

PS#3 Solutions :

Q.1

$$a) E(\hat{\mu}_1) = E\left[\frac{1}{2}(y_1 + y_2)\right] = \frac{1}{2}[E(y_1) + E(y_2)] = \frac{1}{2}(\mu + \mu) = \frac{1}{2} \cdot 2\mu = \mu.$$

$$\begin{aligned} E(\hat{\mu}_2) &= E\left[\frac{1}{4}y_1 + \frac{y_2 + \dots + y_{n-1}}{2(n-2)} + \frac{1}{4}y_n\right] \\ &= \frac{1}{4}E(y_1) + \frac{1}{2(n-2)} \underbrace{E(y_2 + \dots + y_{n-1})}_{= \sum_{i=2}^{n-1} E(y_i)} + \frac{1}{4}E(y_n) \\ &= \frac{1}{4} \cdot \mu + \frac{(n-2)\mu}{2(n-2)} + \frac{1}{4} \cdot \mu = \mu \end{aligned}$$

$$E(\hat{\mu}_3) = E\left(\frac{1}{n} \sum_{i=1}^n y_i\right) = \frac{1}{n} E(y_1 + \dots + y_n) = \frac{1}{n} \cdot n \cdot \mu = \mu.$$

All estimators are unbiased for μ .

$$b) \frac{V(\hat{\mu}_1)}{V(\hat{\mu}_3)} = ? \quad \frac{V(\hat{\mu}_2)}{V(\hat{\mu}_3)} = ?$$

$$V(\hat{\mu}_1) = V\left(\frac{1}{2}(y_1 + y_2)\right) = \frac{1}{4}[V(y_1) + V(y_2)] = \frac{1}{4} \cdot 2\sigma^2 = \frac{\sigma^2}{2}$$

$$V(\hat{\mu}_2) = V\left(\frac{1}{4}y_1 + \frac{y_2 + \dots + y_{n-1}}{2(n-2)} + \frac{1}{4}y_n\right)$$

$$= \frac{1}{16}V(y_1) + \frac{1}{4(n-2)^2} \cdot V\left(\sum_{i=2}^{n-1} y_i\right) + \frac{1}{16}V(y_n)$$

$$= \frac{\sigma^2}{16} + \frac{n-2}{4(n-2)^2} \cdot \sigma^2 + \frac{\sigma^2}{16} = \frac{(n-1)\sigma^2}{8(n-2)}$$

$$V(\hat{\mu}_3) = V\left(\frac{1}{n} \sum_{i=1}^n y_i\right) = \frac{1}{n^2} V\left(\sum_{i=1}^n y_i\right) = \frac{1}{n^2} \cdot n\sigma^2 = \frac{\sigma^2}{n}$$

$$\frac{V(\hat{\mu}_1)}{V(\hat{\mu}_3)} = \frac{\sigma^2/2}{\sigma^2/n} = \frac{n}{2} \quad \frac{V(\hat{\mu}_2)}{V(\hat{\mu}_3)} = \frac{\frac{(n-1)\sigma^2}{8(n-2)}}{\sigma^2/n} = \frac{n(n-1)}{n-2}$$

Q.2

In PS#2 / Q.2, we have found that PDF of

$Y(n)$ where $y_1, \dots, y_n \sim U[0, \theta]$ as:

$$g(n)(y) = \frac{n}{\theta} \cdot \left(\frac{y}{\theta}\right)^{n-1} \quad \text{and by using this density fnc,}$$

$$\text{we have found } V(Y(n)) = \frac{n}{(n+2)(n+1)^2} \cdot \theta^2$$

$$V(\hat{\theta}_1) = V\left[\frac{n+1}{n} \cdot Y(n)\right] = \left[\frac{n+1}{n}\right]^2 \cdot V(Y(n))$$

$$= \left(\frac{n+1}{n}\right)^2 \cdot V(Y(n)) = \left(\frac{n+1}{n}\right)^2 \cdot \frac{n}{(n+2)(n+1)^2} \cdot \theta^2$$

$$= \frac{\theta^2}{n(n+2)}$$

$$V(\hat{\theta}_2) = V(2\bar{y}) = 4V(\bar{y}) = 4 \cdot \frac{\sigma_y^2}{n} = \frac{4\theta^2}{12n} = \frac{\theta^2}{3n}$$

$$\frac{V(\hat{\theta}_2)}{V(\hat{\theta}_1)} = \frac{\theta^2/3n}{\theta^2/n(n+2)} = \frac{(n+2)}{3}$$

Q.3

a) CRLB inequality:

$$\text{var}(\hat{\lambda}) \geq \frac{1}{I(\lambda)}$$

where Fisher Information $I(\lambda)$:

$$I(\lambda) = n \cdot E \left[\frac{\partial \ln f(x; \lambda)}{\partial \lambda} \right]^2 = -n \cdot E \left[\frac{\partial^2 \ln f(x; \lambda)}{\partial^2 \lambda} \right]$$

$$f(x; \lambda) = \frac{\lambda^x \cdot e^{-\lambda}}{x!}$$

✓ You can use both if $\ln f(x; \lambda)$ is twice differentiable

$$\ln f(x; \lambda) = x \cdot \ln \lambda - \lambda - \ln x!$$

$$\frac{\partial \ln f(x; \lambda)}{\partial \lambda} = \frac{x}{\lambda} - 1 \Rightarrow \frac{\partial^2 \ln f(x; \lambda)}{\partial^2 \lambda} = -\frac{x}{\lambda^2}$$

$$E \left[-\frac{x}{\lambda^2} \right] = -\frac{1}{\lambda^2} E(x) = -\frac{1}{\lambda}$$

$$\text{Then ; } I(\lambda) = \frac{n}{\lambda}$$

According to Cramer-Rao LB $\text{var}(\hat{\lambda}) \geq \frac{\lambda}{n}$

b) $\text{var}(\bar{y}) = \frac{\lambda}{n}$ is at CRLB. So it's MVUE.

$\lim_{n \rightarrow \infty} \frac{\lambda}{n} = 0 \rightarrow$ Then, it's consistent estimator ($\bar{y}$) for the parameter λ .

Q.4

a) Let $f(x_1, \dots, x_n; \theta)$ is the joint PDF fnc of r.v. $X_1, \dots, X_n$

Then the statistics $\hat{\theta} = u(x_1, x_2, \dots, x_n)$ is sufficient for θ iff

the joint PDF can be factored into two components as:

$$f(x_1, \dots, x_n; \theta) = g(u(x_1, \dots, x_n), \theta) \cdot h(x_1, \dots, x_n) \text{ where}$$

1. $g()$ is a fnc that depends on the data only through the

fnc $u(x)$ and

2. the fnc $h(x)$ does not depend on the parameter θ .

$$f(x; \theta) = \frac{1}{\theta} \cdot e^{-x/\theta} \quad 0 < x < \infty$$

$$f(x_1, \dots, x_n; \theta) = \prod_{i=1}^n \frac{1}{\theta} \cdot e^{-x_i/\theta} = \underbrace{\frac{1}{\theta^n} \cdot e^{-\sum_{i=1}^n x_i/\theta}}_{g(u(x), \theta)} \cdot \underbrace{1}_{h(x)}$$

By factorization theorem; $u(x) = \sum x_i$ is sufficient stat for the parameter θ .

b) $E(\sum x_i) = n \cdot \theta$ is biased estimator.

$E(\frac{\sum x_i}{n}) = \theta$ is an unbiased estimator.

Since $\bar{X}$ is one-to-one fnc of $\sum x_i$, it's also sufficient. If we know $\sum_{i=1}^n x_i$, we can also find $\frac{\sum x_i}{n} = \bar{X}$.

We can also write the fnc components as:

$$f(x_1, \dots, x_n; \theta) = \underbrace{\frac{1}{\theta^n} \cdot e^{-\frac{n\bar{X}}{\theta}}}_{g(u(x), \theta)} \cdot \underbrace{1}_{h(x)} \text{ where } u(x) = \bar{X}$$

c) $\bar{X}$ is an unbiased estimator and sufficient. So, by Rao-Blackwell theorem

$E(\bar{X} | \sum x_i) = \bar{X}$ is the unique UMVUE estimator for θ .

d)

$$\begin{aligned} I(\theta) &= -E \left[\frac{\partial^2 \ln L(\theta)}{\partial^2 \theta} \right] = -E \left[\frac{\partial^2 \ln \prod f(x_i; \theta)}{\partial^2 \theta} \right] \\ &= -E \left[\frac{\partial^2 \sum \ln f(x_i; \theta)}{\partial^2 \theta} \right] \\ &= -\sum_{i=1}^n E \left[\frac{\partial^2 \ln f(x_i; \theta)}{\partial^2 \theta} \right] \\ &= -n \cdot E \left[\frac{\partial^2 \ln f(x; \theta)}{\partial^2 \theta} \right] \end{aligned}$$

$$I(\theta) = -n E \left[\frac{\partial^2 \ln f(x; \theta)}{\partial^2 \theta} \right] = n E \left[\left(\frac{\partial \ln f(x; \theta)}{\partial \theta} \right)^2 \right]$$

$$f(x; \theta) = \frac{1}{\theta} \cdot e^{-x/\theta} \quad f(x; \theta) = \frac{1}{\theta} \cdot e^{-x/\theta}$$

$$\ln f(x; \theta) = -\ln \theta - \frac{x}{\theta}$$

$$\frac{\partial \ln f(x; \theta)}{\partial \theta} = -\frac{1}{\theta} + \frac{x}{\theta^2} \quad \frac{\partial^2 \ln f(x; \theta)}{\partial^2 \theta} = \frac{1}{\theta^2} - \frac{2x}{\theta^3}$$

$$E \left(\frac{1}{\theta^2} - \frac{2x}{\theta^3} \right) = \frac{1}{\theta^2} - \frac{2}{\theta^3} E(x) = \frac{1}{\theta^2} - \frac{2}{\theta^3} \cdot \theta = -\frac{1}{\theta^2}$$

CRLB inequality.

$$\text{var}(\bar{X}) = \frac{\theta^2}{n} \text{ is at CRLB.}$$

$$\text{var}(\hat{\theta}) \geq \frac{\theta^2}{n}$$

Q.5

$$a) f(x; \theta) = \begin{cases} c \cdot e^{-(x-\theta)} & x \geq \theta \\ 0 & \text{else} \end{cases}$$

$$1 = c \int_0^{\infty} e^{-(x-\theta)} dx = -c \cdot e^{\theta} \cdot e^{-x} \Big|_0^{\infty} = 0 - (-c \cdot e^{\theta} \cdot e^{-\theta}) = c \Rightarrow c = 1$$

$$b) f(\underline{x}; \theta) = \prod_{i=1}^n f(x_i; \theta) = e^{-\sum x_i} \cdot e^{n\theta} \cdot 1(x_{(1)} \geq \theta)$$

$$h(\underline{x}) = e^{-\sum x_i}$$

$$g(T(x); \theta) = e^{n\theta} 1(x_{(1)} \geq \theta)$$

$$f(\underline{x}; \theta) = h(\underline{x}) \cdot g(T(x); \theta)$$

By factorization theorem $T(x) = X_{(1)}$ is a sufficient estimator for θ where $X_{(1)} = \min\{x_1, \dots, x_n\} = T_n$

$$c) F_{X_{(1)}}(x) = P(X_{(1)} \leq x) = 1 - P(X_{(1)} > x)$$

$$= 1 - \left(\int_x^{\infty} e^{-(x-\theta)} dx \right)^n = 1 - \left(e^{\theta} \cdot e^{-x} \Big|_x^{\infty} \right)^n = 1 - e^{-n(x-\theta)}$$

$$f_{X_{(1)}}(x) = \begin{cases} 1 - e^{-n(x-\theta)}, & x \geq \theta \\ 0 & \text{else} \end{cases}$$

$$f_{X_{(1)}}(x) = \frac{d}{dx} \cdot F_{X_{(1)}}(x) = n \cdot e^{-n(x-\theta)}$$

$$f_{X_{(1)}} = \begin{cases} n \cdot e^{-n(x-\theta)}, & x \geq \theta \\ 0 & \text{else} \end{cases}$$

$$E(X_{(1)}) = \int_0^{\infty} n \cdot x \cdot e^{-n(x-\theta)} dx = \frac{-(nx+1)e^{-n(x-\theta)}}{n} \Big|_{x=\theta}^{\infty} = \theta + 1/n$$

$W = X_{(1)} - \frac{1}{n}$ is an unbiased estimator for θ .

d) Since $W_n = X_{(1)} - \frac{1}{n}$ is unbiased

$$\text{Bias}(X_{(1)} - \frac{1}{n}) = 0$$

$$\text{Var}(X_{(1)} - \frac{1}{n}) = \text{Var}(X_{(1)}) = E(X_{(1)}^2) - [E(X_{(1)})]^2$$

$$E(X_{(1)}^2) = \int_{\theta}^{\infty} x^2 \cdot n \cdot e^{-n(x-\theta)} \cdot dx = - \frac{n^2 x^2 + 2nx + 2}{n^2} \cdot e^{-n(x-\theta)} \Big|_{\theta}^{\infty} \\ = \frac{n^2 \theta^2 + 2n\theta + 2}{n^2}$$

$$\text{Var}(X_{(1)}) = E(X_{(1)}^2) - [E(X_{(1)})]^2$$

$$= \frac{n^2 \theta^2 + 2n\theta + 2}{n^2} - \left(\theta + \frac{1}{n}\right)^2 = \frac{1}{n^2}$$

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} = 0 \quad \text{as } n \rightarrow \infty$$

$$n \rightarrow \infty \quad \text{Var}(X_{(1)} - \frac{1}{n}) \rightarrow 0.$$

$$\text{Since } \text{Bias}(W_n) = 0 \quad \& \quad \text{Var}(W_n) = \frac{1}{n^2} \rightarrow 0.$$

The unbiased estimator $W_n = X_{(1)} - \frac{1}{n}$ is consistent for θ .

e) Note that $W_n = X_{(1)} - \frac{1}{n}$ is unbiased and $X_{(1)}$ is sufficient by Rao-Blackwell thm:

$E(X_{(1)} - \frac{1}{n} | X_{(1)}) = X_{(1)} - \frac{1}{n}$ is the unique UMVUE estimator for θ .

Q.6

$$\begin{aligned}
 f(x_1, \dots, x_n; \theta) &= \prod_{i=1}^n \frac{2x_i}{\theta} \cdot e^{-x_i^2/\theta} \\
 &= \frac{2 \prod_{i=1}^n x_i}{\theta^n} \cdot e^{-\sum_{i=1}^n x_i^2/\theta} \\
 &= 2 \prod_{i=1}^n x_i \cdot \frac{1}{\theta^n} \cdot e^{-\frac{1}{\theta} \sum_{i=1}^n x_i^2}
 \end{aligned}$$

Let $u(x) = \sum_{i=1}^n x_i^2 = y$

$$g(u(x), \theta) = \frac{1}{\theta^n} \cdot e^{-\frac{1}{\theta} y}$$

$h(x) = 2 \prod_{i=1}^n x_i$ Then y is sufficient statistics for θ .

Q.7

$$\begin{aligned} f(x_1, \dots, x_n; \theta) &= \prod f(x_i; \theta) \\ &= \frac{1}{\theta^n} \cdot 1_{(x_{(1)} > 0)} \cdot 1_{(x_{(n)} < \theta)} \\ &= \underbrace{1_{(x_{(1)} > 0)}}_{h(x)} \cdot \underbrace{\frac{1}{\theta^n} \cdot 1_{(x_{(n)} < \theta)}}_{g(u(x), \theta)} \end{aligned}$$

By factorization theorem, $x_{(n)}$ is sufficient but biased

$W = \frac{n+1}{n} x_{(n)}$ is unbiased (look ~~many~~ Q.2 in PS#2) estimator for θ .

$x_{(n)}$ is sufficient for θ and $W = \frac{n+1}{n} x_{(n)}$ is unbiased for θ

By Rao-Blackwell theorem:

$\varphi(T) = E(W | x_{(n)}) = W = \frac{n+1}{n} x_{(n)}$ is the unique UMVUE.