

Law of Total Prob. & Bayers Rule

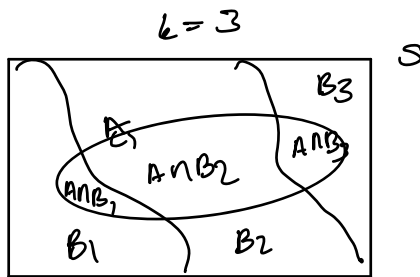
for some positive integer k , let sets $B_1, B_2, \dots, B_k$ such that ① $S = B_1 \cup B_2 \dots \cup B_k$

$$\textcircled{2} B_i \cap B_j = \emptyset$$

collection of sets $[B_1, B_2, \dots, B_k]$ is said to be partition of S

If A is any subset of $S \Rightarrow A$ can be decomposed;

$$A = (A \cap B_1) \cup (A \cap B_2) \cup \dots (A \cap B_k)$$



Assume that these partitions $[B_1, B_2, B_3, \dots, B_k]$ is a partition of S . such that $P(B_i) > 0$, $i=1, 2, \dots, k$ for any event A

$$P(A) = \sum_{i=1}^k P(A|B_i) \cdot P(B_i)$$

Proof: Any subset A of S can be written

$$A = A \cap S = A \cap (B_1 \cup B_2 \dots \cup B_k) = (A \cap B_1) \cup (A \cap B_2) \cup \dots \cup (A \cap B_k)$$

since $[B_1, B_2, \dots, B_k]$ is a partition of S $i \neq j$

therefore

$$(A \cap B_i) \cap (A \cap B_j) = A \cap (B_i \cap B_j) = A \cap \emptyset = \emptyset \Rightarrow (A \cap B_i) \cap (A \cap B_j) = \emptyset$$

is mutually exclusive

$$P(A) = P(A \cap B_1) + P(A \cap B_2) + \dots + P(A \cap B_k) = P(A|B_1) \cdot P(B_1) + P(A|B_2) \cdot P(B_2) + \dots + P(A|B_k) \cdot P(B_k)$$

Bayer's Rule

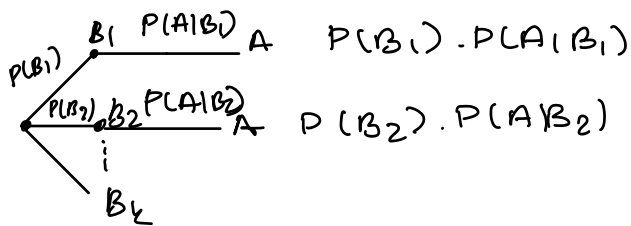
Assume that $\{B_1, B_2, \dots, B_k\}$ is a partition of S . Such that $P(B_i) > 0$, $i = 1, 2, \dots, k$

$$P(B_j|A) = \frac{P(A|B_j) \cdot P(B_j)}{\sum_{i=1}^k P(A|B_i) \cdot P(B_i)}$$

Proof: From definition of conditional prob. & law of Total prob.

$$P(B_j|A) = \frac{P(A \cap B_j)}{P(A)} = \frac{P(A|B_j) \cdot P(B_j)}{\sum_{i=1}^k P(A|B_i) \cdot P(B_i)}$$

Bayer's Tree



Ex1 A firm rents cars from 3 rental agencies

60% of cars from agency 1		9% needs tune up
30% of cars from agency 2		20% needs tune up
10% of cars from agency 3		6% needs tune up

a) what is the probability that a rental car delivered to firm will need tune up B_1, B_2, B_3 3 agencies

$A \Rightarrow$ event that car needs tune up

$$P(B_1) = 0.6 \quad P(B_2) = 0.3 \quad P(B_3) = 0.1$$

$$P(A|B_1) = 0.09 \quad P(A|B_2) = 0.2 \quad P(A|B_3) = 0.06$$

$$P(A) = 0.6 \times 0.09 + 0.3 \times 0.2 + 0.1 \times 0.06 = 0.1$$

b) If a rental car delivered to firm needs tune-up what is the prob. that it comes from agency 1

$$P(B_2|A) = \frac{P(B_2) \cdot P(A|B_2)}{\sum_{i=1}^3 P(B_i) \cdot P(A|B_i)} = \frac{(0.3 \times 0.2)}{(0.6 \times 0.09) + (0.3 \times 0.2) + (0.1 \times 0.06)}$$

(backward reasoning)

Random variables & Sampling

Numerical events

Let $Y \rightarrow$ denote a variable to be measured in an experiment

Because value of Y will vary depending on outcome of an experiment it is called random variable

To each point in $S \rightarrow$ we assign a real number,

denoting value of variable Y

Value assigned to $Y \Rightarrow$ will vary from one sample point to another but some points may be assigned some numerical value

We have defined a variable that is function of sample point in sample space

So, the sample space can be partitioned into subsets so that points within a subset are all assigned the same value of Y . & the subsets are mutually exclusive

A random variable is a real valued func for which the domain is sample space, S

Let y denote an observed value of random variable
 $P(Y=y)$ is sum of probs. of sample points that are assigned value y

Ex: Throwing a pair of dice. Random variable Y takes values 9, $Y=9$ for subset

$\{(6,3) (5,4) (4,5) (3,6)\}$ of S

$Y=9 \rightarrow$ set of elements S for which total is 9

Ex: Two socks are selected at random & removed in succession from a drawer containing 5 brown & 3 green socks. List elements of S , corresponding probs, corresponding values y of random var. Y , where Y is number of brown socks selected

	BB	BG	GB	GG	$P(Y=2) = 5/14$
$P(Y)$	$5/14$	$15/56$	$15/56$	$3/28$	$P(Y=1) = 15/28$
$y =$	2	1	1	0	$P(Y=0) = 3/28$

Discrete Random Variables & Their Probability Distributions
 a random variable is discrete, whose range is finite or countably infinite

We have numerical events corresponding values of a discrete random variables $\rightarrow$ prob. for each value of random var.
 This collection of probs $\rightarrow$ prob distribution of a discrete random variable

Probability that Y takes value y . $P(Y=y)$ is defined as sum of probs. of all sample points in S that are assigned value y

Denoted $P(y)$ instead of $P(Y=y)$

If Y is a discrete random variable, function given by $P(y) = P(Y=y)$ for each y within range of Y is called prob. distr. of Y

$$P(Y) \geq 0 \quad \& \quad P(Y) \leq 1$$

Probability Distribution for a discrete random var.
 Y can be represented by a formula, table, graph
 printing $P(Y) = P(Y=y)$, $\forall y$

Ex: Find a formula for prob. distr. of total number
 of heads obtained in 4 tosses of a coin

Elements of S

HHHH	TTTT	TTHH	HTHH	each case 1/16
HHHT	THTT	THTH	HHTH	
HHTT	TTHT	HTHT	HTTH	
HTTT	TTTH	THTH	THTT	

$y \rightarrow 4, 3, 2, 1, 0$

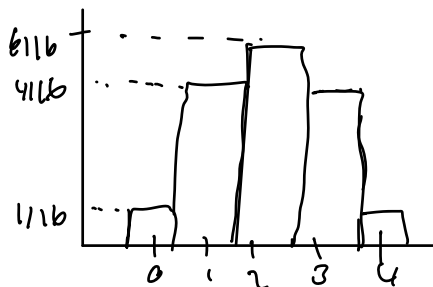
$$P(Y=0) = 1/16 \quad P(Y=3) = 4/16$$

$$P(Y=1) = 4/16 \quad P(Y=4) = 1/16$$

$$P(Y=2) = 6/16$$

Numerators of these
 fractions 1, 4, 6, 4, 1 are
 binomial coefficients

$$\binom{4}{0} \binom{4}{1} \binom{4}{2} \binom{4}{3} \binom{4}{4} \quad P(y) = \frac{\binom{4}{y}}{16}, \quad y=0,1,2,3,4$$



A function can serve as probability distribution of a discrete function Y iff its value $P(y)$ satisfies conditions

$$\textcircled{1} \quad 0 \leq P(y) \leq 1, \forall y$$

$$\textcircled{2} \quad \sum_y P(y) = 1$$

if Y is a discrete random variable, function given by $F(y) = P(Y \leq y) = \sum_{t=y} p(t)$, $-\infty < y < \infty$ where $p(t)$ is

value of prob. distribution of Y at t .

$F(y) \rightarrow$ cumulative distr. or distr. function of Y

Expected Value of a Random Variable

If Y is a discrete rv. and $P(y)$ is value of its prob. distribution & the expected value of Y is defined as

$$E(Y) = \sum_y y \cdot P(y)$$

If the expected value of a discrete rv. is said to exist

if the sum is absolutely convergent. that is if

$$\sum_y |y| \cdot P(y) < \infty$$

If $P(y)$ is an accurate characterization of population

frequency distribution $E(Y) = \mu$
 $\hookrightarrow$ population mean

Ex: A lot of 12 TV includes 2 with white cords

3 of TVs are chosen for shipment to a hotel. How many

TV's with white cords can shipper expect to send to hotel

y of 2 TV's with white cords

$3-y$ of 10 other TV's can be chosen in $\binom{2}{y} \binom{10}{3-y}$ ways

3 of 12 TV's can be chosen $\binom{12}{3}$ ways

probability distribution of y , number of TV's with white cords is

$$P(y) = \frac{\binom{2}{y} \binom{10}{3-y}}{\binom{12}{3}}$$

y	0	1	2	$E(y) = \sum_y y \cdot P(y)$
$P(y)$	$6/11$	$9/22$	$1/22$	$E(y) = 0 \cdot \frac{6}{11} + 1 \cdot \frac{9}{22} + 2 \cdot \frac{1}{22} = \frac{1}{2}$

If y is a discrete random variable, $P(y)$ is value of its prob. distr. of y

$g(y)$ is a func. of y , then exp value of $g(y)$ is

$$E[g(y)] = \sum_y g(y) \cdot P(y)$$

Proof: Assume y has a finite range, $y_1, y_2, y_3, \dots, y_n$

suppose that $g(y)$ takes on value g_i when y takes values

$y_{i1}, y_{i2}, \dots, y_{in_i}$. Then prob. that $g(y)$ will take on value g_i

$P[g(Y) = g_i] = \sum_{i=1}^{n_i} P(y_{ij})$. If $g(y)$ takes values $g_1, g_2, \dots, g_m$, then

$$E[g(Y)] = \sum_{i=1}^m g_i P[g(Y) = g_i] = \sum_{i=1}^m g_i \cdot \sum_{j=1}^{n_i} P(y_{ij})$$

$$= \sum_{i=1}^m \sum_{j=1}^{n_i} g_i P(y_{ij}) = \sum_y g(y) \cdot P(y)$$

Ex: If Y is # of points rolled with a die. Find expected value of $g(Y) = 2Y^2 + 1$

$$\sum_y [g(y)] = \sum_{y=1}^6 (2y^2 + 1) \cdot \frac{1}{6} = (2 \cdot 1^2 + 1) \cdot \frac{1}{6} + \dots + (2 \cdot 6^2 + 1) \cdot \frac{1}{6}$$

$$= 94/3$$

Variance of a rv Y is defined as expected value of $(Y - \mu)^2$

$$V(Y) = E[(Y - \mu)^2], \quad \sigma^2 \rightarrow \text{population variance}$$

$$\sigma \rightarrow \text{standard deviation}$$

Ex:

Y	$P(Y)$
0	6/11
1	9/22
2	1/22

$$E(Y) = 1/2$$

$$\sigma^2 = E[(Y - \mu)^2] = \sum_{y=0}^2 (y - \mu)^2 \cdot P(y)$$

$$= (0 - \frac{1}{2})^2 \cdot \frac{6}{11} + (1 - \frac{1}{2})^2 \cdot \frac{9}{22} + (2 - \frac{1}{2})^2 \cdot \left(\frac{1}{22}\right)$$

$$= 0.136 + 0.102 + 0.102 = 0.340$$

$$\sigma = \sqrt{0.340} = 0.583$$

Ex:

By using above theorem, calculate variance of Y representing # of coins rolled with a dice

$$\mu = E(Y) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = 7/2$$

$$\begin{aligned} E(Y^2) &= 1^2 \cdot \frac{1}{6} + 2^2 \cdot \frac{1}{6} + 3^2 \cdot \frac{1}{6} + 4^2 \cdot \frac{1}{6} + 5^2 \cdot \frac{1}{6} + 6^2 \cdot \frac{1}{6} \\ &= \sigma^2 = E(Y^2) - \mu^2 = \sigma^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = 35/12 \end{aligned}$$