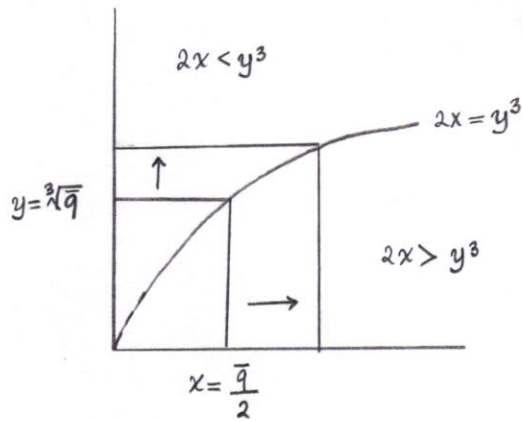


①

(a) $f(x, y) = \max \{2x, y^3\}$

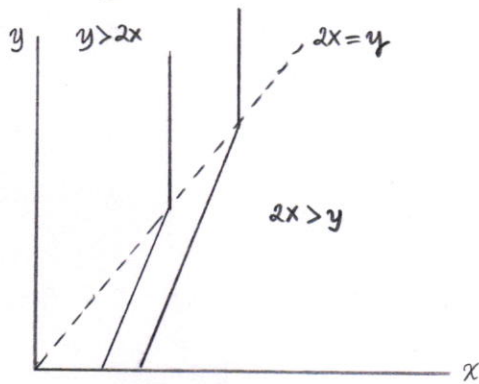


$$\max \{2x, y^3\} = \bar{q}$$

if $2x > y^3 \Rightarrow \bar{q} = 2x \quad x = \frac{\bar{q}}{2}$

if $2x < y^3 \Rightarrow \sqrt[3]{\bar{q}} = y$

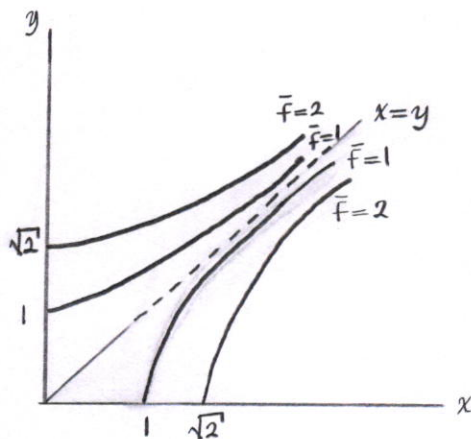
(b) $f(x, y) = 3x - \min \{2x, y\}$



If $2x > y \Rightarrow f(x, y) = 3x - y = \bar{q}$
 $y = 3x - \bar{q}$

If $2x < y \Rightarrow f(x, y) = x$
 $x = \bar{q}$

(c) $f(x, y) = (\max \{x, y\})^2 - (\min \{x, y\})^2$



If $x > y \Rightarrow f(x, y) = x^2 - y^2 = \bar{f}$
 $y^2 = x^2 - \bar{f}$

If $x < y \Rightarrow f(x, y) = y^2 - x^2 = \bar{f}$
 $y^2 = x^2 + \bar{f}$

Note: Isoquant $\Rightarrow$ shows the combinations of inputs that yield the same output.

(2)

(a) $f(L, K) = KL^2 + L$

additional — $MP_L = 2KL + 1$
 output as a $MP_K = L^2$ } $|MRTS| = \frac{2KL+1}{L^2} = \frac{2K}{L} + \frac{1}{L^2}$
 result of using

an additional unit of labor/capital. As L goes up, both terms go down. Thus, $MRTS$ is diminishing. Now, for any $\alpha > 1$,

$f(\alpha L, \alpha K) = \alpha K(\alpha L)^2 + \alpha L = \alpha^3 KL^2 + \alpha L > \alpha KL^2 + \alpha L = \alpha [KL^2 + L] = \alpha f(L, K)$.

Thus, it exhibits $IRTS$.

the level of output increases more than α when all the firm's inputs are increased α times.

(b) $f(L, K) = 7L + 12K - 4$

$MP_L = 7$ } $|MRTS| = 7/12$ which is constant.
 $MP_K = 12$ } Thus, $MRTS$ is not diminishing.

Now, for any $\alpha > 1$, $f(\alpha L, \alpha K) = 7(\alpha L) + 12(\alpha K) - 4 > 7(\alpha L) + 12(\alpha K) - 4\alpha = \alpha f(L, K)$.
 Thus, it exhibits $IRTS$.

(c) $f(L, K) = 3L^{1/3}K$

$MP_L = L^{-2/3}K$ } $|MRTS| = \frac{L^{-2/3}K}{3L^{1/3}} = \frac{K}{3L}$
 $MP_K = 3L^{1/3}$

As L goes up, it decreases. Thus, $MRTS$ is diminishing. Now for any $\alpha > 1$, we have

$f(\alpha L, \alpha K) = 3(\alpha L)^{1/3}(\alpha K) = \alpha^{(1/3+1)} 3L^{1/3}K = \alpha^{4/3} f(L, K) > \alpha f(L, K)$. Thus it exhibits $IRTS$.

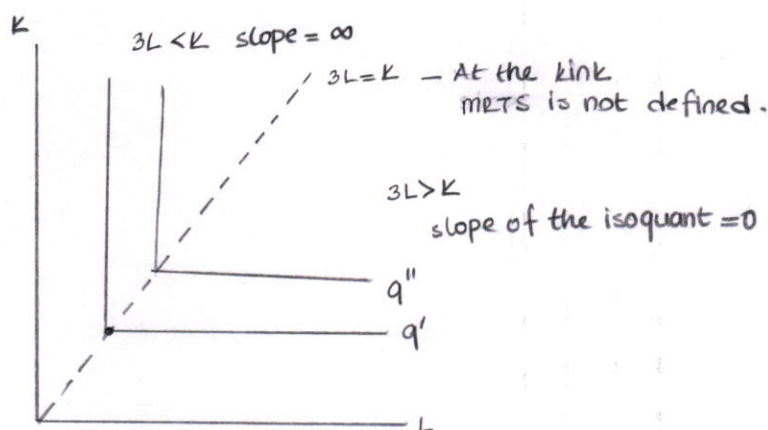
(d) $f(L, K) = \min \{3L, K\}$

If $3L > K$ then $f(L, K) = K$, and $MP_L = 0$ and $MP_K = 1$, where $|MRTS| = 0$. If $3L < K$ then

$f(L, K) = 3L$, and $MP_L = 3$ and $MP_K = 0$, where $|MRTS| = \infty$. If $3L = K$, then $MRTS$ is

not defined. Now, for any $\alpha > 1$, we have $f(\alpha K, \alpha L) = \min \{3\alpha L, \alpha K\} = \alpha \cdot \min \{3L, K\} = \alpha f(L, K)$.

Thus, it exhibits $CRTS$.



3

$$f(x_1, x_2) = (x_1^b + x_2^b)^c ; \quad b > 0, c > 0$$

$$\begin{aligned} f(kx_1, kx_2) &= ((kx_1)^b + (kx_2)^b)^c \\ &= (k^b \cdot x_1^b + k^b \cdot x_2^b)^c \\ &= (k^b (x_1^b + x_2^b))^c \\ &= k^{b \cdot c} (x_1^b + x_2^b)^c = k^{b \cdot c} \cdot f(x_1, x_2) \end{aligned}$$

• If $b \cdot c > 1$, then firm has IRTS ($k^{b \cdot c} \cdot f(x_1, x_2) > k \cdot f(x_1, x_2)$)

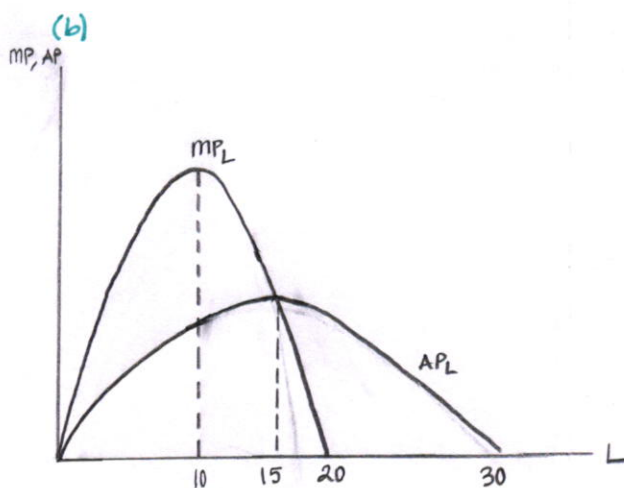
4

$$Q(K, L) = 150K^2L^2 - K^3L^3 ; \quad \bar{K} = 5$$

$$(a) \quad TP_L = 150 \cdot 25 \cdot L^2 - 125 \cdot L^3 = 3750L^2 - 125L^3$$

$$MP_L = \partial TP_L / \partial L = 7500L - 375L^2$$

$$AP_L = TP_L / L = 3750L - 125L^2$$



$$(c) \quad \partial MP_L / \partial L = 0 \Rightarrow 750L = 7500 \\ L = 10$$

$$(d) \quad MP_L = AP_L \Rightarrow 3750L = 125L^2 \\ L = 15$$

$$(e) \quad \partial TP_L / \partial L = 0 \Rightarrow 7500L - 375L^2 = 0 \\ 375L(20 - L) = 0$$

$$L = 0 \text{ or } L = 20$$

$$\left. \begin{aligned} L = 0 &\Rightarrow Q = 0 \\ L = 20 &\Rightarrow Q = 500,000 \end{aligned} \right\} L^* = 20$$

5

$$\pi(x, y) = 48\sqrt{x} + 64\sqrt{y} - x - 4y$$

$$\begin{aligned} \max \pi \\ \text{FOCs: } \left. \begin{aligned} \partial \pi / \partial x &= 0 \Rightarrow \frac{24}{\sqrt{x}} - 1 = 0 \\ \partial \pi / \partial y &= 0 \Rightarrow \frac{32}{\sqrt{y}} - 4 = 0 \end{aligned} \right\} \begin{aligned} x &= 576 \\ y &= 64 \end{aligned} \end{aligned}$$

$$f(x^*, y^*) = 6.24 + 8.8 = 208$$

*(check S.O.C. as well)

6

(a) $\max \pi = p \cdot f(\bar{K}, L) - r\bar{K} - \omega L$

$\Rightarrow \max p \sqrt{8L} - r - \omega L$

(b) From FOC, $\frac{4p}{\sqrt{8L}} - \omega = 0 \Rightarrow L^* = \frac{2p^2}{\omega^2}$

Thus, $f(\bar{K}, L) = \sqrt{8L^*} = \frac{4p}{\omega}$

(c) $\pi = p \cdot \frac{4p}{\omega} - r \cdot 1 - \omega \cdot \frac{2p^2}{\omega^2} = \frac{2p^2 - \omega r}{\omega}$

(d) $\frac{\partial L^*}{\partial \omega} < 0$ — if wage increases, L^* decreases and optimal output level decreases.

$\frac{\partial Q}{\partial \omega} < 0$

$\frac{\partial L^*}{\partial p} > 0$ — if price decreases, L^* and optimal output levels decrease.

$\frac{\partial Q}{\partial p} > 0$

7

$f(x) = \begin{cases} x^{3/2} & \text{where } 0 \leq x \leq 4 \\ x+4 & \text{where } x > 4 \end{cases}$

$\pi(x) = f(x) - 3x = \begin{cases} x^{3/2} - 3x & \text{where } 0 \leq x \leq 4 \\ 4 - 2x & \text{where } x > 4 \end{cases}$

• Notice that when $x > 4 \Rightarrow 4 - 2x < 0 \Rightarrow$ optimum is not here!

$\frac{\partial (x^{3/2} - 3x)}{\partial x} = \frac{3}{2} x^{1/2} - 3 = 0 \Rightarrow x=4$ is a critical point for the first case.

$\frac{\partial^2 (x^{3/2} - 3x)}{\partial x^2} = \frac{3}{4} x^{-1/2} > 0$ Then $x=4$ is a minimum point.

• check the other boundary $x=0$

$\pi(0)=0$ is the optimum.

8

- (i) $f(K, L) = K^2 L^2$ (Cobb-Douglas)
 (ii) $f(K, L) = K + 0.5L$ (Perfect substitutes)
 (iii) $f(K, L) = \sqrt{\min\{K, L\}}$ (Perfect complements)

(a) Returns to Scale:

- (i) $f(tK, tL) = (tK)^2 (tL)^2 = t^4 K^2 L^2 = t^4 f(K, L) > t \cdot f(K, L)$ (IRS)
 (ii) $f(tK, tL) = tK + 0.5tL = t[K + 0.5L] = t f(K, L)$ (CRTS)
 (iii) $f(tK, tL) = \sqrt{\min\{tK, tL\}} = t^{1/2} \sqrt{\min\{K, L\}} = t^{1/2} f(K, L) < t f(K, L)$ (DRTS)

(b) Cost functions:

- (i) For Cobb-Douglas cases, cost minimization requires:

$$MRTS = - \frac{w_K}{w_L}$$

$$\left(- \frac{MU_K}{MU_L} = - \frac{2KL^2}{2K^2L} = - \frac{L}{K} = -1 \right)$$

$K = L \Rightarrow$ Plug $K = L$ into production function.

$$f(K, L) = K^4 \Rightarrow y = K^4 \Rightarrow K = y^{1/4} = L$$

Hence, cost function is $C(y) = w_K \cdot K + w_L \cdot L = 1 \cdot K + 1 \cdot L = y^{1/4} + y^{1/4} = 2y^{1/4}$

- (ii) Notice that the firm will only use K (see the input price) and no L .

$$MP_1 / MP_2 = 1/0.5 = 2 > w_1 / w_2 = 1$$

$y = K + 0.5L = K \Rightarrow$ Plug this into the cost function.

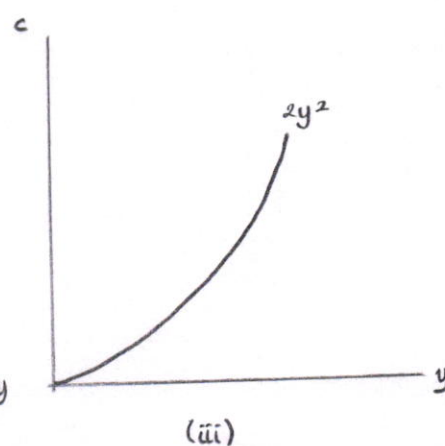
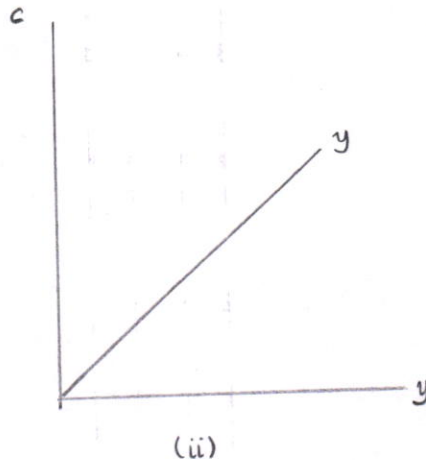
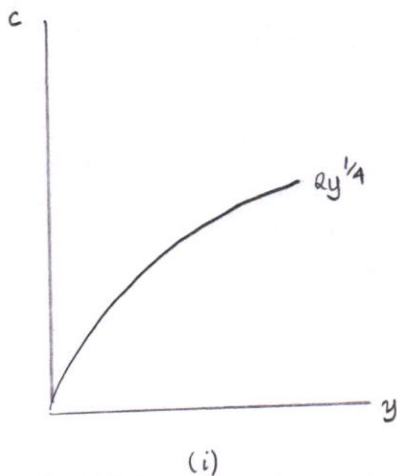
$$C(y) = w_K \cdot K + w_L \cdot L = 1 \cdot y + 1 \cdot 0 = y$$

- (iii) Notice that the cost-minimizing combination is $K = L$, $y = f(K, L) = \sqrt{\min\{K, L\}}$
 $y = \sqrt{K} \Rightarrow K = L = y^2$

Plug this into the cost function.

$$C(y) = w_K \cdot K + w_L \cdot L = K + L = 2y^2$$

(c) Cost function Graphs:

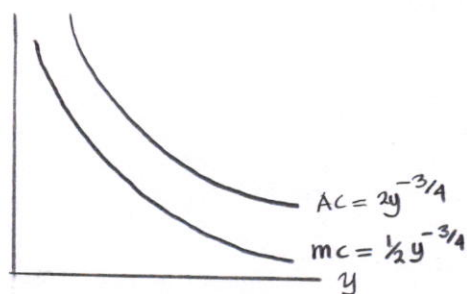


(d) AC & MC functions:

Recall that $AC(y) = \frac{C(y)}{y}$ and $MC(y) = \frac{dC(y)}{dy}$

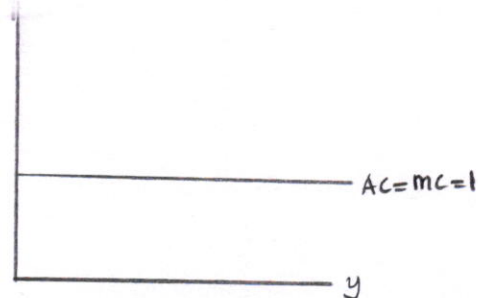
(i) $AC(y) = \frac{2y^{1/4}}{y} = 2y^{-3/4}$

$$MC(y) = \frac{d(2y^{1/4})}{dy} = \frac{1}{2} y^{-3/4}$$



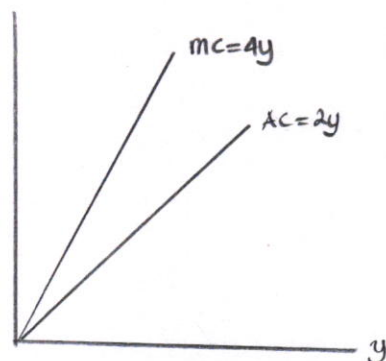
(ii) $AC = \frac{y}{y} = 1$

$$MC = \frac{dy}{dy} = 1$$



(iii) $AC = \frac{2y^2}{y} = 2y$

$$MC = \frac{d(2y^2)}{dy} = 4y$$



9

$$f(x, y) = x^{1/2} + y^{1/2}$$

(a) To find the unconditional demand, we need to solve the profit max. problem.

$$\pi = p \cdot (x^{1/2} + y^{1/2}) - p_x \cdot x - p_y \cdot y$$

$$\text{FOCs: } \partial \pi / \partial x = 0 \Rightarrow \frac{1}{2} p_x^{-1/2} - p_x = 0$$

$$x = \left(\frac{p}{2p_x} \right)^2$$

$$\partial \pi / \partial y = 0 \Rightarrow \frac{1}{2} p_y^{-1/2} - p_y = 0$$

$$y = \left(\frac{p}{2p_y} \right)^2$$

• where p is output price and p_x and p_y are input prices for x & y .

Another way to find supply fn.

$$MC = P = 2\bar{q} \frac{p_x p_y}{p_x + p_y} \Rightarrow \bar{q} = \frac{P}{2p_x} + \frac{P}{2p_y}$$

(b) supply fnc: $q(P, p_x, p_y) = \frac{P}{2p_x} + \frac{P}{2p_y}$

(c) $\pi(P, p_x, p_y) = P \left(\frac{P}{2p_x} + \frac{P}{2p_y} \right) - p_x \left(\frac{P}{2p_x} \right)^2 - p_y \left(\frac{P}{2p_y} \right)^2$

(d) To find the conditional demand, we need to solve the cost min. problem.

$$\min p_x \cdot x + p_y \cdot y \quad \text{s.t.} \quad \bar{q} = x^{1/2} + y^{1/2}$$

$$\mathcal{L}(x, y, \lambda) = p_x \cdot x + p_y \cdot y + \lambda (\bar{q} - x^{1/2} - y^{1/2})$$

$$\text{FOCs: } \begin{cases} \partial \mathcal{L} / \partial x = p_x - \frac{1}{2} \lambda x^{-1/2} = 0 \\ \partial \mathcal{L} / \partial y = p_y - \frac{1}{2} \lambda y^{-1/2} = 0 \end{cases} \Rightarrow \frac{y^{1/2}}{x^{1/2}} = \frac{p_x}{p_y} \Rightarrow y = x \frac{p_x^2}{p_y^2}$$

$$\partial \mathcal{L} / \partial \lambda = \bar{q} - x^{1/2} - y^{1/2} = 0$$

← Plug this into the production function:

$$\bar{q} = x^{1/2} + x^{1/2} \frac{p_x}{p_y} = x^{1/2} \left(1 + \frac{p_x}{p_y} \right)$$

$$\bar{q} = x^{1/2} \left(1 + \frac{p_x}{p_y} \right) \Rightarrow x^*(P, p_x, p_y) = \frac{\bar{q}^2}{\left(1 + \frac{p_x}{p_y} \right)^2} = \frac{p_y^2 \bar{q}^2}{(p_x + p_y)^2}$$

$$y^*(P, p_x, p_y) = \frac{\bar{q}^2}{\left(1 + \frac{p_x}{p_y} \right)^2} \cdot \frac{p_x^2}{p_y^2} = \frac{p_x^2 \bar{q}^2}{(p_x + p_y)^2}$$

(e) $c(p_x, p_y, \bar{q}) = p_x \cdot x^*(p_x, p_y, \bar{q}) + p_y \cdot y^*(p_x, p_y, \bar{q})$

$$= \frac{p_x p_y^2 \bar{q}^2 + p_y p_x^2 \bar{q}^2}{(p_x + p_y)^2} = \frac{p_x p_y \bar{q}^2 (p_x + p_y)}{(p_x + p_y)^2}$$

$$= \frac{p_x p_y \bar{q}}{p_x + p_y}$$

(10)

$$f(x_1, x_2, x_3, x_4) = \min \{2x_1 + x_2, x_3 + 2x_4\}$$

(a) conditional factor demands; $w = \{1, 2, 3, 4\}$

To produce 1 unit of output, cost-min. behavior is $2x_1 + x_2 = x_3 + 2x_4 = 1$ ($q=1$)

Notice that there are two cases to consider:

Case 1: $2x_1 + x_2 = 1 \Rightarrow x_1^* = \frac{1}{2} \quad x_2^* = 0$

$2 > \frac{w_1}{w_2} = \frac{1}{2}$ see the factor prices $w_1=1, w_2=2$
firm will only use x_1 , then

$$\text{Cost} = \frac{1}{2} \cdot 1 + 0 \cdot 2 = \frac{1}{2}$$

Case 2: $x_3 + 2x_4 = 1 \Rightarrow x_3^* = 0 \quad x_4^* = \frac{1}{2}$

$\frac{1}{2} < \frac{w_3}{w_4} = \frac{3}{4}$ Again, similar argument
due to factor prices, $w_3=3 \wedge w_4=4$

$$\text{Cost} = \frac{1}{2} \cdot 4 + 0 \cdot 3 = 2$$

Given input prices, conditional factor demand is: $(x_1, x_2, x_3, x_4) = (\frac{1}{2}, 0, 0, \frac{1}{2})$

(b) Cost function:

At optimum $2x_1 + x_2 = x_3 + 2x_4 = q$

• If $\frac{w_1}{w_2} > 2 \Rightarrow x_1 = 0$ and $x_2 = q$

• If $\frac{w_1}{w_2} < 2 \Rightarrow x_2 = 0$ and $x_1 = \frac{q}{2}$

• If $\frac{w_3}{w_4} > \frac{1}{2} \Rightarrow x_3 = 0$ and $x_4 = \frac{q}{2}$

• If $\frac{w_3}{w_4} < \frac{1}{2} \Rightarrow x_4 = 0$ and $x_3 = q$

• For the sake of simplicity, we will not consider here the inequalities $w_1/w_2 = 2$ and $w_3/w_4 = \frac{1}{2}$ which would yield 9 conditions instead of 4.

So,

- If $\frac{w_1}{w_2} > 2$ and

- (i) if $\frac{w_3}{w_4} > \frac{1}{2} \Rightarrow x_1 = x_3 = 0, x_2 = q \wedge x_4 = \frac{q}{2}$
- (ii) if $\frac{w_3}{w_4} < \frac{1}{2} \Rightarrow x_1 = x_4 = 0, x_2 = x_3 = q$

- If $\frac{w_1}{w_2} < 2$ and

- (iii) if $\frac{w_3}{w_4} > \frac{1}{2} \Rightarrow x_2 = x_3 = 0, x_1 = x_4 = \frac{q}{2}$
- (iv) if $\frac{w_3}{w_4} < \frac{1}{2} \Rightarrow x_2 = x_4 = 0, x_1 = \frac{q}{2} \wedge x_3 = q$

• Hence the cost functions for these 4 cases are:

$$\left. \begin{array}{l} \text{(i)} \quad q w_2 + \frac{q}{2} w_4 \\ \text{(ii)} \quad q w_2 + q w_3 \\ \text{(iii)} \quad \frac{q}{2} w_1 + \frac{q}{2} w_4 \\ \text{(iv)} \quad \frac{q}{2} w_1 + q w_3 \end{array} \right\}$$

From these informations, we can write the cost function:

$$C(w, q) = q \cdot \left(\min \left\{ \frac{w_1}{2}, w_2 \right\} + \min \left\{ w_3, \frac{w_4}{2} \right\} \right)$$

$$\begin{aligned}
 (c) \quad f(\lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4) &= \min \{ \lambda 2x_1 + \lambda x_2, \lambda x_3 + \lambda 2x_4 \} \\
 &= \min \{ \lambda (2x_1 + x_2), \lambda (x_3 + 2x_4) \} \\
 &= \lambda \cdot \min \{ 2x_1 + x_2, x_3 + 2x_4 \} \\
 &= \lambda \cdot f(x_1, x_2, x_3, x_4) \quad (\text{CRTS})
 \end{aligned}$$

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$$f(x_1, x_2, x_3, x_4) = 2 \min \{ x_1, x_2 \} + 3 \min \{ x_3, x_4 \}$$

$$(w_1, w_2, w_3, w_4) = (8, 5, 3, 4) \quad ; \quad \bar{q} = 70$$

Notice that at optimum, $x_1 = x_2$ and $x_3 = x_4$.

Let's say $x_1 = x_2 = a$ and $x_3 = x_4 = b$ where a and b are constant.

So, production function will look like $(a, b) = \underbrace{2a + 3b}_{\text{perfect substitutes}}$

$$\frac{MP_a}{MP_b} = \frac{2}{3} < \frac{13}{7} = \frac{P_a}{P_b}$$

└ input price ratios

$$\text{where } - P_a = 8 + 5 = 13$$

(cost of 1 unit of a)

$$- P_b = 3 + 4 = 7$$

(cost of 1 unit of b)

So, firm will use only b in the production process, and $a = 0$.

Since quantity is given as 70;

$$3x_4 = 3x_3 = 70 \Rightarrow x_3^* = x_4^* = \frac{70}{3} \quad \text{and} \quad x_1^* = x_2^* = 0$$

$$\text{Total cost} = \frac{70}{3} \cdot 3 + \frac{70}{3} \cdot 4 = \frac{490}{3}$$

(12)

• $f(K, L) = K^{0.5} L^{0.5}$; $P_K = 10$; $P_L = 40$

(a) $\mathcal{L}(K, L, \lambda) = P_K \cdot K + P_L \cdot L + \lambda (\bar{q} - K^{0.5} L^{0.5})$

(b) FOCs:

$$\left. \begin{aligned} \frac{\partial \mathcal{L}}{\partial K} &= P_K - \frac{1}{2} \lambda K^{-\frac{1}{2}} L^{-\frac{1}{2}} = 0 \\ \frac{\partial \mathcal{L}}{\partial L} &= P_L - \frac{1}{2} \lambda K^{\frac{1}{2}} L^{-\frac{1}{2}} = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= \bar{q} - K^{\frac{1}{2}} L^{\frac{1}{2}} = 0 \end{aligned} \right\} \begin{aligned} \frac{L}{K} &= \frac{P_K}{P_L} \\ \frac{K}{L} &= \frac{P_L}{P_K} \end{aligned}$$

(c) $K = L \cdot \frac{P_L}{P_K} \Rightarrow \bar{q} = \left(L \cdot \frac{P_L}{P_K} \right)^{\frac{1}{2}} L^{\frac{1}{2}} \Rightarrow \bar{q} = L \left(\frac{P_L}{P_K} \right)^{\frac{1}{2}}$

$$L(P_K, P_L, \bar{q}) = \left(\frac{P_K}{P_L} \right)^{\frac{1}{2}} \bar{q} \quad \wedge \quad K(P_K, P_L, \bar{q}) = \left(\frac{P_L}{P_K} \right)^{\frac{1}{2}} \bar{q}$$

$$\left. \begin{aligned} P_L &= 40 \\ P_K &= 10 \end{aligned} \right\} \bar{q} = 1000 ; L^* = 500 \quad \wedge \quad K^* = 2000$$

$$C_{SR}(10, 40, 1000) = 10 \cdot 2000 + 40 \cdot 500 = 40.000$$

(d) $\bar{K} = 2000$; L can vary ; $\bar{q} = 2000$

$$f(L) = 2000^{\frac{1}{2}} L^{\frac{1}{2}} = 2000 \Rightarrow L^* = 2000$$

$$\text{Total cost} = 2000 \cdot 10 + 2000 \cdot 40 = 100.000$$

(e) use conditional input demand functions that we found in part (c);

$$L(10, 40, 2000) = 1000 \Rightarrow L^* = 1000$$

$$K(10, 40, 2000) = 4000 \Rightarrow K^* = 4000$$

$$C_{LR}(10, 40, 2000) = 10 \cdot 4000 + 40 \cdot 1000 = 80.000$$

(f) In the S/R, in order to produce 2000 cakes, Kerem has to use exactly 2000 L and 2000 K. In the S/R, he does not solve a cost minimization problem. In the L/R, however, he has the ability to adjust amount of labor and capital in the production process so that he can choose levels of labor and capital in order to minimize production costs.

(13)

• $f(L, K) = \left(\min\left(\frac{K}{a}, \frac{L}{b}\right) \right)^c$; $a, b, c > 0$

- (a) Let us pick some arbitrary pair (K, L) and assume for now that $\min(2K, 3L) = 2K$. Then, for any $\alpha > 1$, $\min\{2\alpha K, 3\alpha L\} = 2\alpha K$. Then, we have that $f(\alpha K, \alpha L) = (\min\{2\alpha K, 3\alpha L\})^c = (2\alpha K)^c = \alpha^c (2K)^c = \alpha^c (\min\{2K, 3L\})^c = \alpha^c f(K, L)$.

Note that this calculation would have worked exactly the same way if we had assumed $\min(2K, 3L) = 3L$. This production function exhibits decreasing, constant, and increasing returns to scale when α^c is less than, equal to, and greater than 1, respectively. That is, it exhibits DRTS if $c < 1$, CRTS if $c = 1$ and IRTS if $c > 1$.

- (b) Recall first that the conditional factor demands $K(q, w, r)$, $L(q, w, r)$ are the solutions, $K \geq 0$ and $L \geq 0$, to the cost min. problem:

$$\min_{\{L, K\}} rK + wL \quad \text{s.t.} \quad \left(\min\left(\frac{K}{a}, \frac{L}{b}\right) \right)^c \geq q$$

In this case, because only the min. of $\frac{K}{a}$ and $\frac{L}{b}$ contributes to production, the firm minimizes cost to produce where $\frac{K}{a} = \frac{L}{b}$. We therefore have that:

$$q = \left(\min\left(\frac{K}{a}, \frac{L}{b}\right) \right)^c = \left(\frac{K}{a}\right)^c = \left(\frac{L}{b}\right)^c$$

We can solve for K, L as follows:

$$\begin{aligned} q^{1/c} = \frac{K}{a} &\Leftrightarrow K = a q^{1/c} & K(q, w, r) &= a q^{1/c} \\ \text{and } q^{1/c} = \frac{L}{b} &\Leftrightarrow L = b q^{1/c} & L(q, w, r) &= b q^{1/c} \end{aligned}$$

The resulting cost function is:

$$\begin{aligned} c(q, w, r) &= rK(q, w, r) + wL(q, w, r) = r a q^{1/c} + w b q^{1/c} \\ &= (ra + wb) q^{1/c} \end{aligned}$$

- (c) We observe that the conditional factor demands, $K(q, w, r) = a q^{1/c}$ and $L(q, w, r) = b q^{1/c}$, are both independent of the input prices r and w , while the cost function is increasing in both r and w . The reason for this is as follows. Because $(a \text{ units of})$ capital and $(b \text{ units of})$ labor act as perfect complements for this "fixed proportion" production function (meaning the isoquants look exactly like indifference curves for perfect complements, with the slope being either zero or infinity everywhere), the firm will always choose a pair K, L such that $\frac{K}{a} = \frac{L}{b}$ in order to minimize cost. This equation thus fixes the r/s b/w the optimal level of K and the optimal level of L .

(No tradeoff exists, input prices do not affect the cost of minimizing input levels K and L . However, once K and L are chosen, the firm still has to pay for them, which of course affects its cost function, which is why the cost function is increasing in r and w .)

(d) The cost function can now be written as:

$$C(q) = Aq^{1/c}$$

The firm's marginal cost is given by: $MC(q) = C'(q) = \left(\frac{A}{c}\right) q^{(1-c)/c}$

The firm's average total cost is given by:

$$ATC(q) = \frac{C(q)}{q} = Aq^{(1-c)/c} \\ = c \cdot MC(q)$$

The slope of the cost curves is given by: $\partial MC(q) / \partial q = \left(\frac{A}{c^2}\right) (1-c) q^{(1-2c)/c}$

$$\text{and } \frac{\partial ATC(q)}{\partial q} = \frac{\partial [c \cdot MC(q)]}{\partial q} = c \cdot \frac{\partial MC(q)}{\partial q} \\ = \underbrace{\frac{A}{c}}_{>0} (1-c) \underbrace{q^{(1-2c)/c}}_{>0}$$

(e) Recall that $f(L, K)$ exhibits: $\frac{\partial^2 f}{\partial L^2} > 0$ (IRTS), $\frac{\partial^2 f}{\partial L^2} = 0$ (CRTS), $\frac{\partial^2 f}{\partial L^2} < 0$ (DRTS).

DRTS if $c < 1$

CRTS if $c = 1$

IRTS if $c > 1$.

Notice the sign of the slope of both MC and ATC is the same as the sign of $1-c$. When f exhibits IRTS (CRTS) (DRTS), we have that $c > 1$ ($c = 1$) ($c < 1$) and therefore both MC and ATC will be downward sloping (constant) (upward sloping). Furthermore, when f has CRTS, we have that $MC = ATC$. When it has IRTS, ATC will be above MC , and when it has DRTS, MC will be above ATC , confirming the r/s b/w the slope of the ATC and whether it is above or below MC .

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• $f(L, K, m) = 3L^{1/3} K^{1/3} + m$

Firm's cost min. problem to produce q units is given by:

$$\begin{aligned} \min_{L, K, m \geq 0} \quad & \omega L + rK + m \\ & 3L^{1/3} K^{1/3} + m = q \end{aligned}$$

$$\mathcal{L}(L, K, m, \omega, r, q, \lambda) = \omega L + rK + M + \lambda(q - 3L^{1/3} K^{1/3} - m)$$

$$\text{FOCs: } \left. \begin{aligned} \partial \mathcal{L} / \partial L &= \omega - \lambda \frac{1}{3} L^{-2/3} K^{1/3} = 0 \\ \partial \mathcal{L} / \partial K &= r - \lambda \frac{1}{3} L^{1/3} K^{-2/3} = 0 \end{aligned} \right\} \frac{L^{-2/3} K^{1/3}}{L^{1/3} K^{-2/3}} = \frac{\omega}{r} \quad L = \frac{rK}{\omega}$$

$$\partial \mathcal{L} / \partial m = 1 - \lambda = 0 \quad \leadsto \quad \lambda = 1$$

$$\partial \mathcal{L} / \partial \lambda = q - 3L^{1/3} K^{1/3} - m = 0$$

• From the first one:

$$\lambda = 1 \Rightarrow \omega = \frac{K^{1/3}}{L^{2/3}} \quad K = \omega^3 L^2 \quad \text{or} \quad rK = \omega$$

$$K = \omega^3 \frac{r^2 K^2}{\omega^2} \Rightarrow K = \frac{1}{\omega r^2} \quad \text{and} \quad L = \frac{1}{\omega^2 r}$$

$$\begin{aligned} m &= q - 3L^{1/3} K^{1/3} \Rightarrow m = q - 3 \cdot \left(\frac{1}{\omega^2 r}\right)^{1/3} \left(\frac{1}{\omega r^2}\right)^{1/3} \\ &= q - \frac{3}{\omega r} \end{aligned}$$

$$\bullet (L(q, \omega, r), K(q, \omega, r), m(q, \omega, r)) = \left(\frac{1}{\omega^2 r}, \frac{1}{\omega r^2}, q - \frac{3}{\omega r}\right)$$

• The cost function is given by:

$$c(q, \omega, r) = \omega L(q, \omega, r) + r K(q, \omega, r) + m(q, \omega, r)$$

$$= \omega \frac{1}{\omega^2 r} + r \cdot \frac{1}{\omega r^2} + q - \frac{3}{\omega r}$$

$$= \frac{1}{\omega r} + \frac{1}{\omega r} + q - \frac{3}{\omega r}$$

$$= q - \frac{1}{\omega r}$$

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(a) $TC(q) = 2q^2 - q + 4$

$$AC = 2q - 1 + 4/q ; AVC = 2q - 1 ; AFC = 4/q ; MC = 4q - 1$$

(b) $TC(q) = 1000 + 50q^2$

$$AC = 1000/q + 50q ; AVC = 50q ; AFC = 1000/q ; MC = 100q$$

(c) $TC(q) = q^3 + 1$

$$AC = q^2 + 1/q ; AVC = q^2 ; AFC = 1/q ; MC = 3q^2$$

• To verify that the min. of AC is on the MC curve, you need to find the q that minimizes AC and the minimized value of AC; and show that this q and the minimized value is on the MC curve.