

Econ 205

Multivariate (Joint) Probability Distributions – 1

Course Outline

- **Probability**

- Probability and Inference
- The Discrete Case
- Calculating the Probability of an Event
- Conditional Probability and the Independence of Events
- Laws of Probability
- The Law of Total Probability and Bayes' Rule
- Introduction to Random Variables and their distributions

- **Discrete Random Variables and Their Probability Distributions**

- Discrete Random Variable
- The Probability Distribution for Discrete Random Variable
- The Expected Value of Random Variable
- Moments and Moment-Generating Functions
- The Bernoulli Probability Distribution
- The Binomial Probability Distribution
- The Geometric Probability Distribution
- The Negative Binomial Probability Distribution
- The Hypergeometric Probability Distribution
- The Poisson Probability Distribution
- Probability-Generating Functions

- **Continuous Random Variables and Their Probability Densities**

- Continuous Random Variable
- The Probability Distribution for Continuous Random Variable
- The Expected Value for Continuous Random Variable
- Moments and Moment-Generating Functions
- The Uniform Probability Distribution
- The Normal Probability Distribution
- Exponential Probability Distribution
- Chi-Square Probability Distribution
- Tchebysheff's Theorem

Course Outline (cont'd)

- **Multivariate Probability Distributions**

- Bivariate and Multivariate Probability Distributions
- Independent Random Variables
- Marginal and Conditional Probability Distributions
- The Expected Value of a Function of Random Variables
- Special Theorems
- The Covariance of Two Random Variables
- The Expected Value and Variance of Linear Functions of Random Variables
- The Multinomial Probability Distribution
- The Bivariate Normal Distribution
- Conditional Expectations

- **Functions of Random Variables**

- Finding the Probability Distribution of a Function of Random Variables
- The Method of Distribution Functions
- The Method of Transformations
- The Method of Moment-Generating Functions
- Multivariable Transformations Using Jacobians

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Content of this presentation

- Bivariate and Multivariate (Joint) Probability Distributions
 - Joint Probability Mass Functions
 - Joint Probability Density Functions
 - Joint Cumulative Distribution Functions
- Marginal Probability Distributions
 - Marginal Probability Mass functions
 - Marginal Probability Density functions
- Independent Random Variables
- Examples

Joint Probability Mass Functions

(Discrete) Bivariate case: $p(x,y)$

Let X and Y be two discrete rv's defined on the sample space $\mathcal{S}$ of an experiment. The **joint probability mass function** $p(x,y)$ is defined for each pair of numbers (x, y) by

$$p(x,y) = P(X = x \text{ and } Y = y)$$

Let A be any set consisting of pairs of (x, y) values. Then the probability that the random pair (X, Y) lies in A is obtained by summing the joint pmf over pairs in A :

$$P[(X, Y) \in A] = \sum_{(x,y) \in A} p(x,y)$$

Compare the joint case with the earlier case: (Discrete) Univariate case: $p(x)$

The **probability distribution** or **probability mass function** (pmf) of a discrete rv is defined for every number x by $p(x) = P(X = x) = P(\text{all } s \in \mathcal{S}: X(s) = x)$.¹

Let A be any set consisting of x values. Then the probability that x lies in A is obtained by summing the pmf over x in A :

$$P[X \in A] = \sum_{x \in A} p(x)$$

Joint Probability Mass Functions

(Discrete) Multivariate case: $p(x_1, x_2, \dots, x_n)$

If $X_1, X_2, \dots, X_n$ are all discrete random variables, the **joint pmf** of the variables is the function

$$p(x_1, x_2, \dots, x_n) = P(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n)$$

Joint Probability Density Functions

(Continuous) Bivariate case: $f(x,y)$

Let X and Y be continuous rv's. Then $f(x, y)$ is the **joint probability density function** for X and Y if for any two-dimensional set A

$$P[(X, Y) \in A] = \iint_A f(x, y) dx dy$$

In particular, if A is the two-dimensional rectangle $\{(x, y) : a \leq x \leq b, c \leq y \leq d\}$, then

$$P[(X, Y) \in A] = P(a \leq X \leq b, c \leq Y \leq d) = \int_a^b \int_c^d f(x, y) dy dx$$

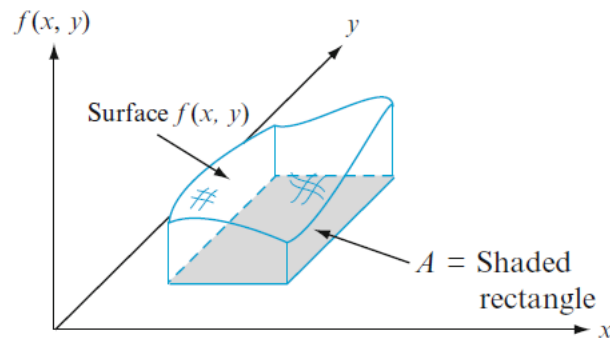


Figure 5.1 $P[(X, Y) \in A] = \text{volume under density surface above } A$

Compare the joint case with the earlier case: (Continuous) Univariate case: $f(x)$

Let X be a continuous rv. Then a **probability distribution** or **probability density function** (pdf) of X is a function $f(x)$ such that for any two numbers a and b with $a \leq b$,

$$P(a \leq X \leq b) = \int_a^b f(x) dx$$

Joint Probability Density Functions

(Continuous) Multivariate case: $f(x_1, x_2, \dots, x_n)$

If the variables are continuous, the **joint pdf** of $X_1, X_2, \dots, X_n$ is the function $f(x_1, x_2, \dots, x_n)$ such that for any n intervals $[a_1, b_1], \dots, [a_n, b_n]$,

$$P(a_1 \leq X_1 \leq b_1, \dots, a_n \leq X_n \leq b_n) = \int_{a_1}^{b_1} \dots \int_{a_n}^{b_n} f(x_1, \dots, x_n) dx_n \dots dx_1$$

The Joint Cumulative Distribution function

The Joint (Bivariate) Cumulative Distribution function

The **joint cumulative distribution function** of two random variables X and Y , denoted by $F(x, y)$, is defined by

$$F(x, y) = P[(X \leq x) \cap (Y \leq y)]$$
$$-\infty < x < \infty, \quad -\infty < y < \infty$$

Compare the joint case with the earlier univariate cases:

The **cumulative distribution function** (cdf) $F(x)$ of a discrete rv X with pmf $p(x)$ is defined for every number x by

$$F(x) = P(X \leq x) = \sum_{y: y \leq x} p(y) \quad (3.3)$$

For any number x , $F(x)$ is the probability that the observed value of X will be at most x .

The **cumulative distribution function** $F(x)$ for a continuous rv X is defined for every number x by

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(y) dy$$

For each x , $F(x)$ is the area under the density curve to the left of x .

Marginal Probability Distributions (Bivariate case)

Discrete case: Marginal pmf s

The **marginal probability mass functions** of X and of Y , denoted by $p_X(x)$ and $p_Y(y)$, respectively, are given by

$$p_X(x) = \sum_y p(x, y) \quad p_Y(y) = \sum_x p(x, y)$$

Continuous case: Marginal pdf s

The **marginal probability density functions** of X and Y , denoted by $f_X(x)$ and $f_Y(y)$, respectively, are given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy \quad \text{for } -\infty < x < \infty$$
$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx \quad \text{for } -\infty < y < \infty$$

Independent Random Variables

Independence of random variables

Two random variables X and Y are said to be **independent** if for every pair of x and y values,

$$p(x, y) = p_X(x) \cdot p_Y(y) \quad \text{when } X \text{ and } Y \text{ are discrete}$$

or

(5.1)

$$f(x, y) = f_X(x) \cdot f_Y(y) \quad \text{when } X \text{ and } Y \text{ are continuous}$$

If (5.1) is not satisfied for all (x, y) , then X and Y are said to be **dependent**.

The random variables $X_1, X_2, \dots, X_n$ are said to be **independent** if for every subset $X_{i_1}, X_{i_2}, \dots, X_{i_k}$ of the variables (each pair, each triple, and so on), the joint pmf or pdf of the subset is equal to the product of the marginal pmf's or pdf's.

Earlier (in Chapter 2): Independence of events

Two events A and B are **independent** if $P(A | B) = P(A)$ and are **dependent** otherwise.

A and B are independent if and only if

$$P(A \cap B) = P(A) \cdot P(B)$$

Events $A_1, \dots, A_n$ are **mutually independent** if for every k ($k = 2, 3, \dots, n$) and every subset of indices $i_1, i_2, \dots, i_k$,

$$P(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}) = P(A_{i_1}) \cdot P(A_{i_2}) \cdot \dots \cdot P(A_{i_k})$$

EXAMPLES

Example 1: Joint Probability Mass Functions

EXAMPLE 5.1 A local supermarket has three checkout counters. Two customers arrive at the counters at different times when the counters are serving no other customers. Each customer chooses a counter at random, independently of the other. Let Y_1 denote the number of customers who choose counter 1 and Y_2 , the number who select counter 2. Find the joint probability function of Y_1 and Y_2 .

Solution We might proceed with the derivation in many ways. The most direct is to consider the sample space associated with the experiment. Let the pair $\{i, j\}$ denote the simple event that the first customer chose counter i and the second customer chose counter j , where $i, j = 1, 2$, and 3 . Using the *mn* rule, the sample space consists of $3 \times 3 = 9$ sample points. Under the assumptions given earlier, each sample point is equally likely and has probability $1/9$. The sample space associated with the experiment is

$$S = [\{1, 1\}, \{1, 2\}, \{1, 3\}, \{2, 1\}, \{2, 2\}, \{2, 3\}, \{3, 1\}, \{3, 2\}, \{3, 3\}].$$

Notice that sample point $\{1, 1\}$ is the only sample point corresponding to $(Y_1 = 2, Y_2 = 0)$ and hence $P(Y_1 = 2, Y_2 = 0) = 1/9$. Similarly, $P(Y_1 = 1, Y_2 = 1) = P(\{1, 2\} \text{ or } \{2, 1\}) = 2/9$. Table 5.1 contains the probabilities associated with each possible pair of values for Y_1 and Y_2 —that is, the joint probability function for Y_1 and Y_2 .

Table 5.1 Probability function for Y_1 and Y_2 , Example 5.1

y_2	y_1		
	0	1	2
0	1/9	2/9	1/9
1	2/9	2/9	0
2	1/9	0	0

Definition of joint pmf: $p(x,y)$

Let X and Y be two discrete rv's defined on the sample space $\mathcal{S}$ of an experiment. The **joint probability mass function** $p(x,y)$ is defined for each pair of numbers (x, y) by

$$p(x,y) = P(X = x \text{ and } Y = y)$$



Example 2: The Joint Cumulative Distribution function (for the discrete case)

EXAMPLE 5.2 Consider the random variables Y_1 and Y_2 of Example 5.1. Find $F(-1, 2)$, $F(1.5, 2)$, and $F(5, 7)$.

Solution Using the results in Table 5.1, we see that

$$F(-1, 2) = P(Y_1 \leq -1, Y_2 \leq 2) = P(\emptyset) = 0.$$

Further,

$$\begin{aligned} F(1.5, 2) &= P(Y_1 \leq 1.5, Y_2 \leq 2) \\ &= p(0, 0) + p(0, 1) + p(0, 2) + p(1, 0) + p(1, 1) + p(1, 2) = 8/9. \end{aligned}$$

Similarly,

$$F(5, 7) = P(Y_1 \leq 5, Y_2 \leq 7) = 1.$$

Notice that $F(y_1, y_2) = 1$ for all y_1, y_2 such that $\min\{y_1, y_2\} \geq 2$. Also, $F(y_1, y_2) = 0$ if $\min\{y_1, y_2\} < 0$. ■

Definition of the Joint (Bivariate) Cumulative Distribution function

The **joint cumulative distribution function** of two random variables X and Y , denoted by $F(x, y)$, is defined by

$$\begin{aligned} F(x, y) &= P[(X \leq x) \cap (Y \leq y)] \\ &\quad -\infty < x < \infty, \quad -\infty < y < \infty \end{aligned}$$

Example 3: Joint Probability Density Functions

Example 5.3

A bank operates both a drive-up facility and a walk-up window. On a randomly selected day, let X = the proportion of time that the drive-up facility is in use (at least one customer is being served or waiting to be served) and Y = the proportion of time that the walk-up window is in use. Then the set of possible values for (X, Y)

is the rectangle $D = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$. Suppose the joint pdf of (X, Y) is given by

$$f(x, y) = \begin{cases} \frac{6}{5}(x + y^2) & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

To verify that this is a legitimate pdf, note that $f(x, y) \geq 0$ and

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy &= \int_0^1 \int_0^1 \frac{6}{5}(x + y^2) dx dy \\ &= \int_0^1 \int_0^1 \frac{6}{5}x dx dy + \int_0^1 \int_0^1 \frac{6}{5}y^2 dx dy \\ &= \int_0^1 \frac{6}{5}x dx + \int_0^1 \frac{6}{5}y^2 dy = \frac{6}{10} + \frac{6}{15} = 1 \end{aligned}$$

The probability that neither facility is busy more than one-quarter of the time is

$$\begin{aligned} P\left(0 \leq X \leq \frac{1}{4}, 0 \leq Y \leq \frac{1}{4}\right) &= \int_0^{1/4} \int_0^{1/4} \frac{6}{5}(x + y^2) dx dy \\ &= \frac{6}{5} \int_0^{1/4} \int_0^{1/4} x dx dy + \frac{6}{5} \int_0^{1/4} \int_0^{1/4} y^2 dx dy \\ &= \frac{6}{20} \cdot \frac{x^2}{2} \Big|_{x=0}^{x=1/4} + \frac{6}{20} \cdot \frac{y^3}{3} \Big|_{y=0}^{y=1/4} = \frac{7}{640} \\ &= .0109 \end{aligned}$$

Definition of the joint (bivariate) pdf: $f(x, y)$

Let X and Y be continuous rv's. Then $f(x, y)$ is the **joint probability density function** for X and Y if for any two-dimensional set A

$$P[(X, Y) \in A] = \iint_A f(x, y) dx dy$$

In particular, if A is the two-dimensional rectangle $\{(x, y) : a \leq x \leq b, c \leq y \leq d\}$, then

$$P[(X, Y) \in A] = P(a \leq X \leq b, c \leq Y \leq d) = \int_a^b \int_c^d f(x, y) dy dx$$

Example 4: Joint Probability Density Functions

EXAMPLE 5.4 Gasoline is to be stocked in a bulk tank once at the beginning of each week and then sold to individual customers. Let Y_1 denote the proportion of the capacity of the bulk tank that is available after the tank is stocked at the beginning of the week. Because of the limited supplies, Y_1 varies from week to week. Let Y_2 denote the proportion of the capacity of the bulk tank that is sold during the week. Because Y_1 and Y_2 are both proportions, both variables take on values between 0 and 1. Further, the amount sold, y_2 , cannot exceed the amount available, y_1 . Suppose that the joint density function for Y_1 and Y_2 is given by

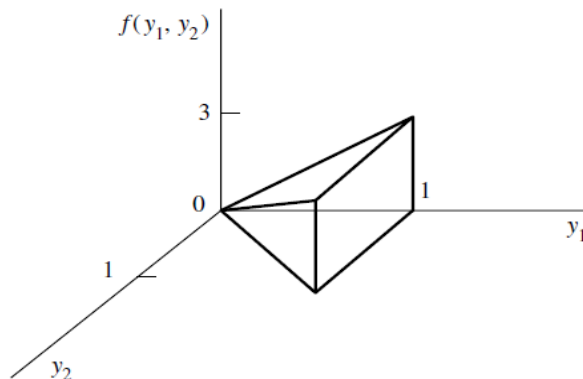
$$f(y_1, y_2) = \begin{cases} 3y_1, & 0 \leq y_2 \leq y_1 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

A sketch of this function is given in Figure 5.4.

Find the probability that less than one-half of the tank will be stocked and more than one-quarter of the tank will be sold.

Solution We want to find $P(0 \leq Y_1 \leq .5, Y_2 > .25)$. For any continuous random variable, the probability of observing a value in a region is the volume under the density function above the region of interest. The density function $f(y_1, y_2)$ is positive only in the

FIGURE 5.4
The joint density
function for
Example 5.4



Definition of the joint (bivariate) pdf: $f(x,y)$

Let X and Y be continuous rv's. Then $f(x, y)$ is the **joint probability density function** for X and Y if for any two-dimensional set A

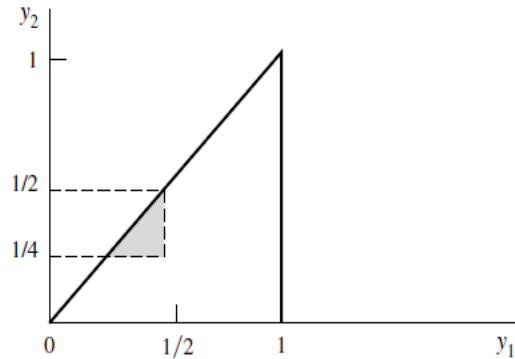
$$P[(X, Y) \in A] = \iint_A f(x, y) dx dy$$

In particular, if A is the two-dimensional rectangle $\{(x, y) : a \leq x \leq b, c \leq y \leq d\}$, then

$$P[(X, Y) \in A] = P(a \leq X \leq b, c \leq Y \leq d) = \int_a^b \int_c^d f(x, y) dy dx$$

Example 4 (cont'd):

FIGURE 5.5
Region of integration
for Example 5.4



large triangular portion of the (y_1, y_2) plane shown in Figure 5.5. We are interested only in values of y_1 and y_2 such that $0 \leq y_1 \leq .5$ and $y_2 > .25$. The intersection of this region and the region where the density function is positive is given by the small (shaded) triangle in Figure 5.5. Consequently, the probability we desire is the volume under the density function of Figure 5.4 above the shaded region in the (y_1, y_2) plane shown in Figure 5.5.

Thus, we have

$$\begin{aligned}
 P(0 \leq Y_1 \leq .5, .25 \leq Y_2) &= \int_{1/4}^{1/2} \int_{1/4}^{y_1} 3y_1 dy_2 dy_1 \\
 &= \int_{1/4}^{1/2} 3y_1 \left(y_2 \right)_{1/4}^{y_1} dy_1 \\
 &= \int_{1/4}^{1/2} 3y_1 (y_1 - 1/4) dy_1 \\
 &= \left[y_1^3 - (3/8)y_1^2 \right]_{1/4}^{1/2} \\
 &= [(1/8) - (3/8)(1/4)] - [(1/64) - (3/8)(1/16)] \\
 &= 5/128.
 \end{aligned}$$

Definition of the joint (bivariate) pdf: $f(x,y)$

Let X and Y be continuous rv's. Then $f(x, y)$ is the **joint probability density function** for X and Y if for any two-dimensional set A

$$P[(X, Y) \in A] = \iint_A f(x, y) dx dy$$

In particular, if A is the two-dimensional rectangle $\{(x, y) : a \leq x \leq b, c \leq y \leq d\}$, then

$$P[(X, Y) \in A] = P(a \leq X \leq b, c \leq Y \leq d) = \int_a^b \int_c^d f(x, y) dy dx$$

Example 5: Marginal Pdf s

Example The marginal pdf of X , which gives the probability distribution of busy time for the drive-up facility without reference to the walk-up window, is
(Example 5.3 continued)

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy = \int_0^1 \frac{6}{5}(x + y^2) dy = \frac{6}{5}x + \frac{2}{5}$$

for $0 \leq x \leq 1$ and 0 otherwise. The marginal pdf of Y is

$$f_Y(y) = \begin{cases} \frac{6}{5}y^2 + \frac{3}{5} & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$P\left(\frac{1}{4} \leq Y \leq \frac{3}{4}\right) = \int_{1/4}^{3/4} \left(\frac{6}{5}y^2 + \frac{3}{5}\right) dy = \frac{37}{80} = .4625.$$

Recall Example 3:

Example 5.3 A bank operates both a drive-up facility and a walk-up window. On a randomly selected day, let X = the proportion of time that the drive-up facility is in use (at least one customer is being served or waiting to be served) and Y = the proportion of time that the walk-up window is in use. Then the set of possible values for (X, Y)

is the rectangle $D = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$. Suppose the joint pdf of (X, Y) is given by

$$f(x, y) = \begin{cases} \frac{6}{5}(x + y^2) & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Definition of the Marginal pdf s (continuous case):

■ The **marginal probability density functions** of X and Y , denoted by $f_X(x)$ and $f_Y(y)$, respectively, are given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy \quad \text{for } -\infty < x < \infty$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx \quad \text{for } -\infty < y < \infty$$

Example 6: Marginal Pdf s

Example 5.5

A nut company markets cans of deluxe mixed nuts containing almonds, cashews, and peanuts. Suppose the net weight of each can is exactly 1 lb, but the weight contribution of each type of nut is random. Because the three weights sum to 1, a joint probability model for any two gives all necessary information about the weight of the third type. Let X = the weight of almonds in a selected can and Y = the weight of cashews. Then the region of positive density is $D = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1, x + y \leq 1\}$, the shaded region pictured in Figure 5.2.

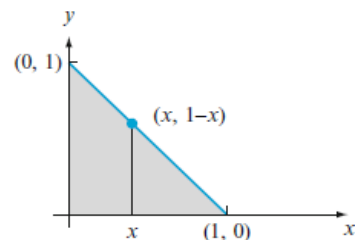


Figure 5.2 Region of positive density for Example 5.5

Now let the joint pdf for (X, Y) be

$$f(x, y) = \begin{cases} 24xy & 0 \leq x \leq 1, 0 \leq y \leq 1, x + y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

For any fixed x , $f(x, y)$ increases with y ; for fixed y , $f(x, y)$ increases with x . This is appropriate because the word *deluxe* implies that most of the can should consist of almonds and cashews rather than peanuts, so that the density function should be large near the upper boundary and small near the origin. The surface determined by $f(x, y)$ slopes upward from zero as (x, y) moves away from either axis.

Clearly, $f(x, y) \geq 0$. To verify the second condition on a joint pdf, recall that a double integral is computed as an iterated integral by holding one variable fixed (such as x as in Figure 5.2), integrating over values of the other variable lying along

Definition of the Marginal pdf s (continuous case):

The **marginal probability density functions** of X and Y , denoted by $f_X(x)$ and $f_Y(y)$, respectively, are given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy \quad \text{for } -\infty < x < \infty$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx \quad \text{for } -\infty < y < \infty$$

Example 6 (cont'd):

the straight line passing through the value of the fixed variable, and finally integrating over all possible values of the fixed variable. Thus

$$\begin{aligned}\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dy dx &= \iint_D f(x, y) dy dx = \int_0^1 \left\{ \int_0^{1-x} 24xy dy \right\} dx \\ &= \int_0^1 24x \left\{ \frac{y^2}{2} \Big|_{y=0}^{y=1-x} \right\} dx = \int_0^1 12x(1-x)^2 dx = 1\end{aligned}$$

To compute the probability that the two types of nuts together make up at most 50% of the can, let $A = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1, \text{ and } x + y \leq .5\}$, as shown in Figure 5.3. Then

$$P[(X, Y) \in A] = \iint_A f(x, y) dx dy = \int_0^{.5} \int_0^{.5-x} 24xy dy dx = .0625$$

The marginal pdf for almonds is obtained by holding X fixed at x and integrating $f(x, y)$ along the vertical line through x :

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy = \begin{cases} \int_0^{1-x} 24xy dy = 12x(1-x)^2 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

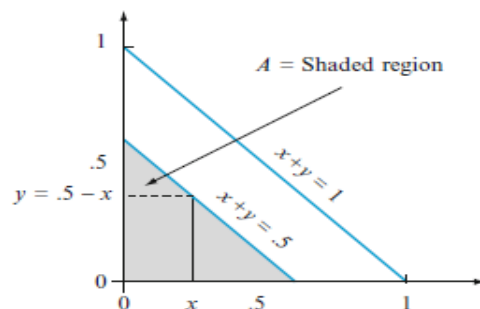


Figure 5.3 Computing $P[(X, Y) \in A]$ for Example 5.5

By symmetry of $f(x, y)$ and the region D , the marginal pdf of Y is obtained by replacing x and X in $f_X(x)$ by y and Y , respectively. ■

Definition of the Marginal pdf s (continuous case):

The **marginal probability density functions** of X and Y , denoted by $f_X(x)$ and $f_Y(y)$, respectively, are given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy \quad \text{for } -\infty < x < \infty$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx \quad \text{for } -\infty < y < \infty$$

Example 7: Marginal pmf s

EXAMPLE From a group of three Republicans, two Democrats, and one independent, a committee of two people is to be randomly selected. Let Y_1 denote the number of Republicans and Y_2 denote the number of Democrats on the committee. Find the joint probability function of Y_1 and Y_2 and then find the marginal probability function of Y_1 .

Solution The probabilities sought here are similar to the hypergeometric probabilities of Chapter 3. For example,

$$P(Y_1 = 1, Y_2 = 1) = p(1, 1) = \frac{\binom{3}{1}\binom{2}{1}\binom{1}{0}}{\binom{6}{2}} = \frac{3(2)}{15} = \frac{6}{15}$$

because there are 15 equally likely sample points; for the event in question we must select one Republican from the three, one Democrat from the two, and zero independents. Similar calculations lead to the other probabilities shown in Table 5.2.

To find $p_1(y_1)$, we must sum over the values of Y_2 , as Definition 5.4 indicates. Hence, these probabilities are given by the column totals in Table 5.2. That is,

$$p_1(0) = p(0, 0) + p(0, 1) + p(0, 2) = 0 + 2/15 + 1/15 = 3/15.$$

Similarly,

$$p_1(1) = 9/15 \quad \text{and} \quad p_1(2) = 3/15.$$

Analogously, the marginal probability function of Y_2 is given by the row totals.

Table 5.2 Joint probability function for Y_1 and Y_2 , Example 5.5

y_2	y_1			<i>Total</i>
	0	1	2	
0	0	3/15	3/15	6/15
1	2/15	6/15	0	8/15
2	1/15	0	0	1/15
<i>Total</i>	3/15	9/15	3/15	1

Definition of the Marginal pmf s (discrete case):

The **marginal probability mass functions** of X and of Y , denoted by $p_X(x)$ and $p_Y(y)$, respectively, are given by

$$p_X(x) = \sum_y p(x, y) \quad p_Y(y) = \sum_x p(x, y)$$

Example 8: Independent Random Variables

Example 5.11

If $X_1, \dots, X_n$ represent the lifetimes of n components, the components operate independently of each other, and each lifetime is exponentially distributed with parameter λ , then

$$\begin{aligned} f(x_1, x_2, \dots, x_n) &= (\lambda e^{-\lambda x_1}) \cdot (\lambda e^{-\lambda x_2}) \cdot \dots \cdot (\lambda e^{-\lambda x_n}) \\ &= \begin{cases} \lambda^n e^{-\lambda \sum x_i} & x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

If these n components are connected in series, so that the system will fail as soon as a single component fails, then the probability that the system lasts past time t is

$$\begin{aligned} P(X_1 > t, \dots, X_n > t) &= \int_t^\infty \dots \int_t^\infty f(x_1, \dots, x_n) dx_1 \dots dx_n \\ &= \left(\int_t^\infty \lambda e^{-\lambda x_1} dx_1 \right) \dots \left(\int_t^\infty \lambda e^{-\lambda x_n} dx_n \right) \\ &= (e^{-\lambda t})^n = e^{-n\lambda t} \end{aligned}$$

Therefore,

$$P(\text{system lifetime} \leq t) = 1 - e^{-n\lambda t} \quad \text{for } t \geq 0$$

which shows that *system* lifetime has an exponential distribution with parameter $n\lambda$; the expected value of system lifetime is $1/n\lambda$. ■

Definition of independence of random variables

Two random variables X and Y are said to be **independent** if for every pair of x and y values,

$$p(x, y) = p_X(x) \cdot p_Y(y) \quad \text{when } X \text{ and } Y \text{ are discrete}$$

or

(5.1)

$$f(x, y) = f_X(x) \cdot f_Y(y) \quad \text{when } X \text{ and } Y \text{ are continuous}$$

If (5.1) is not satisfied for all (x, y) , then X and Y are said to be **dependent**.

The random variables $X_1, X_2, \dots, X_n$ are said to be **independent** if for *every* subset $X_{i_1}, X_{i_2}, \dots, X_{i_k}$ of the variables (each pair, each triple, and so on), the joint pmf or pdf of the subset is equal to the product of the marginal pmf's or pdf's.

References

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