

## The Answers of the Review Problem Set II

- 1) Three balanced coins are tossed independently. One of the variables of interest is  $Y_1$ , the number of heads. Let  $Y_2$  denote the amount of money won on a side bet in the following manner. If the first head occurs on the first toss, you win \$1. If the first head occurs on toss 2 or on toss 3, you win \$2 or \$3, respectively. If no heads appear, you lose \$1.
- a) Find the joint probability function for  $Y_1$  and  $Y_2$ .
- b) Find  $F(2,1)$ .

SOLUTION

- a. The sample space for the toss of three balanced coins w/ probabilities are below:

| Outcome      | HHH    | HHT    | HTH    | HTT    | THH    | THT    | TTH    | TTT     |
|--------------|--------|--------|--------|--------|--------|--------|--------|---------|
| $(y_1, y_2)$ | (3, 1) | (2, 1) | (2, 1) | (1, 1) | (2, 2) | (1, 2) | (1, 3) | (0, -1) |
| probability  | 1/8    | 1/8    | 1/8    | 1/8    | 1/8    | 1/8    | 1/8    | 1/8     |

|       |    | $y_1$ |     |     |     |
|-------|----|-------|-----|-----|-----|
|       |    | 0     | 1   | 2   | 3   |
| $y_2$ | -1 | 1/8   | 0   | 0   | 0   |
|       | 1  | 0     | 1/8 | 2/8 | 1/8 |
|       | 2  | 0     | 1/8 | 1/8 | 0   |
|       | 3  | 0     | 1/8 | 0   | 0   |

- b.  $F(2, 1) = p(0, -1) + p(1, 1) + p(2, 1) = 1/2$ .

- 2) The joint density function of  $Y_1$  and  $Y_2$  is given as:

$$f(y_1, y_2) = \begin{cases} 6y_1^2 y_2, & 0 \leq y_1 \leq y_2, y_1 + y_2 \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

- a) Find the marginal density function for  $Y_1$ .
- b) Find the conditional density function of  $Y_2$  given that  $Y_1 = y_1$ .
- c) Find  $P(Y_2 < 1.1 | Y_1 = 0.6)$ .

### SOLUTION

a.  $f_1(y_1) = \int_{y_1}^{2-y_1} 6y_1^2 y_2 dy_2 = 12y_1^2(1-y_1), 0 \leq y_1 \leq 1.$

b. For  $0 < y_1 < 1$ :

$$f(y_2 | y_1) = \frac{1}{2} y_2 / (1 - y_1), y_1 \leq y_2 \leq 2 - y_1.$$

c.

$$P(Y_2 < 1.1 | Y_1 = 0.6) = \frac{\int_{0.6}^{1.1} \frac{y_2}{0.4} dy_2}{2} = 0.53$$

- 3) Let  $Y_1$  and  $Y_2$  be independently exponentially distributed random variables, each with mean 1. Find  $P(Y_1 > Y_2 | Y_1 < 2Y_2)$ .

### SOLUTION

$$P(Y_1 > Y_2, Y_1 < 2Y_2) = \int_0^\infty \int_{y_1/2}^{y_1} e^{-(y_1+y_2)} dy_2 dy_1 = \frac{1}{6} \text{ and } P(Y_1 < 2Y_2) = \int_0^\infty \int_{y_1/2}^\infty e^{-(y_1+y_2)} dy_2 dy_1 = \frac{2}{3}. \text{ So,}$$
$$P(Y_1 > Y_2 | Y_1 < 2Y_2) = 1/4.$$

- 4) The joint density function of  $Y_1$  and  $Y_2$  is given as:

$$f(y_1, y_2) = \begin{cases} e^{-(y_1+y_2)}, & y_1 > 0, y_2 > 0 \\ 0, & \text{elsewhere} \end{cases}$$

- a) Show that  $Y_1$  and  $Y_2$  are independent.
- b) Find  $E(Y_1+Y_2)$  and  $V(Y_1+Y_2)$ .
- c) Find  $E(Y_1-Y_2)$  and  $V(Y_1-Y_2)$ .
- d) Find  $P(Y_1-Y_2 > 3)$ .

### SOLUTION

a)

$$\text{Note that } f_1(y_1) = \int_0^\infty e^{-(y_1+y_2)} dy_2 = e^{-y_1}, y_1 > 0 \text{ and } f_2(y_2) = \int_0^\infty e^{-(y_1+y_2)} dy_1 = e^{-y_2}, y_2 > 0.$$

Thus,  $f(y_1, y_2) = f_1(y_1)f_2(y_2)$  so that  $Y_1$  and  $Y_2$  are independent.

- b) The marginal distributions for  $Y_1$  and  $Y_2$  are exponential with  $\beta=1$ . Also note that  $Y_1$  and  $Y_2$  are independent.

$E(Y_1+Y_2)=E(Y_1)+E(Y_2)=2$ ,  $V(Y_1+Y_2)=2$ . Note that  $\text{Cov}(Y_1, Y_2)=0$  as  $Y_1$  and  $Y_2$  are independent.

- c)  $E(Y_1-Y_2)=E(Y_1)-E(Y_2)=0$ ,  $V(Y_1-Y_2)=2$ . Note that  $\text{Cov}(Y_1, Y_2)=0$  as  $Y_1$  and  $Y_2$  are independent.

- d)  $P(Y_1-Y_2>3)=P(Y_1>3+Y_2)=\int_0^\infty \int_{3+y_2}^\infty e^{-(y_1+y_2)} dy_1 dy_2 = \frac{1}{2}e^{-3} = 0.0249$ .

- 5) If  $c$  is any constant and  $Y$  is a random variable such that  $E(Y)$  exists, show that  $\text{Cov}(c, Y)=0$ .

### SOLUTION

$$\text{Since } E(c) = c, \text{Cov}(c, Y) = E[(c - c)(Y - \mu)] = 0.$$

Note that  $\mu$  denotes  $E(Y)$ .

- 6) Show that  $\text{Cov}(X, X)=\text{Var}(X)$ .

### SOLUTION

$$\text{Cov}(X, X) = E[(X - \mu_X)(X - \mu_X)] = E[(X - \mu_X)^2] = \text{Var}(X)$$

7)

Suppose we have a bowl with 10 marbles - 2 red marbles, 3 green marbles, and 5 blue marbles. We randomly select 4 marbles from the bowl, with replacement. What is the probability of selecting 2 green marbles and 2 blue marbles?

### SOLUTION

Let's define  $Y_1 = \# \text{ of red marbles}$ ,  $Y_2 = \# \text{ of green marbles}$ ,  $Y_3 = \# \text{ of blue marbles}$ .

Therefore, we have that  $(Y_1, Y_2, Y_3)$  is multinomial with  $p_1 = 0.2$ ,  $p_2 = 0.3$ ,  $p_3 = 0.5$ .

$$P(Y_1 = 0, Y_2 = 2, Y_3 = 2) = \frac{4!}{0! * 2! * 2!} (0.2)^0 (0.3)^2 (0.5)^2 \quad (1)$$

$$P(Y_1 = 0, Y_2 = 2, Y_3 = 2) = 0.135 \quad (2)$$

- 8) Let the random variable  $Y$  possess a uniform distribution on the interval  $(0, 1)$ . Find the cumulative distribution function of the random variable  $W = \sqrt{Y}$ .

**SOLUTION**

$$F_W(w) = P(W \leq w) = P(\sqrt{Y} \leq w) = P(Y \leq w^2) = F_Y(w^2) = w^2, 0 \leq w \leq 1.$$

$$F_W(w) = \begin{cases} 0, & w < 0 \\ w^2, & 0 \leq w \leq 1 \\ 1, & w > 1 \end{cases}$$

(Remark: The third equality above holds for  $0 \leq w \leq 1$  since  $w$  is nonnegative. In general,  $P(\sqrt{Y} \leq w) + P(w \leq -\sqrt{Y}) = P(Y \leq w^2)$ .)

9)

Consider the random variable  $Y$ . Its distribution function is given by

$$F(Y) = \begin{cases} 0, & \text{if } y < 0 \\ y(1 - 0.25y), & \text{if } 0 \leq y \leq 2 \\ 1, & \text{if } y > 2 \end{cases}$$

- a.) What is the density of  $Y$ ?  
b.) Find the density of  $e^{-Y}$ .

**SOLUTION**

- a) Note that  $\frac{dF(Y)}{dY} = f(y)$ .

$$f(y) = \begin{cases} 1 - 0.5y, & 0 \leq y \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

- b)  $u = h(y) = e^{-y}, y = h^{-1}(u) = -\ln u, \left| \frac{dh^{-1}(u)}{du} \right| = \frac{1}{u}$

Note that  $u$  is a strictly decreasing function of  $y$ .

$$f_u(u) = \begin{cases} \frac{(1 + 0.5 \ln u)}{u} & \text{for } e^{-2} \leq u \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

10) The probability density function of Y is given by

$$f(y) = \begin{cases} \left(\frac{2y}{\theta}\right) e^{\frac{-y^2}{\theta}}, & y \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

Find the density function for  $U=Y^2$ .

Note that U is a strictly increasing function of Y for  $y \geq 0$ .

#### SOLUTION

If  $U = Y^2$ ,  $Y = \sqrt{U}$ . Thus,  $\left|\frac{dy}{du}\right| = \frac{1}{2\sqrt{u}}$  and  $f_U(u) = \frac{1}{\theta} e^{-u/\theta}$ ,  $u \geq 0$ .

$$f_U(u) = \begin{cases} \frac{1}{\theta} e^{-\frac{u}{\theta}} & \text{for } u \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

11) Let  $Y_1, Y_2, \dots, Y_n$  be independent Poisson random variables with means  $\lambda_1, \lambda_2, \dots, \lambda_n$ , respectively. Find the probability distribution function of  $\sum_{i=1}^n Y_i$ .

#### SOLUTION

The moment-generating function for  $U = \sum_{i=1}^n Y_i$  is  $m_U(t) = e^{(e^t - 1)\sum_i \lambda_i}$  which is the moment-generating function for a Poisson with mean  $\sum_i \lambda_i$ . Hence the pmf of U is

$$p(u) = \frac{e^{-(\sum_i \lambda_i)} (\sum_i \lambda_i)^u}{u!}, \quad u=0,1,2,\dots$$

12)

Let  $Y_1$  and  $Y_2$  have joint density function

$$f_{Y_1, Y_2}(y_1, y_2) = \begin{cases} 8y_1 y_2, & 0 \leq y_1 < y_2 \leq 1, \\ 0, & \text{otherwise,} \end{cases}$$

and  $U_1 = Y_1/Y_2$  and  $U_2 = Y_2$

Derive the joint density function for  $(U_1, U_2)$ .

### SOLUTION

$$f_{Y_1, Y_2}(y_1, y_2) = \begin{cases} 8y_1y_2, & 0 \leq y_1 < y_2 \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

$$u_1 = \frac{y_1}{y_2} \text{ and } u_2 = y_2$$

The given transformation is a one-to-one transformation with the following inverse:

$$y_2 = u_2 \text{ and } y_1 = u_1 u_2.$$

Also,  $0 \leq y_1 < y_2 \leq 1$  becomes  $0 \leq u_1 u_2 < u_2 \leq 1$ . Equivalently,  $0 \leq u_1 < 1$  and  $0 < u_2 \leq 1$

The Jacobian is:

$$J = \begin{vmatrix} u_2 & u_1 \\ 0 & 1 \end{vmatrix} = u_2$$

$$f_{U_1, U_2}(u_1, u_2) = f_{Y_1, Y_2}(u_1 u_2, u_2) |u_2|$$

$$f_{U_1, U_2}(u_1, u_2) = 8(u_1 u_2) u_2 u_2 = \begin{cases} 8u_1 u_2^3 & \text{for } 0 \leq u_1 < 1, 0 < u_2 \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

13)

An electronic system has one each of two different types of components in joint operation. Let  $Y_1$  and  $Y_2$  denote the random lengths of life of the components of type I and type II, respectively. The joint density function is given by

$$f(y_1, y_2) = \begin{cases} (1/8)y_1 e^{-(y_1+y_2)/2}, & y_1 > 0, y_2 > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

(Measurements are in hundreds of hours.) Find  $P(Y_1 > 1, Y_2 > 1)$ .

### SOLUTION

$$P(Y_1 > 1, Y_2 > 1) = \int_1^\infty \int_1^\infty \frac{1}{8} y_1 e^{-(y_1+y_2)/2} dy_1 dy_2 = \left[ \int_1^\infty \frac{1}{4} y_1 e^{-y_1/2} dy_1 \right] \left[ \int_1^\infty \frac{1}{2} e^{-y_2/2} dy_2 \right] = \frac{3}{2} e^{-\frac{1}{2}} \left( e^{-\frac{1}{2}} \right) = \frac{3}{2} e^{-1}$$

14) For the following joint probability density function

$$f(y_1, y_2) = \begin{cases} 2, & 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1, 0 \leq y_1 + y_2 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

a Find  $P(Y_1 \geq 1/2 | Y_2 \leq 1/4)$ .

b Find  $P(Y_1 \geq 1/2 | Y_2 = 1/4)$ .

c. Are  $Y_1$  and  $Y_2$  are independent?

Note that  $f_2(y_2) = \int_0^{1-y_2} 2dy_1 = 2(1 - y_2)$  if  $0 \leq y_2 \leq 1$

### SOLUTION

a.  $P(Y_1 \geq 1/2, Y_2 \leq 1/4) = \int_0^{1/4} \int_{1/2}^{1-y_2} 2dy_1 dy_2 = \frac{3}{16}$ . And,  $P(Y_2 \leq 1/4) = \int_0^{1/4} 2(1 - y_2) dy_2 = \frac{7}{16}$ .

Thus,  $P(Y_1 \geq 1/2 | Y_2 \leq 1/4) = \frac{3}{7}$ .

b. For  $0 \leq y_2 \leq 1$ :  $f(y_1 | y_2) = \frac{1}{1-y_2}$ ,  $0 \leq y_1 \leq 1 - y_2$ . Thus,  $f(y_1 | 1/4) = 4/3$ ,  $0 \leq y_1 \leq 3/4$ .

Thus,  $P(Y_1 \geq 1/2 | Y_2 = 1/4) = \int_{1/2}^{3/4} 4/3 dy_1 = 1/3$ .

c.

The ranges of  $y_1$  and  $y_2$  depend on each other so  $Y_1$  and  $Y_2$  cannot be independent.

15)

Suppose that  $Y_1$  and  $Y_2$  are independent  $\chi^2$  random variables with  $v_1$  and  $v_2$  degrees of freedom, respectively. Find

a  $E(Y_1 + Y_2)$ .

b  $V(Y_1 + Y_2)$ .

### SOLUTION

Note that a random variable  $Y$  has a  $\chi^2$  distribution with  $v$  degrees of freedom iff  $Y$  is a gamma distributed random variable with parameters  $\alpha=v/2$  and  $\beta=2$ .

Then,  $E(Y)=\alpha\beta=v$  and  $V(Y)=\alpha\beta^2=2v$ .

a.  $E(Y_1+Y_2)=E(Y_1)+E(Y_2)=v_1+v_2$ .

b.  $V(Y_1+Y_2)=V(Y_1)+V(Y_2)+2\text{Cov}(Y_1, Y_2)$ . As  $Y_1$  and  $Y_2$  are independent,  $2\text{Cov}(Y_1, Y_2)=0$ .  
 $V(Y_1+Y_2)=2v_1+2v_2$ .

16) The joint density of  $Y_1$  and  $Y_2$  is given by

$$f(y_1, y_2) = \begin{cases} y_1 + y_2, & 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

Find  $E(30Y_1 + 25Y_2)$  and  $\text{Var}(30Y_1 + 25Y_2)$ .

**SOLUTION**

$$f_1(y_1) = \int_0^1 (y_1 + y_2) dy_2 = y_1 + \frac{1}{2}, \quad 0 \leq y_1 \leq 1. \quad \text{Similarly } f_2(y_2) = y_2 + \frac{1}{2}, \quad 0 \leq y_2 \leq 1.$$

$$E(Y_1) = 7/12 \text{ and } E(Y_2) = 7/12. \text{ So, } E(30Y_1 + 25Y_2) = 30(7/12) + 25(7/12) = 32.08.$$

We have  $E(Y_1) = E(Y_2) = 7/12$ . Intermediate calculations give  $V(Y_1) = V(Y_2) = 11/144$ .

$$\text{Thus, } E(Y_1 Y_2) = \int_0^1 \int_0^1 y_1 y_2 (y_1 + y_2) dy_1 dy_2 = 1/3, \quad \text{Cov}(Y_1, Y_2) = 1/3 - (7/12)^2 = -1/144.$$

$$E(30Y_1 + 25Y_2) = 32.08, \text{ so}$$

$$V(30Y_1 + 25Y_2) = 900V(Y_1) + 625V(Y_2) + 2(30)(25)\text{Cov}(Y_1, Y_2) = 106.08.$$

17)

The amount of flour used per day by a bakery is a random variable  $Y$  that has an exponential distribution with mean equal to 4 tons. The cost of the flour is proportional to  $U = 3Y + 1$ .

- Find the probability density function of  $U$  via the method of distribution functions.
- Find the probability density function of  $U$  via the method of transformations.

**SOLUTION**

a)

The distribution function of  $Y$  is  $F_Y(y) = 1 - e^{-y/4}, \quad 0 \leq y$ .

$$F_U(u) = P(U \leq u) = P(3Y + 1 \leq u) = P(Y \leq \frac{u-1}{3}) = F_Y(\frac{u-1}{3}) = 1 - e^{-(u-1)/12}. \quad \text{Thus,}$$

$$f_U(u) = F'_U(u) = \frac{1}{12} e^{-(u-1)/12}, \quad u \geq 1.$$

$$\text{if } \frac{u-1}{3} \geq 0$$

$$F_Y(\frac{u-1}{3}) = 0 \text{ if } \frac{u-1}{3} < 0$$

b) Note that  $U = 3Y + 1$  is strictly monotonic.

If  $U = 3Y + 1$ , then  $Y = \frac{U-1}{3}$ . Thus,  $\frac{dy}{du} = \frac{1}{3}$ . With  $f_Y(y) = \frac{1}{4} e^{-y/4}$ , we have that

$$f_U(u) = \frac{1}{3} \left[ \frac{1}{4} e^{-(u-1)/12} \right] = \frac{1}{12} e^{-(u-1)/12}, \quad 1 \leq u.$$

**18)** For the given joint density function, find the probability density function of  $U=Y_1-Y_2$ .

$$f(y_1, y_2) = \begin{cases} e^{-y_1}, & 0 \leq y_2 \leq y_1 < \infty \\ 0, & \text{elsewhere} \end{cases}$$

**SOLUTION**

$$F_U(u) = P(U \leq u) = P(Y_1 - Y_2 \leq u) = P(Y_1 \leq u + Y_2)$$

$$\text{For } u \geq 0, P(Y_1 \leq u + Y_2) = \int_0^\infty \int_{y_2}^{u+y_2} e^{-y_1} dy_1 dy_2 = 1 - e^{-u}$$

$$\text{For } u < 0, P(Y_1 \leq u + Y_2) = 0$$

$$f_U(u) = F'_U(u) = e^{-u}, u \geq 0$$

$$f_U(u) = \begin{cases} e^{-u}, & u \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

**19)**

The joint probability density function of  $X$  and  $Y$  is given by

$$f(x, y) = \frac{6}{7} \left( x^2 + \frac{xy}{2} \right), \quad 0 < x < 1, \quad 0 < y < 2$$

- Verify that is indeed a joint density function.
- Compute the marginal density function of  $X$ .
- Find  $P(X > Y)$ .

**SOLUTION**

**a)**

By the property of the joint pdf, it must be true that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$$

Let's look at our joint pdf.

$$\int_0^2 \int_0^1 \left( \frac{6}{7} \left( x^2 + \frac{xy}{2} \right) \right) dx dy = \frac{6}{7} \left( \int_0^1 \int_0^2 \left( x^2 + \frac{xy}{2} \right) dy dx \right)$$

$$\frac{6}{7} \left( \int_0^1 \left[ x^2 y + \frac{xy^2}{4} \right]_{y=0}^{y=2} dx \right) = \frac{6}{7} \left( \int_0^1 (2x^2 + x) dx \right)$$

$$\frac{6}{7} \left[ \frac{2x^3}{3} + \frac{x^2}{2} \right]_{x=0}^{x=1} = \frac{6}{7} \left( \frac{2}{3} + \frac{1}{2} \right) = \frac{6}{7} * \frac{7}{6} = 1$$

b)

To find the marginal pdf of X,

For  $0 < x < 1$ ,  $f_X(x) = \int_0^2 f(x, y) \cdot dy = \int_0^2 \left( \frac{6}{7} \left( x^2 + \frac{xy}{2} \right) \right) dy$

$$f_X(x) = \frac{6}{7} \left[ x^2 y + \frac{xy^2}{4} \right]_{y=0}^{y=2} = \frac{6}{7} (2x^2 + x) = \frac{12x^2}{7} + \frac{6x}{7} \text{ for } 0 < x < 1$$

So  $f_X(x) = \begin{cases} \frac{12x^2}{7} + \frac{6x}{7}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

c)

$$P(X > Y) = \int_0^1 \int_0^x \left( \frac{6}{7} \left( x^2 + \frac{xy}{2} \right) \right) dy dx = \int_0^1 \frac{6}{7} \left[ x^2 y + \frac{xy^2}{4} \right]_{y=0}^{y=x} dx$$

$$P(X > Y) = \int_0^1 \left( \frac{6x^3}{7} + \frac{3x^3}{14} \right) dx = \int_0^1 \left( \frac{15x^3}{14} \right) dx$$

$$P(X > Y) = \left[ \frac{15x^4}{56} \right]_{x=0}^{x=1} = \frac{15}{56}$$

20)

Assume that  $(X, Y)$  has the following joint probability function,

$$f(x, y) = \begin{cases} e^{-y}, & 0 < x < y < \infty \\ 0, & \text{elsewhere} \end{cases}$$

a) Calculate  $P(Y > 4 | X = 5)$ .

b) Calculate  $E(Y | X=x)$ .

### SOLUTION

a)

First, we need to find the marginal pdf of X.

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy = \int_x^{\infty} e^{-y} dy = [-e^{-y}]_x^{\infty} = e^{-x}$$

$$f_X(x) = \begin{cases} e^{-x}, & 0 < x < \infty \\ 0, & \text{otherwise} \end{cases}$$

For  $x > 0$ :

$$f_{Y|X}(y|x) = \frac{f(x, y)}{f_X(x)} = \begin{cases} e^{-y+x}, & \text{if } x < y \\ 0, & \text{otherwise} \end{cases}$$

$$P(Y > 4 | X = 5) = \int_5^{\infty} f(y|x=5) dy = \int_5^{\infty} e^{-y+5} dy$$

$$P(Y > 4 | X = 5) = [-e^{-y+5}]_5^{\infty} = e^0 = 1$$

b)

$$E(Y|X = x) = \int_x^{\infty} yf(y|x)dy = \int_x^{\infty} ye^{-y+x}dy$$

Integration by parts: with  $u = y$ ,  $du = dy$ ,  $dv = e^{-y+x}dy$ ,  $v = -e^{-y+x}$

$$E(Y|X = x) = y(-e^{-y+x})|_{y=x}^{y=\infty} - \int_x^{\infty} -e^{-y+x}dy$$

$$E(Y|X = x) = x - (e^{-y+x})|_{y=x}^{y=\infty} = x - (0 - 1) = x + 1$$

21)

Let  $X$  be such that  $\ln(X) \sim N(\mu, \sigma^2)$ . Then prove that

a)  $E(X) = e^{\mu + (\frac{\sigma^2}{2})}$

b)  $E(X^2) = e^{2\mu + 2\sigma^2}$

c)  $V(X) = e^{2\mu + \sigma^2} (e^{\sigma^2} - 1)$

#### SOLUTION

a)  $\ln X \sim N(\mu, \sigma^2)$ .

Then, show that  $E(X) = e^{\mu + \sigma^2/2}$

Suppose that  $Y = \ln X$ . As  $Y$  has a normal distribution by using moment generating function of normal distribution

$$m_Y(t) = E[e^{tY}] = \exp(\mu t + \frac{\sigma^2 t^2}{2})$$

Set  $t=1$ .

$$E[e^Y] = \exp(\mu + \frac{\sigma^2}{2})$$

$$E[X] = E[e^{\ln X}] = E[e^Y] = \exp(\mu + \frac{\sigma^2}{2})$$

b)  $E[X^2] = E[e^{\ln X^2}] = E[e^{2\ln X}] = E[e^{2Y}]$

Let  $t=2$ .

$$E[e^{2Y}] = \exp(2\mu + 2\sigma^2)$$

c)

$$\begin{aligned} V(X) &= E[X^2] - [E[X]]^2 = \exp(2\mu + 2\sigma^2) - [\exp(\mu + \frac{\sigma^2}{2})]^2 = \exp(2\mu + 2\sigma^2) - \exp(2\mu + \sigma^2) \\ &= \exp(2\mu + \sigma^2) (\exp(\sigma^2) - 1). \end{aligned}$$

22)

Let the probability density function of  $X_1$  and  $X_2$  be given by

$$f(x_1, x_2) = \begin{cases} e^{-(x_1+x_2)}, & \text{for } x_1 > 0, \quad x_2 > 0 \\ 0, & \text{otherwise} \end{cases}$$

Consider two random variables  $Y_1$  and  $Y_2$  are given by

$$Y_1 = X_1 + X_2$$

$$Y_2 = \frac{X_1}{X_1 + X_2}$$

Find the joint density of  $Y_1$  and  $Y_2$

### SOLUTION

$$Y_1 = X_1 + X_2$$

$$Y_2 = \frac{X_1}{X_1 + X_2}$$

$$X_1 = Y_1 Y_2$$

$$X_2 = Y_1 - Y_1 Y_2$$

$$J = \begin{vmatrix} y_2 & y_1 \\ 1 - y_2 & -y_1 \end{vmatrix} = -y_1 y_2 - (y_1(1 - y_2)) = -y_1$$

$$|J| = |-y_1| = y_1$$

$$f(y_1, y_2) = \exp(-(y_1 y_2) + y_1 - y_1 y_2) y_1$$

$$f(y_1, y_2) = \begin{cases} e^{-y_1 y_2}, & y_1 > 0 \text{ and } 0 < y_2 < 1 \\ 0, & \text{elsewhere} \end{cases}$$