pi$$

The components have continuous first derivatives and  $|\mathbf{v}(t)| = \sqrt{(\cos t)^2 + (\sin t)^2 + (1)^2} = \sqrt{2}$  is never 0, so the parametrization is smooth.

2<sup>nd</sup> step: Evaluate the integral as

$$\begin{aligned} \int_C y \sin z ds &= \int_a^b f(x, y, z) \cdot |\mathbf{v}(t)| dt = \int_0^{2\pi} (\sin t)(\sin t) \cdot (\sqrt{2}) dt = \sqrt{2} \int_0^{2\pi} \sin^2 t dt = \sqrt{2} \int_0^{2\pi} \sin^2 t dt \\ &= \sqrt{2} \int_0^{2\pi} (1 - \cos 2t) dt = \frac{\sqrt{2}}{2} \left[ t - \frac{1}{2} \sin 2t \right]_0^{2\pi} = \sqrt{2}\pi \end{aligned}$$

NOTE 7.2.1

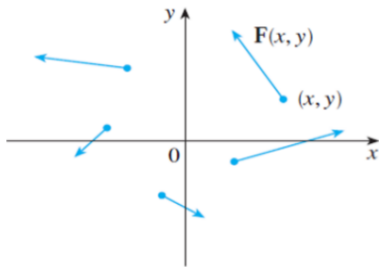
Vector Field on  $\mathbb{R}^2$

**Definition:** Let  $D$  be a set in  $\mathbb{R}^2$  (a plane region). A **vector field** on  $\mathbb{R}^2$  is a function  $\mathbf{F}$  that assigns to each point  $(x, y)$  in  $D$  a two-dimensional vector  $\mathbf{F}(x, y)$ .

Vector Field Function in  $\mathbb{R}^2$  is

$$\mathbf{F}(x, y) = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j}$$

Where  $P(x, y)$ ,  $Q(x, y)$  are **scalar** (real-valued) functions (fields) of two variables.



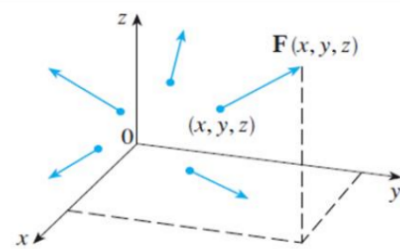
Vector Field on  $\mathbb{R}^3$

**Definition:** Let  $E$  be a subset in  $\mathbb{R}^3$ . A **vector field** on  $\mathbb{R}^3$  is a function  $\mathbf{F}$  that assigns to each point  $(x, y, z)$  in  $E$  a three-dimensional vector  $\mathbf{F}(x, y, z)$ .

Vector Field Function in  $\mathbb{R}^3$  is

$$\mathbf{F}(x, y, z) = P(x, y, z) \mathbf{i} + Q(x, y, z) \mathbf{j} + R(x, y, z) \mathbf{k}$$

Where  $P(x, y, z)$ ,  $Q(x, y, z)$ , and  $R(x, y, z)$  are **scalar** (real-valued) functions (fields) of three variables.

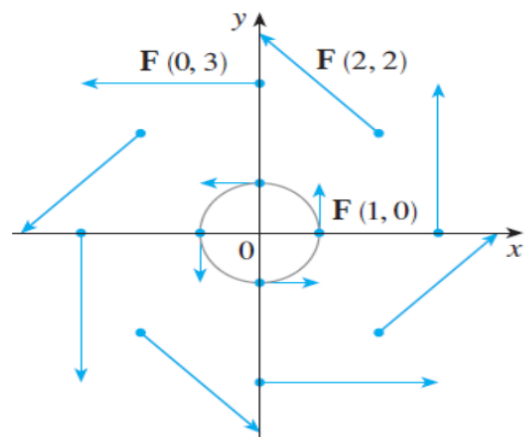


Example 1

Sketch the vector field on  $\mathbb{R}^2$  is defined by  $\mathbf{F}(x, y) = -y \mathbf{i} + x \mathbf{j}$

Answer

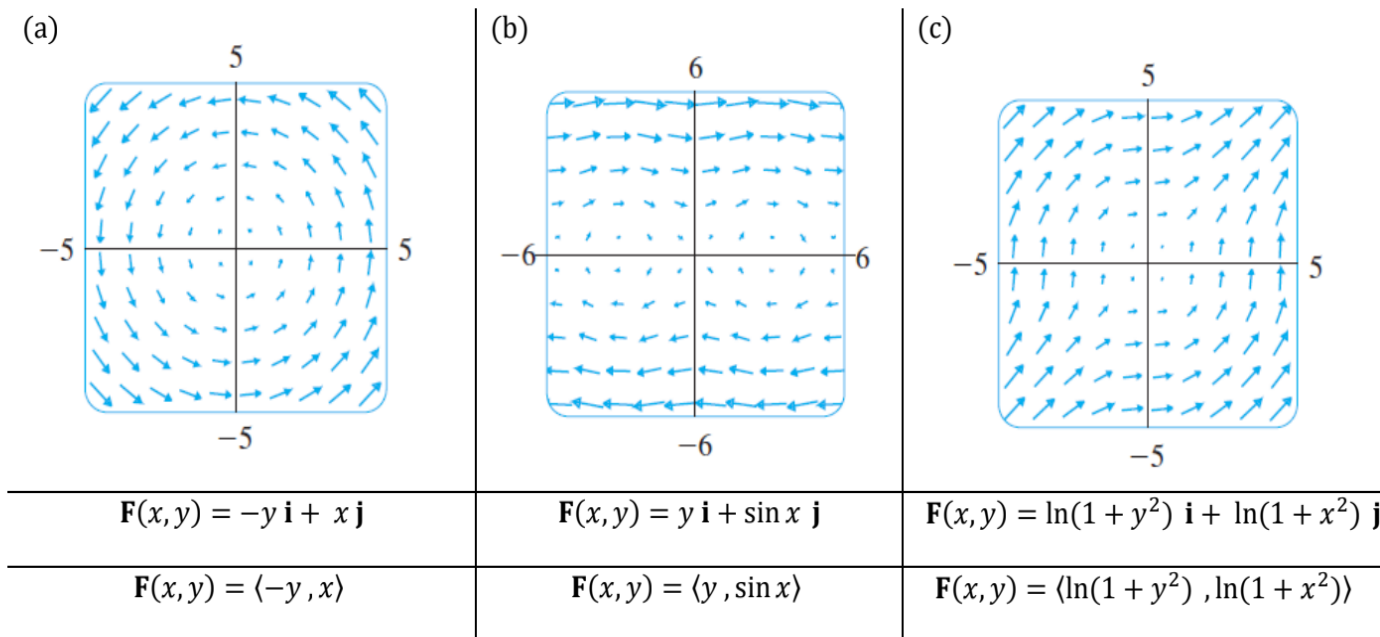
| $(x, y)$   | $\mathbf{F}(x, y)$       |
|------------|--------------------------|
| $(1, 0)$   | $\langle 0, 1 \rangle$   |
| $(2, 2)$   | $\langle -2, 2 \rangle$  |
| $(3, 0)$   | $\langle 0, 3 \rangle$   |
| $(0, 1)$   | $\langle -1, 0 \rangle$  |
| $(-2, 2)$  | $\langle -2, -2 \rangle$ |
| $(0, 3)$   | $\langle -3, 0 \rangle$  |
| $(-1, 0)$  | $\langle 0, -1 \rangle$  |
| $(-2, -2)$ | $\langle 2, -2 \rangle$  |
| $(-3, 0)$  | $\langle 0, -3 \rangle$  |
| $(0, -1)$  | $\langle 1, 0 \rangle$   |
| $(2, -2)$  | $\langle 2, 2 \rangle$   |
| $(0, -3)$  | $\langle 3, 0 \rangle$   |



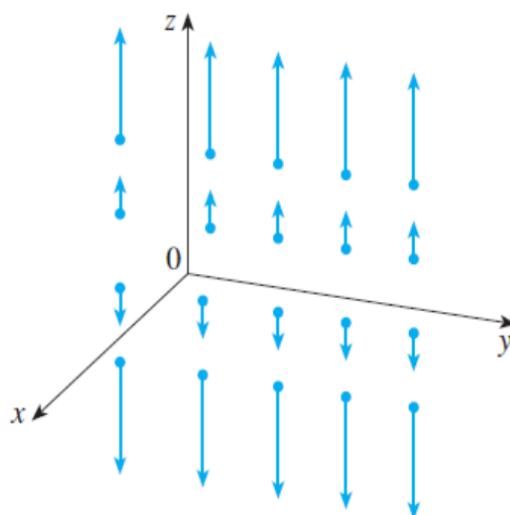
**Example 2**

Sketch the vector fields on  $\mathbb{R}^2$  is defined by

(a)  $\mathbf{F}(x, y) = -y \mathbf{i} + x \mathbf{j}$       (b)  $\mathbf{F}(x, y) = y \mathbf{i} + \sin x \mathbf{j}$       (c)  $\mathbf{F}(x, y) = \ln(1 + y^2) \mathbf{i} + \ln(1 + x^2) \mathbf{j}$

**Answer****Example 3**

Sketch the vector field on  $\mathbb{R}^3$  is defined by  $\mathbf{F}(x, y, z) = z \mathbf{k}$

**Answer**



We were able to draw the vector field in Example 3 by hand because of its particularly simple formula. Most three-dimensional vector fields, however, are virtually impossible to sketch by hand and so we need to resort to a computer algebra system as shown in the next example.

### Example 3

Sketch the vector fields on  $\mathbb{R}^3$  is defined by

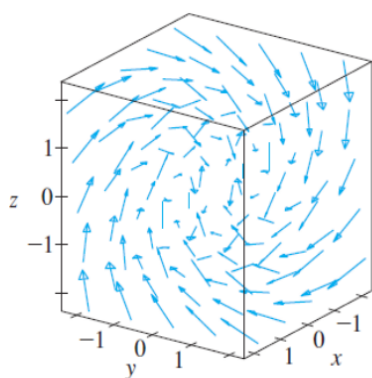
(a)  $\mathbf{F}(x, y, z) = y \mathbf{i} + z \mathbf{j} + x \mathbf{k}$

(b)  $\mathbf{F}(x, y, z) = y \mathbf{i} - 2z \mathbf{j} + x \mathbf{k}$

(c)  $\mathbf{F}(x, y, z) = \frac{y}{z} \mathbf{i} + \frac{x}{z} \mathbf{j} + \frac{z}{4} \mathbf{k}$

### Answer

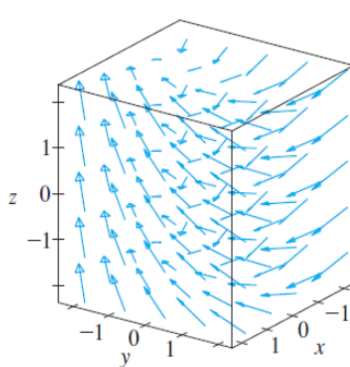
(a)



$$\mathbf{F}(x, y, z) = y \mathbf{i} + z \mathbf{j} + x \mathbf{k}$$

$$\mathbf{F}(x, y, z) = \langle y, z, x \rangle$$

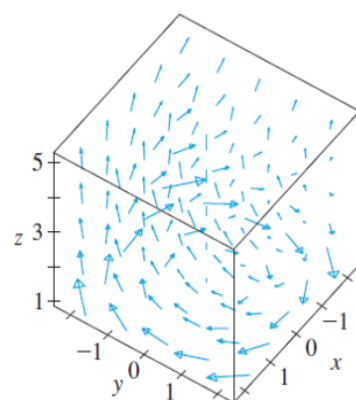
(b)



$$\mathbf{F}(x, y, z) = y \mathbf{i} - 2z \mathbf{j} + x \mathbf{k}$$

$$\mathbf{F}(x, y, z) = \langle y, -2z, x \rangle$$

(c)



$$\mathbf{F}(x, y, z) = \frac{y}{z} \mathbf{i} + \frac{x}{z} \mathbf{j} + \frac{z}{4} \mathbf{k}$$

$$\mathbf{F}(x, y, z) = \left\langle \frac{y}{z}, \frac{x}{z}, \frac{z}{4} \right\rangle$$



## NOTE 7.3.1

We introduce the vector differential (gradient) operator  $\nabla$  ("del") as  $\nabla = \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k}$

- If  $\mathbf{F} = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k}$  is a vector field on  $\mathbb{R}^3$  and derivatives of  $P, Q, R$  and all exist, then the **curl** of  $\mathbf{F}$  is the vector field on  $\mathbb{R}^3$  defined by

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}$$

- If  $f$  is a function of three variables that has continuous second-order partial derivatives, then  $\text{curl } (\nabla f) = \mathbf{0}$
- If  $\mathbf{F}$  is conservative, then  $\text{curl } \mathbf{F} = \mathbf{0}$ .
- If  $\mathbf{F}$  is a vector field defined on all of  $\mathbb{R}^3$  whose component functions have continuous partial derivatives and  $\text{curl } \mathbf{F} = \mathbf{0}$ , then  $\mathbf{F}$  is a conservative vector field.
- If  $\mathbf{F} = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k}$  is a vector field on  $\mathbb{R}^3$  and derivatives of  $P, Q, R$  and all exist, then the **divergence** of  $\mathbf{F}$  is the scalar field on  $\mathbb{R}^3$  defined by

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

## Example 1

- (a) Show that  $\mathbf{F}(x, y, z) = y^2 z^3 \mathbf{i} + 2xyz^3 \mathbf{j} + 3xy^2 z^2 \mathbf{k}$  is a conservative vector field.
- (b) Find a function  $f$  (scalar/ real-valued/ potential function) such that  $\mathbf{F} = \nabla f$ .

## Answer

$$(a) \text{ curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 z^3 & 2xyz^3 & 3xy^2 z^2 \end{vmatrix} = (6xyz^2 - 6xyz^2) \mathbf{i} + (3y^2 z^2 - 3y^2 z^2) \mathbf{j} + (2yz^3 - 2yz^3) \mathbf{k} = \mathbf{0}$$

Since  $\text{curl } \mathbf{F} = \mathbf{0}$  and the domain of  $\mathbf{F}$  is  $\mathbb{R}^3$ ,  $\mathbf{F}$  is a conservative vector field.

$$(b) \text{ we have } f_x(x, y, z) = y^2 z^3 \quad f_y(x, y, z) = 2xyz^3 \quad f_z(x, y, z) = 3xy^2 z^2$$

Integrating  $f_x(x, y, z) = y^2 z^3$  with respect to  $x$ , we obtain

$$f(x, y, z) = xy^2 z^3 + g(y, z)$$

Differentiating with respect to  $y$ , we get  $f_y(x, y, z) = 2xyz^3 + g_y(y, z)$

so, comparison with  $f_y(x, y, z) = 2xyz^3$  gives  $g_y(y, z) = 0$ . Thus  $g(y, z) = h(z)$  and

Differentiating with respect to  $z$ , we get  $f_z(x, y, z) = 3xy^2z^2 + h'(z)$

so, comparison with  $f_z(x, y, z) = 3xy^2z^2$  gives  $h'(z) = 0$ . Therefore,  $f(x, y, z) = xy^2z^3 + c$ ,  $c$  is constant.

### Example 2

If  $\mathbf{F}(x, y, z) = xz \mathbf{i} + xyz \mathbf{j} - y^2 \mathbf{k}$ , find  $\text{div } \mathbf{F}$ .

### Answer

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle xy, xyz, -y^2 \rangle = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(-y^2) = z + xz$$

## Section 7.4 Conservative Fields

### NOTE 7.4.1

If  $\mathbf{F}(x, y, z) = \nabla f(x, y, z) = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}$  in a domain  $D$ , then we can say that  $\mathbf{F}$  is a **conservative vector field** in  $D$ . We call the function  $f$  a (scalar/ real-valued) potential for  $\mathbf{F}$  on  $D$ .

#### Necessary condition for a conservative plane vector field

If  $\mathbf{F}(x, y) = F_1(x, y) \mathbf{i} + F_2(x, y) \mathbf{j}$  is a conservative vector field in a domain  $D$  of  $xy$  -plane, the condition

$$\frac{\partial}{\partial y} F_1(x, y) = \frac{\partial}{\partial x} F_2(x, y) \text{ must satisfied all points of } D.$$

#### Necessary conditions for a conservative vector field in 3-space

If  $\mathbf{F}(x, y, z) = F_1(x, y, z) \mathbf{i} + F_2(x, y, z) \mathbf{j} + F_3(x, y, z) \mathbf{k}$  is a conservative vector field in a domain  $D$  in 3-space, then we must have, everywhere in  $D$ ,

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x} \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$$

Note that: In the next section 7.5, we will show that more than just the above **necessary condition** is sufficient to guarantee that  $\mathbf{F}$  is conservative on  $D$ .

### Example 1

Determine whether or not the vector field  $\mathbf{F} = (3 + 2xy) \mathbf{i} + (x^2 - 3y^2) \mathbf{j}$  is conservative.

If the vector field is conservative, find a function such that  $\mathbf{F} = \nabla f$ .

### Answer

Let  $P(x, y) = 3 + 2xy$  and  $Q(x, y) = x^2 - 3y^2$ . Then  $\frac{\partial P}{\partial y} = 2x = \frac{\partial Q}{\partial x}$

Also, the domain of is the entire plane  $D = \mathbb{R}^2$ , which is open and simply connected. Therefore, we can conclude that the given vector field is conservative.

since  $P(x, y) = 3 + 2xy = F_1$  and  $Q(x, y) = x^2 - 3y^2 = F_2$

Integrating  $P(x, y) = 3 + 2xy = F_1$  with respect to  $x$ , we obtain  $f(x, y) = 3x + x^2y + g(y)$

Differentiating with respect to  $y$ , we get  $f_y(x, y) = x^2 + g'(y)$

so, comparison with  $Q(x, y) = x^2 - 3y^2 = F_2$  gives  $g'(y) = -3y^2$ .

Integrating  $g'(y) = -3y^2$  with respect to  $y$ , we obtain  $g(y) = -y^3 + c$  where  $c$  is a constant

Therefore,  $f(x, y) = 3x + x^2y - y^3 + c$ ,  $c$  is a constant as the desired potential function.

### Example 2

Show that  $\mathbf{F} = (e^x \cos y + yz) \mathbf{i} + (xz - e^x \sin y) \mathbf{j} + (xy + z) \mathbf{k}$  is conservative over its natural domain and find a potential function for it.

### Answer

The natural domain of  $\mathbf{F}$  is all of space, which is connected and simply connected.

We apply the test in Equations  $M = \frac{\partial f}{\partial x} = e^x \cos y + yz$        $N = \frac{\partial f}{\partial y} = xz - e^x \sin y$        $P = \frac{\partial f}{\partial z} = xy + z$

$$\frac{\partial M}{\partial y} = -e^x \sin y + z = \frac{\partial N}{\partial x} \quad \frac{\partial M}{\partial z} = y = \frac{\partial P}{\partial x} \quad \frac{\partial N}{\partial z} = x = \frac{\partial P}{\partial y}$$

The partial derivatives are continuous, so these equalities tell us that  $\mathbf{F}$  is conservative, so there is a function  $f$  with  $\mathbf{F} = \nabla f$

Since  $M = \frac{\partial f}{\partial x} = e^x \cos y + yz$        $N = \frac{\partial f}{\partial y} = xz - e^x \sin y$        $P = \frac{\partial f}{\partial z} = xy + z$

We integrate the first equation with respect to  $x$ , holding  $y$  and  $z$  fixed, to get

$$f(x, y, z) = e^x \cos y + xyz + g(y, z)$$

Differentiating with respect to  $y$ , we get  $f_y(x, y, z) = -e^x \sin y + xz + g_y(y, z)$

so, comparison with  $f_y(x, y, z) = \frac{\partial f}{\partial y} = xz - e^x \sin y$  gives  $g_y(y, z) = 0$ .

Thus  $g(y, z) = h(z)$  that means

$$f(x, y, z) = e^x \cos y + xyz + h(z)$$

Then, differentiating with respect to  $z$ , we get  $f_z(x, y, z) = xy + h'(z)$

Then  $f_z(x, y, z) = \frac{\partial f}{\partial z} = xy + z$  gives  $h'(z) = z$ .

Then, by Integrating  $h'(z) = z$ , we have  $h(z) = \frac{z^2}{2} + c$  where  $c$  is constant.

Therefore,  $f(x, y, z) = e^x \cos y + xyz + \frac{z^2}{2} + c$  where  $c$  is constant.

We have infinitely many potential functions of  $\mathbf{F}$ , one for each value of  $c$ .

## NOTE 7.5.1

## Line Integrals of Vector Fields

- Let  $\mathbf{F}$  be a **vector field** with continuous components defined along a smooth curve  $C$  parametrized by  $\mathbf{r}(t)$ ,  $a \leq t \leq b$ . Then the **line integral** of  $\mathbf{F}$  along  $C$  is

$$\int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_C \left( \mathbf{F} \cdot \frac{d\mathbf{r}}{ds} \right) ds = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

The tangent vector  $\mathbf{T}$  is a unit vector tangent to the path.

- Let  $\mathbf{F}$  be a **force vector field** with continuous components defined along a smooth curve  $C$  parametrized by  $\mathbf{r}(t)$ ,  $a \leq t \leq b$ . Then the **work done** moving an object from the point  $A = \mathbf{r}(a)$  to  $B = \mathbf{r}(b)$  the point **along**  $C$  is

$$W = \int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_a^b \left( \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt} \right) dt$$

## Example 1

Find the work done by the force field  $\mathbf{F} = x^2 \mathbf{i} - xy \mathbf{j}$  in moving a particle along the quarter-circle

$$\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j}, \quad 0 \leq t \leq 2\pi$$

## Answer

Since  $x = \cos t$  and  $y = \sin t$ , we have  $\mathbf{F}(\mathbf{r}(t)) = \cos^2 t \mathbf{i} - \cos t \sin t \mathbf{j}$

and  $\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = -\sin t \mathbf{i} + \cos t \mathbf{j}$

Therefore, the work done is  $\int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_0^{2\pi} \left( \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt} \right) dt = \int_0^{2\pi} (-2 \cos^2 t \sin t) dt = 2 \left. \frac{\cos^3 t}{3} \right|_0^{2\pi} = -\frac{2}{3}$

## NOTE 7.5.2

Even though  $\int_C \mathbf{F} \cdot \mathbf{T} ds = \int_C \mathbf{F} \cdot d\mathbf{r}$  and integrals with respect to arc length are unchanged when orientation is reversed, it is still true that

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = - \int_C \mathbf{F} \cdot d\mathbf{r}$$

because the unit tangent vector  $\mathbf{T}$  is replaced by its negative when  $C$  is replaced by  $-C$ .

## NOTE 7.5.3

**Independence of Path**

- Let  $D$  be an open, connected domain, let  $\mathbf{F}$  be a smooth vector field defined on  $D$ . Then the following three statements are equivalent in the sense that, if any one of them is true, so are the other two:
  - (a) Given any two points  $P_0$  and  $P_1$  in  $D$ ,  $\int_C \mathbf{F} \cdot d\mathbf{r}$  has the same value for all piecewise smooth curves in  $D$  starting at  $P_0$  and ending at  $P_1$ .
  - (b)  $\mathbf{F}$  is conservative in  $D$ .
  - (c)  $\oint \mathbf{F} \cdot d\mathbf{r} = 0$  for every piecewise smooth, closed curve  $C$  in  $D$ .
- The following diagram summarizes the results of Independence of Path.

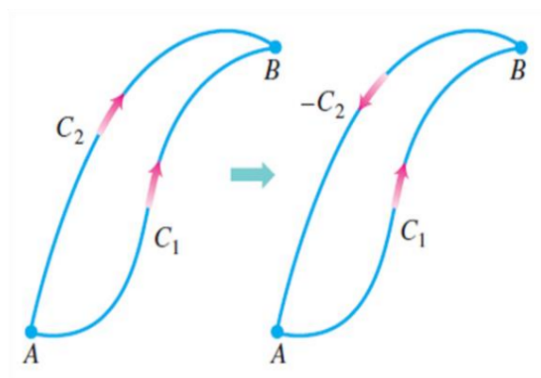
$$\mathbf{F} = \nabla f \text{ on } D \quad \Leftrightarrow \quad \mathbf{F} \text{ is conservative in } D \quad \Leftrightarrow \quad \oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \text{ over any loop in } D$$

- Fundamental Theorem of Line Integrals: Let  $C$  be a smooth curve joining the point  $A$  to the point  $B$  in the plane or in space and parametrized by  $\mathbf{r}(t)$ . Also,  $\mathbf{F}$  is conservative in  $D$ .

$$\text{Then, we have } \int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A)$$

 NOTE 7.5.4

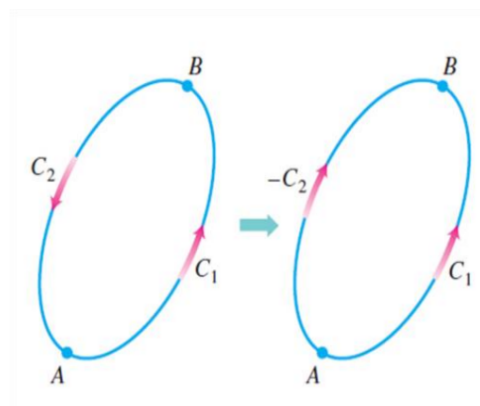
### Loop Property of Conservative Fields



If we have two paths from A to B, one of them can be reversed to make a loop.

$C_1$  and  $-C_2$  make a closed loop.

$$\begin{aligned}\int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r} &= \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} \\ &= \int_C \mathbf{F} \cdot d\mathbf{r} = 0\end{aligned}$$

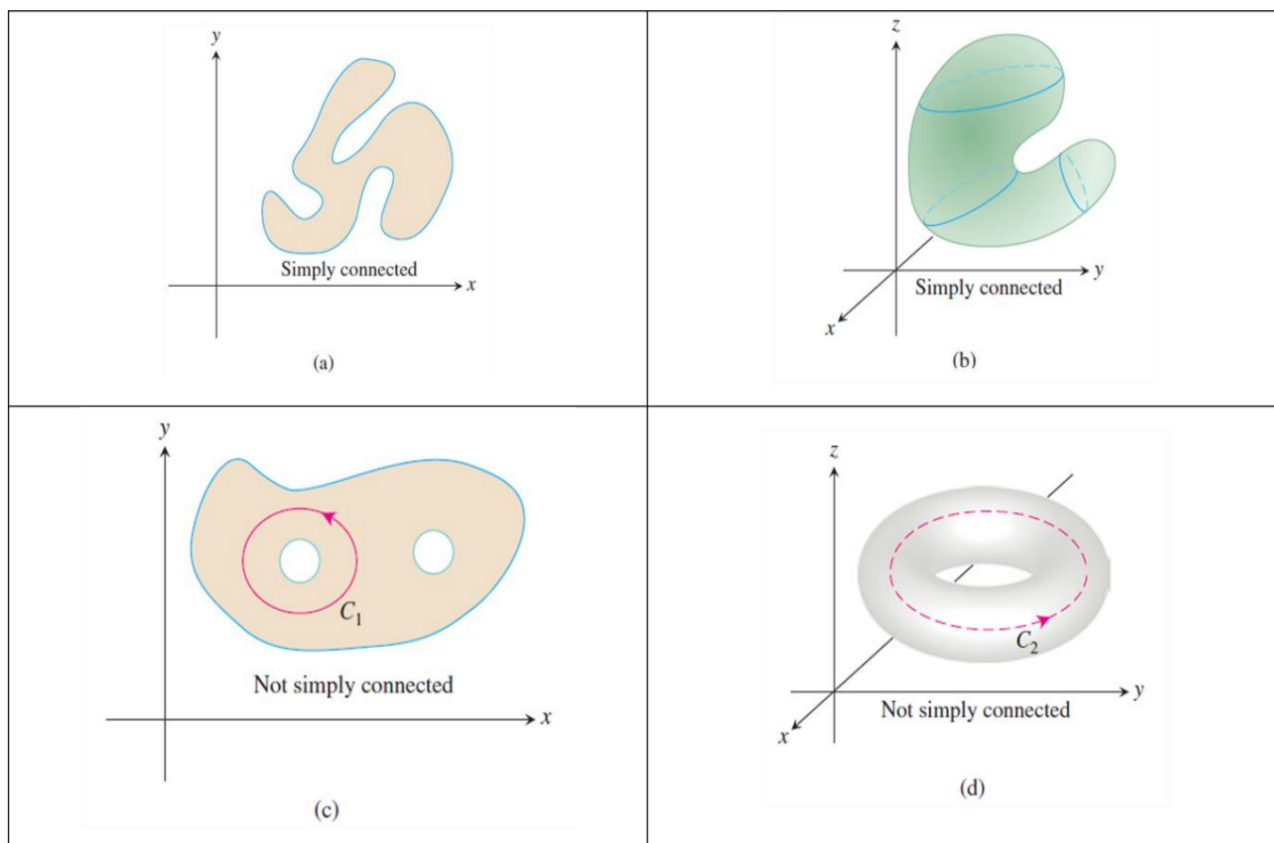


If A and B lie on a loop, we can reverse part of the loop to make two paths from A to B.

$C_1$  and  $C_2$  make a closed loop.

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} \\ &= \int_A^B \mathbf{F} \cdot d\mathbf{r} - \int_A^B \mathbf{F} \cdot d\mathbf{r} = 0\end{aligned}$$

## NOTE 7.5.5



In (c) and (d), the regions are not simply connected because the curves cannot be contracted to a point inside the regions containing them.

**Caution** the test for conservative fields is not valid on domains that are not simply connected (see Example 2).

## Example 2

Show that the vector field  $\mathbf{F} = \frac{-y}{x^2+y^2} \mathbf{i} + \frac{x}{x^2+y^2} \mathbf{j}$  is not conservative over its natural domain. Explain why this is possible.

## Answer

We apply the test in Equations  $M = \frac{\partial f}{\partial x} = \frac{-y}{x^2+y^2}$        $N = \frac{\partial f}{\partial y} = \frac{x}{x^2+y^2}$

$$\frac{\partial M}{\partial y} = \frac{y^2 - x^2}{(x^2+y^2)^2} = \frac{\partial N}{\partial x} \quad \text{for } (x, y) \neq (0, 0)$$

However, the conservative test assumes that the domain of  $\mathbf{F}$  is simply connected, which is not the case.

Since  $x^2 + y^2$  cannot equal zero, the natural domain is the complement of the  $z$ -axis and contains loops that cannot be contracted to a point.

One such loop is the unit circle  $\mathcal{C}$  in the  $xy$ -plane. The circle is parametrized by

$$\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} \quad 0 \leq t \leq 2\pi$$

To show that  $\mathbf{F}$  is not conservative, we compute the line integral  $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$  around the loop  $\mathcal{C}$ . First, we write the field in terms of the parameter  $t$ :



$$\mathbf{F} = \frac{-y}{x^2 + y^2} \mathbf{i} + \frac{x}{x^2 + y^2} \mathbf{j} = \frac{-\sin t}{\sin^2 t + \cos^2 t} \mathbf{i} + \frac{\cos t}{\sin^2 t + \cos^2 t} \mathbf{j} = -\sin t \mathbf{i} + \cos t \mathbf{j}$$

Next, we find  $d\mathbf{r}/dt = -\sin t \mathbf{i} + \cos t \mathbf{j}$  and then calculate the line integral as

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \left( \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} \right) dt = \int_0^{2\pi} \sin^2 t + \cos^2 t \, dt = 2\pi \neq 0$$

Since the line integral of  $\mathbf{F}$  around the loop  $C$  is not zero, the field  $\mathbf{F}$  is not conservative over its natural domain  $\mathbb{R}^2$

### NOTE 7.5.6

Example 2 shows that the Conservative Test does not apply when the domain of the field is not simply connected. However, if we change the domain in the example so that it is restricted to the ball of radius 1 centered at the point  $(2, 2, 2)$ , or to any similar ball-shaped region which does not contain a piece of the  $z$ -axis, then this new domain  $D$  is simply connected.

### Example 3

For what values of the constants  $A$  and  $B$  are the vector field

$$\mathbf{F} = Ax \sin(\pi y) \mathbf{i} + (x^2 \cos(\pi y) + Bye^{-z}) \mathbf{j} + y^2 e^{-z} \mathbf{k} \quad \text{conservative?}$$

For this choice of  $A$  and  $B$ , evaluate  $\int_C \mathbf{F} \cdot d\mathbf{r}$ , where  $C$  is

(a) the curve  $\mathbf{r}(t) = \cos t \mathbf{i} + \sin 2t \mathbf{j} + \sin^2 t \mathbf{k} \quad 0 \leq t \leq 2\pi$

(b) the curve of intersection of the paraboloid  $z = x^2 + 4y^2$  and the plane  $z = 3x - 2y$  from  $(0, 0, 0)$  to  $(1, \frac{1}{2}, 2)$ .

### Answer

If  $\mathbf{F}$  cannot be conservative unless

The natural domain of  $\mathbf{F}$  is all of space, which is connected and simply connected.

We apply the test in Equations  $M = \frac{\partial f}{\partial x} = Ax \sin(\pi y) \quad N = \frac{\partial f}{\partial y} = x^2 \cos(\pi y) + Bye^{-z} \quad P = \frac{\partial f}{\partial z} = y^2 e^{-z}$

$$\frac{\partial M}{\partial y} = A\pi x \cos(\pi y) = 2x \cos(\pi y) = \frac{\partial N}{\partial x} \quad \frac{\partial M}{\partial z} = 0 = 0 = \frac{\partial P}{\partial x} \quad \frac{\partial N}{\partial z} = -Bye^{-z} = 2ye^{-z} = \frac{\partial P}{\partial y}$$

$$A = \frac{2}{\pi} \text{ and } B = -2 \text{ makes the vector field conservative}$$

Since the  $\mathbf{F}$  is conservative, that means there is a function  $f$  with  $\mathbf{F} = \nabla f$

Since  $M = \frac{\partial f}{\partial x} = \frac{2}{\pi} x \sin(\pi y) \quad N = \frac{\partial f}{\partial y} = x^2 \cos(\pi y) - 2ye^{-z} \quad P = \frac{\partial f}{\partial z} = y^2 e^{-z}$

We integrate the second equation with respect to  $y$ , holding  $x$  and  $z$  fixed, to get

$$f(x, y, z) = \frac{x^2 \sin(\pi y)}{\pi} - y^2 e^{-z} + g(x, z)$$



Differentiating with respect to  $x$ , we get  $f_x(x, y, z) = \frac{2x \sin(\pi y)}{\pi} + g_x(x, z)$

so, comparison with  $f_x(x, y, z) = \frac{\partial f}{\partial x} = \frac{2}{\pi} x \sin(\pi y)$  gives  $g_x(x, z) = 0$ .

Thus  $g(x, z) = h(z)$  that means

$$f(x, y, z) = \frac{x^2 \sin(\pi y)}{\pi} - y^2 e^{-z} + h(z)$$

Then, differentiating with respect to  $z$ , we get  $f_z(x, y, z) = y^2 e^{-z} + h'(z)$

Then  $f_z(x, y, z) = \frac{\partial f}{\partial z} = y^2 e^{-z}$  gives  $h'(z) = 0$ .

Then, by Integrating  $h'(z) = 0$ , we have  $h(z) = c$  where  $c$  is constant.

$$\text{Therefore, } f(x, y, z) = \frac{x^2 \sin(\pi y)}{\pi} - y^2 e^{-z} + c \quad \text{where } c \text{ is constant.}$$

(a) the curve  $\mathbf{r}(t) = \cos t \mathbf{i} + \sin 2t \mathbf{j} + \sin^2 t \mathbf{k} \quad 0 \leq t \leq 2\pi$

We have  $\mathbf{r}(0) = \mathbf{r}(2\pi) = \mathbf{i}$ , then this curve is a closed curve. Thus,  $\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \nabla f \cdot d\mathbf{r} = 0$

(b) the curve of intersection of the paraboloid  $z = x^2 + 4y^2$  and the plane  $z = 3x - 2y$  from  $(0,0,0)$  to  $(1, \frac{1}{2}, 2)$ .

$$\text{Thus, } \int_C \mathbf{F} \cdot d\mathbf{r} = f(0,0,0) - f\left(1, \frac{1}{2}, 2\right) = \frac{x^2 \sin(\pi y)}{\pi} - y^2 e^{-z} \Big|_{(0,0,0)}^{\left(1, \frac{1}{2}, 2\right)} = \frac{1}{\pi} - \frac{1}{4e^2}$$

#### Example 4

Evaluate  $\oint_C (e^x \sin y + 3y)dx + (e^x \cos y + 2x - 2y)dy$  counterclockwise around the ellipse  $4x^2 + y^2 = 4$

#### Answer

$$I = \oint_C \nabla f \cdot d\mathbf{r} \text{ where the vector field is } \mathbf{F} = \underbrace{(e^x \sin y + 3y)}_{F_1} \mathbf{i} + \underbrace{(e^x \cos y + 2x - 2y)}_{F_2} \mathbf{j}$$

This vector field is not conservative, but it would be if the  $3y$  term in  $F_1$  were  $2y$  instead

$$I = \oint_C \underbrace{(e^x \sin y + 2y)dx + (e^x \cos y + 2x - 2y)dy}_{\text{conservative part}} + \underbrace{\oint_C y dx}_{\text{nonconservative part}} = 0 + \oint_C y dx$$

The first integral is zero since its vector field is conservative and  $C$  is closed.

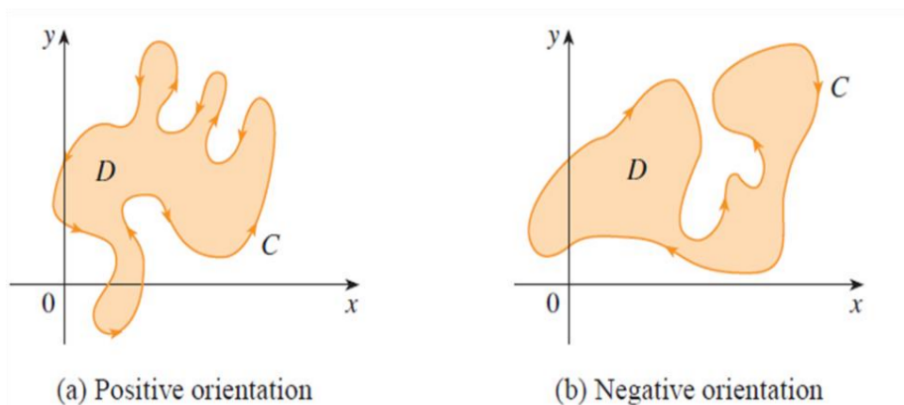
For the second integral, we parametrize  $C$  the ellipse  $4x^2 + y^2 = 4 \rightarrow x^2 + \frac{y^2}{4} = 1$  by

$$x = \cos t, y = 2 \sin t \quad 0 \leq t \leq 2\pi \quad dx = -\sin t$$

$$I = \oint_C y dx = -2 \int_0^{2\pi} \sin^2 t dt = -2 \int_0^{2\pi} \frac{1 - \cos(2t)}{2} dt = -\left(t - \frac{\sin(2t)}{2}\right) \Big|_0^{2\pi} = -2\pi$$

## NOTE 8.1.1

**Green's Theorem** gives the relationship between a line integral around a simple closed curve  $C$  and a double integral over the plane region  $D$  bounded by  $C$ . In stating Green's Theorem, we use the convention that the positive orientation of a simple closed curve  $C$  refers to a single counterclockwise traversal of  $C$ .



**Green's Theorem** Let  $C$  be a positively oriented, piecewise-smooth, regular simple closed curve in the plane and let  $D$  be the region bounded by  $C$ . If  $P$  and  $Q$  have continuous partial derivatives on an open region that contains  $D$ , then

$$\oint_C P dx + Q dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

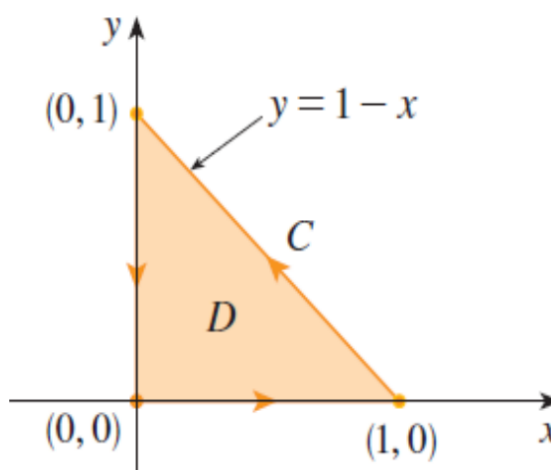
## Example 1

Evaluate  $\int_C x^4 dx + xy dy$ , where  $C$  is the triangular curve consisting of the line segments from  $(0,0)$  to  $(1,0)$ , from  $(1,0)$  to  $(0,1)$ , and from  $(0,1)$  to  $(0,0)$ .

## Answer

By Green's Theorem, we have that the region  $D$  enclosed by  $C$  is simple and  $C$  has positive orientation (see Figure on the right). If we let  $P(x, y) = x^4$  and  $Q(x, y) = xy$ , then we have

$$\begin{aligned} \oint_C x^4 dx + xy dy &= \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \\ &= \int_0^1 \int_0^{1-x} (y - 0) dy dx = \int_0^1 \left[ \frac{1}{2} y^2 \right]_{y=0}^{y=1-x} dx \\ &= \frac{1}{2} \int_0^1 (1-x)^2 dx = \left[ -\frac{(1-x)^3}{6} \right]_0^1 = \frac{1}{6} \end{aligned}$$



**Example 2**

Evaluate  $\oint_C (3y - e^{\sin x}) dx + (7x + \sqrt{y^4 + 1}) dy$ , where  $C$  is the circle  $x^2 + y^2 = 9$

**Answer**

The region  $D$  bounded by  $C$  is the disk  $x^2 + y^2 \leq 9$ , so we can apply Green's Theorem:

If we let  $P(x, y) = 3y - e^{\sin x}$  and  $Q(x, y) = 7x + \sqrt{y^4 + 1}$ , then we have

$$\begin{aligned} \oint_C (3y - e^{\sin x}) dx + (7x + \sqrt{y^4 + 1}) dy &= \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \\ &= \iint_D (7 - 3) dA = 4 \iint_D dA = \text{area of } D = 4 \cdot \pi(3)^2 = 36\pi \end{aligned}$$

**NOTE 8.1.2**

Another application of  $D$  the reverse direction of **Green's Theorem** is in computing **areas**.

Since the **area** of  $D$  is  $\iint_D 1 dA$ , we wish to choose  $P$  and  $Q$  so that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

**Example 3**

Evaluate  $\oint_C y^2 dx + 3xy dy$ , where  $C$  is the boundary of the semi-annular region  $D$  in the upper half-plane between the circles  $x^2 + y^2 = 1$  and  $x^2 + y^2 = 4$ .

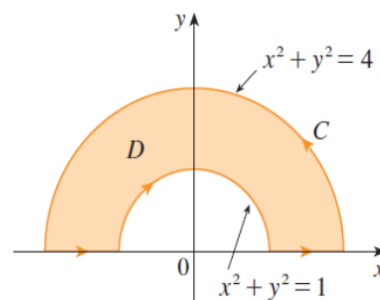
**Answer**

Notice that although is not simple, the  $y$ -axis divides it into two simple regions (see Figure below). In polar coordinates we can write

$$D = \{(r, \theta) | 1 \leq r \leq 2, 0 \leq \theta \leq \pi\}$$

Therefore, Green's Theorem gives

$$\begin{aligned} \oint_C y^2 dx + 3xy dy &= \iint_D \left( \frac{\partial}{\partial x} (3xy) - \frac{\partial P}{\partial y} (y^2) \right) dA \\ &= \iint_D y dA = \int_0^\pi \int_1^2 (r \sin \theta) r dr d\theta = [-\cos \theta]_0^\pi \left[ \frac{1}{3} r^3 \right]_1^2 \\ &= \frac{14}{3} \end{aligned}$$



**Example 4**

If  $\mathbf{F}(x, y) = \frac{-y}{x^2+y^2} \mathbf{i} + \frac{x}{x^2+y^2} \mathbf{j}$ , show that  $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$  for every positively oriented simple closed path that encloses the origin.

**Answer**

Let  $D$  be the region bounded by  $C$  and  $C'$ . Then its positively oriented boundary is  $C \cup (-C')$  and so the general version of Green's Theorem gives

$$\int_C P dx + Q dy + \int_{-C'} P dx + Q dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_D \left( \frac{y^2 - x^2}{(x^2 + y^2)^2} - \frac{y^2 - x^2}{(x^2 + y^2)^2} \right) dA = 0$$

Therefore  $\int_C P dx + Q dy = \int_{C'} P dx + Q dy$

that is,  $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r}$

We now easily compute this last integral using the parametrization given by

$$\mathbf{r}(t) = a \cos t \mathbf{i} + a \sin t \mathbf{j}, \quad 0 \leq t \leq 2\pi$$

Since  $C$  is an *arbitrary* closed path that encloses the origin, it's difficult to compute the given integral directly. So, let's consider a counterclockwise-oriented circle with center the origin and radius  $a$ , where  $a$  is chosen to be small enough that  $C'$  lies inside  $C$ . (See Figure below.)

Thus,  $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$

$$= \int_0^{2\pi} \frac{(-a \sin t)(-a \sin t) + (a \cos t)(a \cos t)}{a^2 \cos^2 t + a^2 \sin^2 t} dt = \int_0^{2\pi} dt = 2\pi.$$

