

Review Problem Set 1 - Answers

- 1) Show that for any three events A, B and C with $P(C) > 0$, $P(A \cup B | C) = P(A | C) + P(B | C) - P(A \cap B | C)$.

$$\begin{aligned} P(A \cup B | C) &= \frac{P[(A \cup B) \cap C]}{P(C)} = \frac{P[(A \cap C) \cup (B \cap C)]}{P(C)} \\ &= \frac{P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)}{P(C)} \\ &= P(A | C) + P(B | C) - P(A \cap B | C) \end{aligned}$$

- 2) A deck consists of seven cards, marked 1, 2, ..., 7. Three of these cards are selected at random. Define a random variable W as the sum of the resulting numbers. Compute the probability mass function (pmf) of W, and find its expected value $E(W)$ and variance $V(W)$.

SOLUTION

Each 3-card draw combination can be expressed as $\{C_1, C_2, C_3\}$. The number of outcomes is calculated as the number of 3-card combinations from a 7-card deck.

Outcomes are (1,2,3) (1,2,4) (1,2,5) ... (5,6,7); there are 35 such outcomes, each having probability $\frac{1}{35}$. The W values for these outcomes are 6 ($= 1 + 2 + 3$), 7, 8, ..., 18. Since there is just one outcome with W value 6, $p(6) = P(W=6) = \frac{1}{35}$. Similarly, there are three outcomes with W value 9 [(1,2,6) (1,3,5) and (2,3,4)], so $p(9) = \frac{3}{35}$. Continuing in this manner yields the following distribution:

w	6	7	8	9	10	11	12	13	14	15	16	17	18
$p(w)$	$\frac{1}{35}$	$\frac{1}{35}$	$\frac{2}{35}$	$\frac{3}{35}$	$\frac{4}{35}$	$\frac{4}{35}$	$\frac{5}{35}$	$\frac{4}{35}$	$\frac{4}{35}$	$\frac{3}{35}$	$\frac{2}{35}$	$\frac{1}{35}$	$\frac{1}{35}$

Since the distribution is symmetric about 12, $\mu = 12$, and $\sigma^2 = \sum_{w=6}^{18} (w-12)^2 p(w) =$

$$\frac{1}{35} [(-6)^2(1) + (-5)^2(1) + \dots + (5)^2(1) + (6)^2(1)] = 8.$$

- 3) Suppose that of all individuals buying a new iPhone, 60% include a heavy-duty phone case, 40% include a portable battery, and 30% include both. If an iPhone buyer is randomly selected, what is the probability they purchased a heavy-duty case given that they purchased a portable battery? What is the probability they purchased a portable battery given that they purchased a heavy-duty case?

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{.30}{.40} = .75$$

That is, of all those purchasing a portable battery, 75% purchased a heavy-duty phone case. Similarly,

$$P(\text{battery}|\text{case}) = P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{.30}{.60} = .50$$

Notice that $P(A|B) \neq P(A)$ and $P(B|A) \neq P(B)$. Notice also that $P(A|B) \neq P(B|A)$: these represent two different probabilities computed using different pieces of “given” information. ■

- 4) A university receives a shipment of 25 laptops, with 10 AMD and 15 Intel processors. If 6 laptops are selected at random, what is the probability that exactly 3 have Intel processors?

Let $D_3 = \{\text{exactly 3 of the 6 selected have Intel processors}\}$. Assuming that any particular set of 6 laptops is as likely to be chosen as is any other set of 6, we have equally likely outcomes, so $P(D_3) = N(D_3)/N$, where N is the number of ways of choosing 6 laptops from the 25 and $N(D_3)$ is the number of ways of choosing 3 with AMD processors and 3 with Intel chips. Thus $N = \binom{25}{6}$. To obtain $N(D_3)$, think of first choosing 3 of the 15 Intel laptops and then 3 of the AMD laptops. There are $\binom{15}{3}$ ways of choosing the 3 with Intel processors, and there are $\binom{10}{3}$ ways of choosing the 3 with AMD processors; by the Fundamental Counting Principle, $N(D_3)$ is the product of these two numbers. So,

$$P(D_3) = \frac{N(D_3)}{N} = \frac{\binom{15}{3} \binom{10}{3}}{\binom{25}{6}} = \frac{\frac{15!}{3!12!} \cdot \frac{10!}{3!7!}}{\frac{25!}{6!19!}} = .3083$$

- 5) Let X = the number of days of sick leave taken by a randomly selected employee of a large company during a particular year. Assume the maximum number of allowable sick days per year is 14. Assume also that $F(0)=.58$, $F(1)=.72$, $F(2)=.76$, $F(3)=0.81$, $F(4)=.88$ and $F(5)=0.94$. Calculate the following:

- $P(2 \leq X \leq 5)$
- $P(X=3)$

SOLUTION

- $P(2 \leq X \leq 5) = P(X = 2, 3, 4 \text{ or } 5) = F(5) - F(1) = .22$
- $P(X=3)=F(3)-F(2)=0.05$

- 6) The lifetime of an instrument produced by a machine has a normal distribution with a mean of 12 months and standard deviation of 2 months. Find the probability that an instrument produced by this machine will last

- less than 7 months
- between 7 and 12 months
- more than 10 months

SOLUTION

a) For $x=7$, $z = \frac{x-\mu}{\sigma} = \frac{7-12}{2} = -2.5$

$$P(X < 7) = P(Z < -2.5) = P(Z > 2.5) = 1 - P(Z < 2.5)$$

z	.00	.01	.02
2.5	.9938	.9940	.9941
2.6	.9953	.9955	.9956
2.7	.9965	.9966	.9967

$$P(X < 7) = 1 - 0.9938 = 0.0062$$

b)

$$P(7 < X < 12) = P\left(\frac{7-12}{2} < Z < \frac{12-12}{2}\right) = P(-2.5 < Z < 0) \\ = P(Z < 0) - P(Z < -2.5)$$

z	.00	.01	.02
0.0	.5000	.5040	.5080
0.1	.5398	.5438	.5478
0.2	.5793	.5832	.5871

z	.00	.01	.02	.03
-3.4	.0003	.0003	.0003	.0003
-3.3	.0005	.0005	.0005	.0004
-3.2	.0007	.0007	.0006	.0006
-3.1	.0010	.0009	.0009	.0009
-3.0	.0013	.0013	.0013	.0012
-2.9	.0019	.0018	.0017	.0017
-2.8	.0026	.0025	.0024	.0023
-2.7	.0035	.0034	.0033	.0032
-2.6	.0047	.0045	.0044	.0043
-2.5	.0062	.0060	.0059	.0057

$$P(7 < X < 12) = 0.5 - 0.0062 = 0.4938$$

c) $P(X > 10) = P\left(Z > \frac{10-12}{2}\right) = P(Z > -1) = 1 - P(Z < -1)$

$$\text{As } P(Z > -1) = P(Z < 1)$$

$$P(X > 10) = 1 - 0.1587 = 0.8413$$

z	.00	.01	.02
-1.0	.1587	.1562	.1539
-0.9	.1841	.1814	.1788
-0.8	.2119	.2090	.2061

7) The amount of time, in minutes, that Alice must wait for a bus is uniformly distributed between 0 and 15 minutes, inclusive.

- a) What is the probability that Alice waits less than 12.5 minutes?
- b) Find μ and σ .
- c) Ninety percent of the time, the time Alice must wait falls below what value?

SOLUTION

a) $X \sim U[0,15]$

$$f(x) = \begin{cases} \frac{1}{15}, & 0 \leq X \leq 15 \\ 0, & \text{otherwise} \end{cases}$$

$$P(X < 12.5) = \frac{12.5-0}{15} = 0.83$$

b) $\mu = \frac{15+0}{2} = 7.5$

$$\sigma = \sqrt{\frac{15^2}{12}} = 4.3$$

c) $P(X < k) = \frac{k}{15} = 0.9$

$$k = 13.5$$

The 90% of the time, Alice must wait at most 13.5 minutes.

- 8) Suppose the time spent by a randomly selected student at a campus computer laboratory has a gamma distribution with a mean of 20 minutes and a variance of 80 minutes squared.
- a. What are the values of the parameters α and β ?
 - b. What is the probability that a student uses the laboratory for at most 24 minutes?
 - c. What is the probability that a student spends between 20 and 40 minutes at the laboratory?

a. $\alpha\beta = 20, \alpha\beta^2 = 80 \Rightarrow \beta = 4 \Rightarrow \alpha = 5 .$

b. Let $X \sim \text{Gamma}(5, 4)$. $P(X \leq 24) = G(24 / 4; 5) = G(6; 5) = .715 .$

c. $P(20 \leq X \leq 40) = G(40 / 4; 5) - G(20 / 4; 5) = G(10; 5) - G(5; 5) = .411 .$

9) Let X = the time between two successive arrivals at the drive-up window of a local bank. If X has an exponential distribution with $\lambda = 1$ (which is identical to a standard gamma distribution with $\alpha = 1$), compute the following:

- a. The expected time between two successive arrivals.
- b. The standard deviation of the time between successive arrivals.
- c. $P(X \leq 4)$.
- d. $P(2 \leq X \leq 5)$.

a. $E(X) = \frac{1}{\lambda} = 1$.

b. $\sigma = \frac{1}{\lambda} = 1$.

c. Using the exponential cdf, $P(X \leq 4) = 1 - e^{-(1)(4)} = 1 - e^{-4} = .982$.

d. Similarly, $P(2 \leq X \leq 5) = (1 - e^{-(1)(5)}) - (1 - e^{-(1)(2)}) = e^{-2} - e^{-5} = .129$.

10) A theoretical justification based on a material failure mechanism underlies the assumption that ductile strength X of a material has a lognormal distribution. Suppose the parameters are $\mu = 5$ and $\sigma = .1$.

- a. Compute $E(X)$ and $V(X)$.
- b. Compute $P(X > 125)$.
- c. Compute $P(110 \leq X \leq 125)$.
- d. What is the value of median ductile strength?
- e. If ten different samples of an alloy steel of this type were subjected to a strength test, how many would you expect to have strength of at least 125?
- f. If the smallest 5% of strength values were unacceptable, what would the minimum acceptable strength be?

- a.** $E(X) = e^{5+(.01)/2} = e^{5.005} = 149.157$; $V(X) = e^{10+(.01)} \cdot (e^{.01} - 1) = 223.594$.
- b.** $P(X > 125) = 1 - P(X \leq 125) = 1 - \Phi\left(\frac{\ln(125) - 5}{.1}\right) = 1 - \Phi(-1.72) = .9573$.
- c.** $P(110 \leq X \leq 125) = \Phi(-1.72) - \Phi\left(\frac{\ln(110) - 5}{.1}\right) = .0427 - .0013 = .0414$.
- d.** $\tilde{\mu} = e^{\mu} = e^5 = 148.41$.
- e.** $P(\text{any particular one has } X > 125) = .9573 \Rightarrow \text{expected \#} = 10(.9573) = 9.573$.
- f.** We want the 5th percentile, which is $e^{5+(-1.645)(.1)} = 125.90$.

Table A.4 The incomplete gamma function
$$G(x; \alpha) = \int_0^x \frac{1}{\Gamma(\alpha)} y^{\alpha-1} e^{-y} dy$$

x	α									
	1	2	3	4	5	6	7	8	9	10
1	.632	.264	.080	.019	.004	.001	.000	.000	.000	.000
2	.865	.594	.323	.143	.053	.017	.005	.001	.000	.000
3	.950	.801	.577	.353	.185	.084	.034	.012	.004	.001
4	.982	.908	.762	.567	.371	.215	.111	.051	.021	.008
5	.993	.960	.875	.735	.560	.384	.238	.133	.068	.032
6	.998	.983	.938	.849	.715	.554	.394	.256	.153	.084
7	.999	.993	.970	.918	.827	.699	.550	.401	.271	.170
8	1.000	.997	.986	.958	.900	.809	.687	.547	.407	.283
9		.999	.994	.979	.945	.884	.793	.676	.544	.413
10		1.000	.997	.990	.971	.933	.870	.780	.667	.542
11			.999	.995	.985	.962	.921	.857	.768	.659
12			1.000	.998	.992	.980	.954	.911	.845	.758
13				.999	.996	.989	.974	.946	.900	.834
14				1.000	.998	.994	.986	.968	.938	.891
15					.999	.997	.992	.982	.963	.930

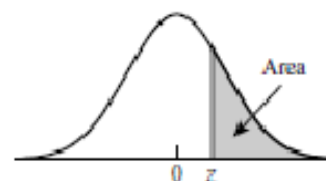
Table A.3 Standard Normal Curve Areas (*cont.*)

$\Phi(z) = P(Z \leq z)$

<i>z</i>	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9278	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Normal Curve Areas

Standard normal probability in right-hand tail (for negative values of z , areas are found by symmetry)



Second decimal place of z										
z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641
0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0722	.0708	.0694	.0681
1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
1.8	.0359	.0352	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
2.9	.0019	.0018	.0017	.0017	.0016	.0016	.0015	.0015	.0014	.0014