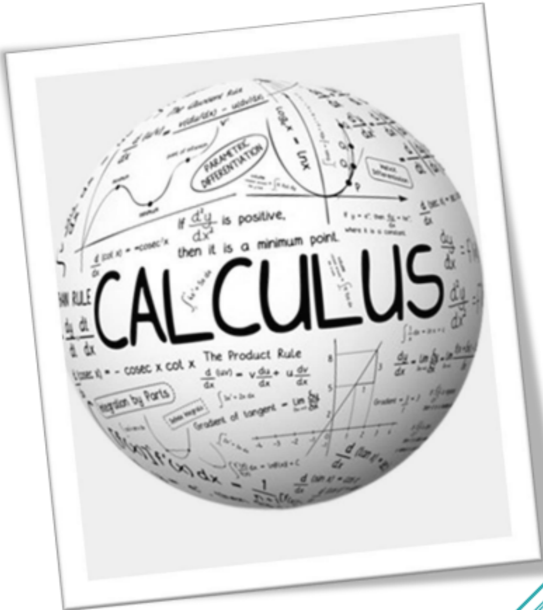
SOLVING PROBLEMS (SORUNLARI ÇÖZMEK)

[MATH 120 BOOKLET]



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ANSWER KEY

(CEVAP ANAHTARI)

RISHA BOOKLET

Road To Success

CALCULUS OF FUNCTIONS OF SEVERAL VARIABLES

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INTRODUCTION

This booklet is for those who are taking Math120 (Calculus of Functions of Several Variables).

What does the booklet offer?

1. Sufficient way to **read** the calculus subjects.
2. Organized way to **understand** calculus theorems and some proves as well as some necessary rules.
3. Sample examples to **understand** how to solve the related problems
4. Some suggested problems to **practice**
5. Appendix which contains functions and their graphs Geometric formulas
6. Short answer key of the suggested problems as well as full solution in the answer key booklet.

$$Read \rightarrow Understand \rightarrow Practice \rightarrow AA$$

What are the sources used on this booklet?

1. Calculus a Complete Course Textbook
2. METU Past Exams

To have chance of having mock/sample exams as well as other online extra recourses and support, you may join MATH120 Course page via https://lms.rishabooklet.com

If you have any inquiries, please do not hesitate to contact me via

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I hope such a work helps you to reach AA letter grade.

Best Regards,

Risha

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The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: Determine whether the given sequence is (a) bounded (above or below), (b) positive or negative (ultimately), (c) increasing, decreasing, or alternating, and (d) convergent, divergent, divergent to ∞ or $-\infty$.

(a) $\left\{\frac{e^n}{\pi^n}\right\}$

(b) $\left\{\frac{(-1)^n n}{e^n}\right\}$

(c) $\dfrac{(n!)^2}{(2n)!}$

Question 2: Evaluate, wherever possible, the limit of the sequence $\{a_n\}$.

(a) $a_n = (-1)^n \frac{n}{n^3+1}$

(b) $a_n = \dfrac{n^2 - 2\sqrt{n} + 1}{1 - n - 3n^2}$

(c) $a_n = \frac{e^n - e^{-n}}{e^n + e^{-n}}$

(d) $a_n = n - \sqrt{n^2 - 4n}$

(e) $a_n = \left(\dfrac{n-1}{n+1}\right)^n$

(f) $a_n = \dfrac{\pi^n}{1 + 2^{2n}}$

Question 3: Let $a_1 = 1$ and $a_{n+1} = \sqrt{1 + 2a_n}$ $(n = 1,2,3,\ldots)$. Show that $\{a_n\}$ is increasing and bounded above. (Hint: Show that 3 is an upper bound.) Hence, conclude that the sequence converges, and find its limit.

The Following Questions Are Obtained From PAST EXAMS

Question 1: Is the sequence $\left\{\frac{1\cdot 4\cdots(3n+1)}{2\cdot 5\cdots(3n+2)}\right\}_{n=1}^{\infty}$ convergent or divergent? Explain your answer. PE. SPRING2017

Question 2: Determine whether the given sequence $\{a_n\}$ is convergent or divergent: PE. SUMMER2016

(a) $a_n = \dfrac{(-4)^n}{2^{2n} + 1}$

(b) $a_n = \dfrac{\sin(2n) + \cos(3n) + 2}{n^{1/2}}$

(c) $a_n = \dfrac{3 + \sqrt{n}}{\ln\left(n + \sqrt{n}\right)}$

<u>Question 3:</u> Find the limit of the given sequence if it exists. PE. SUMMER2015

(a) $a_n = \frac{sin(5n) - 3\,ln\,n}{ln\,n}$

(b) $b_n = \dfrac{\sqrt{119} + \sqrt{n}}{\sqrt{\ln\left(\sqrt{n}\right) + 120}}$

(c) $c_n = \dfrac{3n+5}{\sqrt{4n^2+n-2}}$

Question 4: Suppose that $r \in \mathbb{R}$ and $a_n = nr^n$. For what values of r is the sequence $\{a_n\}$ convergent?

PE. SPRING2014

Question 5: Find the limit of the given sequence if it exists. PE. SUMMER2014

(a) $a_n = \dfrac{n+119}{\ln(120n)}$

(b) $b_n = \dfrac{n\sin(n) + \sin(n)}{n^2} = \dfrac{(n+1)\sin(n)}{n^2}$

Question 6: Is the sequence $\left\{\frac{(-1)^n n^3}{2^n}\right\}_{n=1}^{\infty}$ convergent or divergent? Explain your answer. PE. SPRING2013

Question 7: Let $\{a_n\}_{n=1}^{\infty}$ be the sequence defined recursively by PE. SPRING2011

$$a_1 = 3$$

$$a_{n+1} = \sqrt{15 + 2a_n}\,, n = 1,2,3, \ldots$$

a) Show that $\{a_n\}$ is increasing.

b) Show that $\{a_n\}$ is bounded.

c) Does $\lim\limits_{n\to\infty} a_n$ exist? Explain. If yes, find it.

Section 1.2 Infinite Series

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: Determine whether the following series is convergent or divergent. If convergent, find its sum.

(a) $\sum_{n=0}^{\infty} \frac{5}{10^{3n}}$

(b) $\sum_{n=0}^{\infty} \frac{1}{e^n}$

(c) $\sum_{j=1}^{\infty} \pi^{j/2} \cos(j\pi)$

(e) $\sum_{n=0}^{\infty} \frac{3+2^n}{3^{n+2}}$

(f) $\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \cdots$

(g) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{1\times 2\times 3} + \frac{1}{2\times 3\times 4} + \frac{1}{3\times 4\times 5} + \cdots$

Question 2: decide whether the given statement is TRUE or FALSE. If it is true, prove it. If it is false, give a counterexample showing the falsehood.

1. If $a_n = 0$ for every n, then $\sum a_n$ converges.

2. If $\sum a_n$ converges, then $\sum(1/a_n)$ diverges to infinity.

3. If $\sum a_n$ and $\sum b_n$ both diverge, then so does $\sum(a_n + b_n)$.

4. If $a_n \geq c > 0$ for every n, then $\sum a_n$ diverges to infinity.

5. If $\sum a_n$ diverges and $\{b_n\}$ is bounded, then $\sum a_n b_n$ diverges.

6. If $a_n > 0$ and $\sum a_n$ converges, then $\sum (a_n)^2$ converges.

The Following Questions Are Obtained From PAST EXAMS

Question 1: Determine whether the following statement is true or false. Explain. PE. SPRING2017

If $\lim_{n\to\infty} a_n = 0$, then the series $\sum_{n=1}^{\infty} a_n$ is convergent.

Question 2: Find the series $\sum_{n=1}^{\infty} a_n$, where $a_1 = ln\,2\,, a_n = \ln\left(1 - \frac{1}{n^2}\right)$ for $n \geq 2$. PE. SPRING2017

Question 3: Let $\sum_{n=1}^{\infty} a_n$ be a series whose n −th partial sum is given by PE. SUMMER2017

$$S_n = n[\pi - 2\arctan(n)].$$

(a) Is $\sum_{n=1}^{\infty} a_n$ convergent? If yes, find its sum.

(b) Is $\{a_n\}$ the sequence convergent?

Question 4: Find the sum of the series $\sum_{n=1}^{\infty} \frac{4}{n^2+2n}$ PE. SPRING2016

Question 5: Determine whether the following series is convergent or divergent. If convergent, find the sum $\sum_{k=1}^{\infty} \ln\left(\frac{\arctan(k+1)}{\arctan(k)}\right)$ PE. SPRING2006

Section 1.3 Convergence Tests for Positive Series

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: Determine whether the followings are convergent or divergent. Show your work.

1. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+n+1}$ since

2. $\sum_{n=8}^{\infty} \frac{1}{\pi^n+5}$

3. $\sum_{n=1}^{\infty} \frac{n^2}{1+n\sqrt{n}}$

4. $\sum_{n=1}^{\infty} \frac{1+(-1)^n}{\sqrt{n}}$

5. $\sum_{n=1}^{\infty} \frac{n^4}{n!}$

6. $\sum_{n=1}^{\infty} \frac{(2n)!6^n}{(3n)!}$

7. $\sum_{n=1}^{\infty} \frac{1+n!}{(1+n)!}$

8. $\sum_{n=1}^{\infty} \frac{n^n}{\pi^n n!}$

Question 2: Use the root test to show that $\sum_{n=1}^{\infty} \frac{2^{n+1}}{n^n}$ converges.

Question 3: Determine whether the series $\sum_{n=1}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2}$ converges. Show that $a_n \geq 1/(2n)$.

The Following Questions Are Obtained From PAST EXAMS

Question 1: Determine whether the followings are convergent or divergent. Show your work. PE. SPRING2005

a) $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}$

b) $\sum_{n=0}^{\infty} \frac{1}{2+3^n}$

Question 2:

a) Prove that for $a_n \geq 0$, if $\sum_{n=0}^{\infty} a_n$ is convergent then $\sum_{n=0}^{\infty} a_n^2$ is also convergent. PE. SPRING2005

b) The terms of a sequence are defined recursively by the equations $a_1 = 2,\ a_{n+1} = \left(\frac{4n+1}{5n+2}\right)a_n$.

Is $(a_n)_{n=1}^{\infty}$ convergent? (Explain, and if so, what is the limit?) PE. SPRING2005

Section 1.4 Absolute and Conditional Convergence

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: Determine whether the following series converge absolutely, converge conditionally, or diverge.

1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+\ln n}$

2. $\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{2^n}$

3. $\sum_{n=0}^{\infty} \frac{-n}{n^2+1}$

4. $\sum_{n=1}^{\infty} \frac{100\cos(n\pi)}{2n+3}$

Question 2: For the following series, find the smallest integer n that ensures that the partial sum s_n approximates the sum s of the series with error less than 0.001 in absolute value.

$$\sum_{n=0}^{\infty}(-1)^n\frac{3^n}{n!}$$

Question 3: Determine the values of x for which the following series converge absolutely, converge conditionally, or diverge.

1. $\sum_{n=1}^{\infty}\frac{1}{2n-1}\left(\frac{3x+2}{-5}\right)^n$

2. $\sum_{n=1}^{\infty}\frac{1}{n}\left(1+\frac{1}{x}\right)^n$

Question 4: Which of the following statements are TRUE and which are FALSE? Justify your assertion of truth or give a counterexample to show falsehood.

(a) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.

(b) If $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} (-1)^n a_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(c) If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges absolutely.

The Following Questions Are Obtained From PAST EXAMS

Question 1: Approximate $\frac{1}{e}$ with an error less than $\frac{1}{119}$. PE. SPRING2005

Question 2: For each series below determine whether it is: absolutely convergent, conditionally convergent, or divergent. You must **verify** your answers, **state** any theorem you use. PE. SPRING2006

a) $\sum_{n=1}^{\infty} e^{a_n}$, where $\sum_{n=120}^{\infty} a_n$ is convergent

b) $\sum_{n=1}^{\infty} \frac{\cos(n)+e^{-n}}{n^3-2n+4}$

c) $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$

Section 1.5 Power Series

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: Determine the center, radius, and interval of convergence of each of the following power series.

1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} x^n$

2. $\sum_{n=1}^{\infty} \frac{(4x-1)^n}{n^n}$

Question 2: Determine the Cauchy product of the series $1 + x + x^2 + x^3 + \cdots$ and $1 - x + x^2 - x^3 + \cdots$. On what interval and to what function does the product series converge?

Question 3: Determine following power series representations for the indicated functions. On what interval is each representation valid?

(a) $\frac{1}{(2-x)^2}$ in powers of x

(b) $\frac{1}{1+2x}$ in powers of x

(c) $\frac{1}{x^2}$ in powers of $x+2$

(d) $\frac{1-x}{1+x}$ in powers of x

Question 4: Determine the interval of convergence and the sum of each of the following series.

a) $3+4x+5x^2+6x^3+\cdots=\sum_{n=0}^{\infty}(n+3)x^n$

b) $1 - \frac{x^2}{2} + \frac{x^4}{3} - \frac{x^6}{4} + \frac{x^8}{5} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n+1}$

Question 5: Find the sums of the following series.

1. $\sum_{n=0}^{\infty} \frac{n+1}{2^n}$

2. $\sum_{n=1}^{\infty} \frac{(-1)^n n(n+1)}{2^n}$

The Following Questions Are Obtained From PAST EXAMS

Question 1: Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$

PE. SPRING2005

Question 2: Give an example of a power series which is convergent in $(1,5)$, and divergent on $(-\infty, 1) \cup (5, \infty)$.

(No condition at $x = 1$ and $x = 5$) PE. SPRING2005

Question 3: For the series $\sum_{n=1}^{\infty} \frac{(3x+2)^n}{\sqrt{n}}$

PE. SPRING2006

a) Find the radius R of convergence

b) Find the interval I of convergence

c) If $f(x) = \sum_{n=1}^{\infty} \frac{(3x+2)^n}{\sqrt{n}}$, $x \in I$, find 2006^{th} derivative $f^{(2006)}\left(\frac{-2}{3}\right)$

Question 4: Find the radius $\mathbf{R}$ and interval $\mathbf{I}$ of convergence of $\sum_{k=1}^{\infty} \frac{1}{k}(x-1)^k$ PE. SPRING2006

Section 1.6 Taylor and Maclaurin Series

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: Find Maclaurin series representations for the following functions. For what values of x is each representation valid?

a. $\cos^2(x/2)$

b. $\tan^{-1}(5x^2)$

c. $\left(e^{2x^2}-1\right)/x^2$

Question 2: Find the required Taylor series representations of the following functions. Where is each series representation valid?

1. $f(x)=\ln\ x$ in powers of $x-3$

2. $f(x)=\cos^2\ x$ about $\frac{\pi}{8}$

3. $f(x) = xe^x$ in powers of $x + 2$

Question 3: Find the sums of the following series.

(a) $x^3 - \frac{x^9}{3! \times 4} + \frac{x^{15}}{5! \times 16} - \frac{x^{21}}{7! \times 64} + \frac{x^{27}}{9! \times 256} - \cdots$

(b) $1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \frac{x^6}{7!} + \frac{x^8}{9!} + \cdots$

The Following Questions Are Obtained From PAST EXAMS

Question 1: Consider $f(x) = \frac{1}{1+x^k}; k \geq 1$ an integer. PE. SPRING2005

a) Find the power series expansion of $f(x)$ in powers of x.

b) Use the series in part (a), to expand $g(x) = x \arctan(x)$ in powers of x.

c) Find the expansion of $h(x) = \sqrt{x+1} \arctan(\sqrt{x+1})$ around $a = -1$. Indicate the interval of convergence.

Question 2: Find an approximation of $\int_0^2 \frac{\sin x}{x} dx$ with an error less than $\frac{1}{1000}$ PE. SPRING2006

Section 1.7 Applications of Taylor and Maclaurin Series

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: Use Maclaurin or Taylor following series to calculate the function values, with error less than 5×10^{-5} in absolute value.

a. $\sin(0.1)$

b. $\cos 5^{\circ}$

c. $\tan^{-1} 0.2$

Question 2: Find Maclaurin series for the following functions.

(a) $J(x) = \int_0^x \frac{e^t - 1}{t} dt$

(b) $L(x) = \int_0^x \cos(t^2) dt$

Question 3: Evaluate the limit (Do NOT use L'Hôpital's rule)

$$\lim_{x\to\infty} \frac{(e^x-1-x)^2}{x^2-\ln(1+x^2)}$$

Section 2.1 Analytic Geometry in Three Dimensions

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: Show that the triangle with vertices $(1,2,3)$, $(4,0,5)$, and $(3,6,4)$ has a right angle.

Question 2: In the following questions, describe (and sketch if possible) the set of points in $\mathbb{R}^3$ that satisfy the given equation or inequality.

1. $y^2 + z^2 \le 4$

2. $z \ge \sqrt{x^2 + y^2}$

Question 3: In the following questions, describe (and sketch if possible) the set of points in $\mathbb{R}^3$ that satisfy the given pair of equations or inequalities.

1. $\begin{cases} x^2 + y^2 + z^2 = 4 \\ x^2 + y^2 + z^2 = 4x \end{cases}$

2. $\begin{cases} x^2 + y^2 + z^2 \leq 1 \\ \sqrt{x^2 + y^2} \leq z \end{cases}$

Question 4: In the following questions, specify the boundary and the interior of the plane sets S whose points (x, y) satisfy the given conditions. Is S open, closed, or neither?

1. $|x| + |y| \leq 1$

2. $x^2 + y^2 < 1, y + z > 2$

Section 2.2 Vectors

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: Use vectors to show that the triangle with vertices $(-1,1)$, $(2,5)$, and $(10,-1)$ is right-angled.

Question 2: For what value of t is the vector $2t\mathbf{i} + 4\mathbf{j} - (10+t)\mathbf{k}$ perpendicular to the vector $\mathbf{i} + t\mathbf{j} + \mathbf{k}$?

Question 3: If a vector $\mathbf{u}$ in $\mathbb{R}^3$ makes angles α, β, and γ with the coordinate axes, show that

$$\hat{\mathbf{u}} = \cos\ \alpha\mathbf{i} + \cos\ \beta\mathbf{j} + \cos\ \gamma\mathbf{k}$$

is a unit vector in the direction of $\mathbf{u}$, so $\cos^2\ \alpha + \cos^2\ \beta + \cos^2\ \gamma = 1$. The numbers $\cos\ \alpha, \cos\ \beta$, and $\cos\ \gamma$ are called the direction cosines of $\mathbf{u}$.

Question 4: Find the three angles of the triangle with vertices $(1,0,0)$, $(0,2,0)$, and $(0,0,3)$.

Question 5: Find two-unit vectors each of which is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

Question 6: If a is a nonzero vector and $\mathbf{w}$ is any vector, find vectors $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{w} = \mathbf{u} + \mathbf{v}$, $\mathbf{u}$ is parallel to $\mathbf{a}$, and $\mathbf{v}$ is perpendicular to $\mathbf{a}$.

Section 2.3 The Cross Product in 3-Space

The Following Questions Are Obtained From **Calculus - A Complete Course 7th ed - R. Adams, C. Essex**

Question 1: Find the area of the triangle with vertices $(1,2,0)$, $(1,0,2)$, and $(0,3,1)$.

Question 2: Find a unit vector perpendicular to the vectors $\mathbf{i} + \mathbf{j}$ and $\mathbf{j} + 2\mathbf{k}$

Question 3: Find the volume of the tetrahedron with vertices $(1,0,0)$, $(1,2,0)$, $(2,2,2)$, and $(0,3,2)$.

Question 4: For what value of k do the four points $(1,1,-1)$, $(0,3,-2)$, $(-2,1,0)$, and $(k,0,2)$ all lie in a plane?

Question 5: The product $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ is called a vector triple product. Since it is perpendicular to $\mathbf{v} \times \mathbf{w}$, it must lie in the plane of $\mathbf{v}$ and $\mathbf{w}$. Show that

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$$

The Following Questions Are Obtained From PAST EXAMS

Question 1: a) Find all vectors $\vec{v}$ that satisfy the equation $(-\vec{\imath} + 2\vec{\jmath} + 3\vec{k}) \times \vec{v} = \vec{\imath} + 5\vec{\jmath} - 3\vec{k}$ PE. SPRING2005

b) Show that the four points $A(1,2,-1), B(0,1,5), C(-1,2,1)$ and $D(2,1,3)$ are coplanar (i.e., are on the same plane).

PE. SPRING2005

Section 2.4 Planes and Lines

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: In following questions, find equations of the planes satisfying the given conditions.

1. Passing through the origin and having normal $\mathbf{i}-\mathbf{j}+2\mathbf{k}$

2. Passing through the three points $(-2,0,0)$, $(0,3,0)$, and $(0,0,4)$

3. Passing through the line $x+y=2, y-z=3$, and perpendicular to the plane $2x+3y+4z=5$

Question 2: find equations of the line specified in vector and scalar parametric forms and in standard form.

Through $(2, -1, -1)$ and parallel to each of the two planes $x + y = 0$ and $x - y + 2z = 0$

Question 3: If $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$, show that the equations

$$\begin{cases} x = x_1 + t(x_2 - x_1) \\ y = y_1 + t(y_2 - y_1) \\ z = z_1 + t(z_2 - z_1) \end{cases}$$

represent a line through P_1 and P_2.

Question 4: Find the required distances in the following questions.

1. From the origin to the plane $x + 2y + 3z = 4$

2. From the origin to the line $x + y + z = 0, 2x - y - 5z = 1$

3. Between the lines

$$\begin{cases} x + 2y = 3 \\ y + 2z = 3 \end{cases} \text{ and } \begin{cases} x + y + z = 6 \\ x - 2z = -5 \end{cases}$$

The Following Questions Are Obtained From PAST EXAMS

Question 1: Find the distance between the lines

$\mathcal{L}_1$: intersection of the planes ($x + 2y = 3$ and $y + 2z = 3$) and

$\mathcal{L}_2$: intersection of the planes ($x + y + z = 6$ and $x - 2z = -5$) PE. SPRING2005

Question 2: Let $L_1 = \begin{cases} x = 1 + t \\ y = -9 + 2t \\ z = 5 - t \end{cases}$ $t \in R$ and $L_2 = \begin{cases} x = 2 - 4s \\ y = -s \\ z = 1 + s \end{cases}$ $s \in R$ PE. SPRING2006

a) Find an equation of the line L which is perpendicular to both L_1 and L_2 and passes through the point of intersection of L_1 and L_2.

b) Find the distance between the line L_1 and the plane containing the point $P_0(1,1,1)$ and the line L_2

Section 2.5 Quadric Surfaces

The Following Questions Are Obtained From **Calculus - A Complete Course 7th ed - R. Adams, C. Essex**

Question 1: Identify the surfaces represented by the equations in following questions and sketch their graphs.

a. $2x^2 + 2y^2 + 2z^2 - 4x + 8y - 12z + 27 = 0$

b. $z = x^2 + 2y^2$

c. $-x^2 + y^2 + z^2 = 4$

d. $x^2 + 4z^2 = 4$

e. $y = z^2$

f. $(z-1)^2 = (x-2)^2 + (y-3)^2$

Question 2: Describe and sketch the geometric objects represented by the systems of equations in following questions

a. $\begin{cases} x^2 + y^2 + z^2 = 4 \\ x + y + z = 1 \end{cases}$

b. $\begin{cases} x^2 + 2y^2 + 3z^2 = 6 \\ y = 1 \end{cases}$

Section 3.1 Functions of Several Variables

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: Specify the domains of the following functions.

1. $f(x,y) = \frac{xy}{x^2-y^2}$

2. $f(x,y) = \sqrt{4x^2+9y^2-36}$

3. $f(x,y) = \sin^{-1}(x+y)$

Question 2: Sketch the graphs of the following functions.

1. $f(x,y) = \sin\ x,\ (0 \le x \le 2\pi,\ 0 \le y \le 1)$

2. $f(x,y) = y^2,\ (-1 \le x \le 1,\ -1 \le y \le 1)$

3. $f(x,y) = 4 - x^2 - y^2$, $(x^2 + y^2 \leq 4, x \geq 0, y \geq 0)$

<u>Question 3:</u> Sketch some of the level curves of the following functions.

1. $f(x,y) = x^2 + 2y^2$

2. $f(x,y) = \frac{y}{x^2+y^2}$

Section 3.2 Limits and Continuity

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: In following questions, evaluate the indicated limit or explain why it does not exist.

1. $\lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2}$

2. $\lim_{(x,y)\to(0,1)} \frac{x^2(y-1)^2}{x^2+(y-1)^2}$

3. $\lim_{(x,y)\to(0,0)} \frac{\sin(x-y)}{\cos(x+y)}$

4. $\lim\limits_{(x,y)\to(1,2)} \frac{2x^2-xy}{4x^2-y^2}$

5. $\lim\limits_{(x,y)\to(0,0)} \frac{x^2y^2}{2x^4+y^4}$

Question 2: How can the function $f(x,y) = \frac{x^3-y^3}{x-y}$, $(x \neq y)$,

be defined along the line $x = y$ so that the resulting function is continuous on the whole xy-plane?

Question 3: What condition must the nonnegative integers m, n, and p satisfy to guarantee that

$\lim\limits_{(x,y)\to(0,0)} x^m y^n/(x^2+y^2)^p$ exists? Prove your answer.

The Following Questions Are Obtained From PAST EXAMS

Question 1: Consider the function $f(x,y)=\frac{x^3}{x^2+y^2}$ if $(x,y)\neq(0,0), f(0,0)=0$. Show that: PE. SPRING2005

a) f is continuous everywhere

b) f is **not** differentiable $(0,0)$

Question 2: Determine whether the following limits exist: PE. SPRING2006

a) $\lim\limits_{(x,y)\to(1,2)} \frac{(x-1)(y-2)}{(x-1)^2+(y-2)^2}$

b) $\lim\limits_{(x,y)\to(0,0)} \frac{xy^5}{x^2+y^2}$

Question 3: Let $f(x,y) = \begin{cases} \frac{x-2y^2}{x-y} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$

PE. SPRING2006

a) Compute $f_x(0,0) = \frac{\partial f}{\partial x}(0,0)$

b) Is $f_x(x,y)$ continuous at $(0,0)$?

Section 3.3 Partial Derivatives

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: In following questions, find all the first partial derivatives of the function specified, and evaluate them at the given point.

1. $g(x,y,z) = \frac{xz}{y+z}$, $(1,1,1)$

2. $z = \tan^{-1}\left(\frac{y}{x}\right)$, $(-1,1)$

3. $w = \ln(1 + e^{xyz}),\ (2,0,-1)$

Question 2: In following questions, calculate the first partial derivatives of the given functions at $(0,0)$. You will have to use limit Definition.

1. $f(x,y) = \begin{cases} \frac{2x^3 - y^3}{x^2 + 3y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$

2. $f(x,y) = \begin{cases} \dfrac{x^2 - 2y^2}{x - y}, & \text{if } x \neq y \\ 0, & \text{if } x = y. \end{cases}$

Question 3: In following questions, find equations of the tangent plane and normal line to the graph of the given function at the point with specified values of x and y.

1. $f(x,y) = e^{xy}$ at $(2,0)$

2. $f(x,y) = \frac{x}{x^2+y^2}$ at $(1,2)$

3. $f(x, y) = \tan^{-1}(y/x)$ at $(1, -1)$

Question 4: Find all horizontal planes that are tangent to the surface with equation $z = xye^{-(x^2+y^2)/2}$. At what points are they tangent?

Question 5: In following questions, show that the given function satisfies the given partial differential equation.

1. $w = x^2 + yz,\qquad x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} + z\dfrac{\partial w}{\partial z} = 2w$

2. $z = f(x^2 - y^2)$, where f is any differentiable function of one variable,

$y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0.$

Question 6: $f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2}, & \textit{if } (x,y) \neq (0,0) \\ 0, & \textit{if } (x,y) = (0,0) \end{cases}$

Question 7: Let $f(x,y)=\begin{cases}\frac{x^3-y^3}{x^2+y^2}, & \text{if } (x,y)\neq(0,0)\\ 0, & \text{if } (x,y)=(0,0)\end{cases}$.

Calculate $f_1(x,y)$ and $f_2(x,y)$ at all points (x,y) in the plane. Is f continuous at $(0,0)$? Are f_1 and f_2 continuous at $(0,0)$?

The Following Questions Are Obtained From PAST EXAMS

Question 1: Let $f(u)$ be a differentiable function with $f(1)=1, f'(1)=2$ and $g(x,y)=xf\left(\frac{y}{x^2}\right)$. $Find\ g_x(1,1)$.

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Section 3.4 Higher-Order Derivatives

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: find all the second partial derivatives of the given function.

$z = \sqrt{3x^2 + y^2}$

Question 2: Show that the function is harmonic in the plane regions indicated.

$f(x, y) = \frac{x}{x^2+y^2}$ everywhere except at the origin.

Question 3: Let $F(x,y) = \begin{cases} \frac{2xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

Calculate $F_1(x,y), F_2(x,y), F_{12}(x,y)$, and $F_{21}(x,y)$ at points $(x,y) \neq (0,0)$. Also calculate these derivatives at $(0,0)$. Observe that $F_{21}(0,0) = 2$ and $F_{12}(0,0) = -2$. Does this result show that the partials F_{12} and F_{21} are continuous at $(0,0)$.? Explain why.

Section 3.5 The Chain Rule

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: write appropriate versions of the Chain Rule for the indicated derivatives.

dw/dt if $w = f(x, y), x = g(r, s), y = h(r, t), r = k(s, t)$, and $s = m(t)$

Question 2: Use two methods to calculate dz/dt given that $z = txy^2, x = t + \ln(y + t^2)$, and $y = e^t$.

Question 3: In following questions, assume that f has continuous partial derivatives of all orders.

1. If $f(x, y)$ is harmonic, show that $f\left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$ is also harmonic, that is, $f_{11}(u, v) + f_{22}(u, v) = 0$.

2. Find $\frac{\partial^3}{\partial x\,\partial y^2} f(2x+3y, xy)$ in terms of partial derivatives of the function f.

Question 4: (a) Show that $F(x,y) = -F(y,x)$ for all (x,y).

(b) Show that $F_1(x,y) = -F_2(y,x)$ and $F_{12}(x,y) = -F_{21}(y,x)$ for $(x,y) \neq (0{,}0)$.

(c) Show that $F_1(0,y) = -2y$ for all y and, hence, that $F_{12}(0{,}0) = -2$.

(d) Deduce that $F_2(x,0) = 2x$ and $F_{21}(0{,}0) = 2$.

$$F(x,y) = \begin{cases} \dfrac{2xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0{,}0) \\ 0 & \text{if } (x,y) = (0{,}0) \end{cases}$$

a)

b)

c)

<u>Question 5:</u> (a) Use question 4 (b) to find $F_{12}(x, x)$ for $x \neq 0$.

(b) Is $F_{12}(x, y)$ continuous at $(0,0)$? Why?

a)

b)

The Following Questions Are Obtained From PAST EXAMS

Question 1: Given that the function $z = f\left(\frac{x}{y}, \frac{y}{x}\right)$ has continuous partial derivatives in the neighborhood of the point $(x, y) = (1, -1)$ with

$$f_1(-1,-1) = 2 = f_2(1,-1), \quad f_1(1,-1) = -2 = f_2(-1,-1)$$
$$f_{11}(-1,-1) = 6 = f_{12}(1,-1), \quad f_{11}(1,-1) = 8 = f_{21}(1,-1)$$
$$f_{12}(-1,-1) = -5 = f_{21}(-1,-1), \quad f_{22}(1,-1) = 1 = f_{22}(-1,-1)$$

Find $z_{12} = \frac{\partial^2 z}{\partial y\, \partial x}$ at the point $(x, y) = (1, -1)$.

PE. SPRING2005

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: In following questions, use suitable linearization to find approximate values for the given functions at the points indicated.

1. $f(x, y) = \frac{24}{x^2+xy+y^2}$ at $(2.1, 1.8)$

2. $f(x, y) = xe^{y+x^2}$ at $(2.05, -3.92)$

Question 2: write the differential of the given function and use it to estimate the value of the function at the given point by starting with a known value at a nearby point.

$$u = x\sin(x+y), \text{ at } x = \frac{\pi}{2} + \frac{1}{20}, y = \frac{\pi}{2} - \frac{1}{30}$$

Question 3: By approximately what percentage will the value of $w = \frac{x^2y^3}{z^4}$ " increases by 2%, and z increases by 3%?

The Following Questions Are Obtained From PAST EXAMS

Question 1: Let $z = f(x, y) = \sqrt{20 - x^2 - 7y^2}$. Use Tangent Plane Approximation to approximate $f(1.95, 1.08)$.

Solution: We have PE. SPRING2005

Question 2: Find $sin\left(0.002\left(\frac{\pi}{2} - 0.01\right)^2\right)$ appraximately using the tangent plane (differential) approximation at a suitable point. PE. SPRING2006

Hint: Consider the function $f(x, y) = \sin\left(x\left(\frac{\pi}{2} - y\right)^2\right)$.

Section 3.7 Gradients and Directional Derivatives

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: find:

(a) the gradient of the given function at the point indicated,

(b) an equation of the plane tangent to the graph of the given function at the point whose x and y coordinates are given, and

(c) an equation of the straight-line tangent, at the given point, to the level curve of the given function passing through that point.

$$f(x,y) = e^{xy} \text{ at } (2,0)$$

Question 2: find an equation of the tangent plane to the level surface of the given function that passes through the given point.

$f(x,y,z) = \cos(x + 2y + 3z)$ at $\left(\frac{\pi}{2}, \pi, \pi\right)$

Question 3: find the rate of change of the given function at the given point in the specified direction.

$$f(x,y) = 3x - 4y \text{ at } (0,2) \text{ in the direction of the vector } -2\mathbf{i}$$

Question 4: In what directions at the point $(2,0)$ does the function $f(x,y) = xy$ have rate of change -1? Are there directions in which the rate is -3? How about -2?

Question 5: In what directions at the point (a,b,c) does the function $f(x,y,z) = x^2 + y^2 - z^2$ increase at half of its maximal rate at that point?

Question 6: Find $\nabla f(a,b)$ for the differentiable function $f(x,y)$ given the directional derivatives

$$D_{(\mathbf{i}+\mathbf{j})/\sqrt{2}} f(a,b) = 3\sqrt{2} \text{ and } D_{(3\mathbf{i}-4\mathbf{j})/5} f(a,b) = 5.$$

Question 7: Find an equation of the curve in the xy-plane that passes through the point $(1,1)$ and intersects all level curves of the function $f(x,y) = x^4 + y^2$ at right angles.

Question 8: Find a vector tangent to the curve of intersection of the two cylinders $x^2 + y^2 = 2$ and $y^2 + z^2 = 2$ at the point $(1,-1,1)$.

Question 9: Let $f(x,y) = \begin{cases} \frac{\sin(xy)}{\sqrt{x^2+y^2}}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$.

(a) Calculate $\nabla f(0,0)$.

(b) Use the definition of directional derivative to calculate $D_{\mathbf{u}} f(0,0)$, where $\mathbf{u} = (\mathbf{i} + \mathbf{j})/\sqrt{2}$.

(c) Is $f(x,y)$ differentiable at $(0,0)$? Why?

The Following Questions Are Obtained From PAST EXAMS

Question 1: Show that for any tangent plane to the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{a}, (a > 0)$ the sum of $x, y,$ and z axis intercepts is constant.

(x_0 is called x intercept if the graph intersects x axis at the point $(x_0, 0,0)$.) PE. SPRING2005

Question 2: Consider the function $f(x,y)=\frac{x^3}{x^2+y^2}$ if $(x,y)\neq(0,0), f(0,0)=0$. Show that; PE. SPRING2005

a) all directional derivatives of f **exist** at $(0,0)$, using limit definition of directionsl derivative

b) find the set of all vectors $\vec{v}$ for which the equation $D_{\vec{v}}f(0,0)=\nabla f(0,0)\cdot\vec{v}$ is satisfied.

Question 3: Find a vector $\vec{v}$ in the xy −plane such that $h(x,y) = 2x^2 - 6xy + 3y^2$ increases most rapidly when (x,y) moves along $\vec{v}$ starting at $(1,1)$. PE. SPRING2006

Section 3.8 Implicit Functions

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: In following questions, calculate the indicated derivative from the given equation(s). What condition on the variables will guarantee the existence of a solution that has the indicated derivative? Assume that any general functions F, G, and H have continuous first partial derivatives.

1. $\frac{\partial x}{\partial y}$ if $xy^3 = y - z$

2. $\frac{\partial x}{\partial w}$ if $x^2y^2 + y^2z^2 + z^2t^2 + t^2w^2 - xw = 0$

3. $\frac{dy}{dx}$ if $F(x, y, x^2 - y^2) = 0$

4. $\left(\frac{\partial x}{\partial y}\right)_z$ if $x^2 + y^2 + z^2 + w^2 = 1$, and $x + 2y + 3z + 4w = 2$

Section 4.1 Extreme Values

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: In following questions, find and classify the critical points of the given functions.

(a) $f(x, y) = x^2 + 2y^2 - 4x + 4y$

(b) $f(x, y) = x^3 + y^3 - 3xy$

(c) $f(x,y)=\cos(x+y)$

(d) $f(x,y)=x\sin\ y$

(e) $f(x,y)=x^2ye^{-(x^2+y^2)}$

(f) $f(x,y) = xe^{-x^3+y^3}$

(g) $f(x,y,z) = xy + x^2z - x^2 - y - z^2$

Question 2: Find the maximum and minimum values of $f(x,y) = xye^{-x^2-y^4}$

Question 3: The material used to make the bottom of a rectangular box is twice as expensive per unit area as the material used to make the top or side walls. Find the dimensions of the box of given volume V for which the cost of materials is minimum.

Question 4: Find the three positive numbers $a, b,$ and c, whose sum is 30 and for which the expression ab^2c^3 is maximum.

The Following Questions Are Obtained From PAST EXAMS

Question 1: Find and classify all the critical points of the function $f(x, y) = \frac{1}{3}x^3 + xy^2 - x$.

PE. SPRING2005

Question 2: a) Find all critical points of $f(x, y) = 2x^3 - 6xy + 3y^2$.

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Section 4.2 Extreme Values of Functions Defined on Restricted Domains

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: Find the maximum and minimum values of $f(x, y) = xy - y^2$ on the disk $x^2 + y^2 \leq 1$.

Question 2: Find the maximum value of $f(x, y) = xy - x^3y^2$ over the square $0 \leq x \leq 1, 0 \leq y \leq 1$.

Question 3: Find the maximum and minimum values of $f(x, y) = \sin\ x\cos\ y$ on the closed triangular region bounded by the coordinate axes and the line $x + y = 2\pi$.

Question 4: Find the maximum value of

$$f(x, y) = \sin x \sin y \sin (x + y)$$

over the triangle bounded by the coordinate axes and the line $x + y = \pi$.

Question 5: The temperature at all points in the disk $x^2 + y^2 \leq 1$ is given by

$$T = (x + y)e^{-x^2-y^2}$$

Find the maximum and minimum temperatures at points of the disk.

Question 6: Find the maximum and minimum values of $xy^2 + yz^2$ over the ball $x^2 + y^2 + z^2 \leq 1$.

Question 7: Maximize $Q(x, y) = 2x + 3y$ subject to the constraints $x \geq 0, y \geq 0, y \leq 5, x + 2y \leq 12$, and $4x + y \leq 12$.

Section 4.3 Lagrange Multipliers

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: Use the Method of Lagrange Multipliers to maximize x^3y^5 subject to the constraint $x + y = 8$.

Question 2: Find the distance from the origin to the plane $x + 2y + 2z = 3$,

(a) using a geometric argument (no calculus),

(b) by reducing the problem to an unconstrained problem in two variables, and

(c) using the method of Lagrange multipliers.

a)

b)

c)

<u>Question 3:</u> Use the Lagrange multiplier method to find the greatest and least distances from the point $(2,1,-2)$ to the sphere with equation $x^2 + y^2 + z^2 = 1$

Question 4: Find a, b, and c so that the volume $V = 4\pi abc/3$ of an ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ passing through the point $(1,2,1)$ is as small as possible.

Question 5: Find the maximum and minimum values of $f(x, y, z) = xyz$ on the sphere $x^2 + y^2 + z^2 = 12$.

Question 6: Find the maximum and minimum values of the function $f(x, y, z) = x$ over the curve of intersection of the plane $z = x + y$ and the ellipsoid $x^2 + 2y^2 + 2z^2 = 8$.

Question 7: Find the maximum volume of a rectangular box with faces parallel to the coordinate planes if one corner is at the origin and the diagonally opposite corner lies on the plane $4x + 2y + z = 2$

Question 8: Find the maximum and minimum values of $xy + z^2$ on the ball $x^2 + y^2 + z^2 \leq 1$. Use Lagrange multipliers to treat the boundary case.

The Following Questions Are Obtained From PAST EXAMS

Question 1: Use the Method of Lagrange Multipliers to find the least and greatest distance between the origin and the ellipse $17x^2 + 12xy + 8y^2 - 100 = 0$. PE. SPRING2005

Question 2: Find the set of all points on the ellipsoid $x^2 + 2y^2 + 4z^2 - 12 = 0$ at which the tangent plane to the ellipsoid is parallel to the plane $y + z = 120$.

PE. SPRING2006

Question 3: Using Lagrange Multipliers Method, find a point of the surface $xyz^2 = 2$ that is closest to the origin.

PE. SPRING2006

Section 5.1 Double Integrals

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: evaluate the following given double integral by inspection.

1. $\iint_R dA$, where R is the rectangle $-1 \leq x \leq 3, -4 \leq y \leq 1$

2. $\iint_T (x+y)dA$, where T is the parallelogram having the points $(2,2), (1,-1), (-2,-2)$, and $(-1,1)$ as vertices

3. $\iint_{x^2+y^2\leq a^2}\sqrt{a^2-x^2-y^2}dA$

4. $\iint_{x^2+y^2\leq a^2}\left(a-\sqrt{x^2+y^2}\right)dA$

Section 5.2 Iteration of Double Integrals in Cartesian Coordinates

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: calculate the given following iterated integrals.

$a)\ \int_0^1 dx\int_0^x (xy+y^2)dy$

$b)\ \int_0^{\pi}\int_{-x}^{x} \cos y\ dydx$

Question 2: evaluate the following double integrals by iteration.

$a)\ \iint_R (x^2+y^2)dA$, where R is the rectangle $0 \le x \le a, 0 \le y \le b$

$b)\ \iint_S (\sin\ x + \cos\ y)dA$, where S is the square $0 \le x \le \pi/2, 0 \le y \le \pi/2$

$c)$ $\iint_R xy^2 dA$, where R is the finite region in the first quadrant bounded by the curves $y = x^2$ and $x = y^2$

$d)$ $\iint_D \ln x \, dA$, where D is the finite region in the first quadrant bounded by the line $2x + 2y = 5$ and the hyperbola $xy = 1$

$e) \iint_R \frac{x}{y} e^y \, dA$, where R is the region $0 \leq x \leq 1, x^2 \leq y \leq x$

<u>Question 3:</u> sketch the domain of the following integrations and evaluate the given following iterated integrals.

a. $\int_0^1 dy \int_y^1 e^{-x^2} dx$

b. $\int_0^1 dx \int_x^1 \frac{y^\lambda}{x^2+y^2} dy \ (\lambda > 0)$

<u>Question 4:</u> Find the volumes of the following indicated solids.

a. Under $z = 1 - x^2$ and above the region $0 \leq x \leq 1, 0 \leq y \leq x$

b. Under $z = 1 - x^2 - y^2$ and above the region $x \geq 0, y \geq 0, x + y \leq 1$

c. Under the surface $z = 1/(x + y)$ and above the region in the xy-plane bounded by $x = 1, x = 2, y = 0$, and $y = x$

d. Above the xy-plane and under the surface $z = 1 - x^2 - 2y^2$

e. Inside the two cylinders $x^2 + y^2 = a^2$ and $y^2 + z^2 = a^2$ Vol $= 8 \times$ part in the first octant

Section 5.3 Double Integrals in Polar Coordinates

The Following Questions Are Obtained From **Calculus - A Complete Course 7th ed - R. Adams, C. Essex**

Question 1: evaluate the given following double integrals over the disk D given by $x^2 + y^2 \leq a^2$, where $a > 0$.

1. $\iint_D (x^2 + y^2)dA$

2. $\iint_D \frac{dA}{\sqrt{x^2 + y^2}}$

3. $\iint_D x^2 dA$

Question 2: evaluate the given following double integrals over the quarter-disk Q given by $x \geq 0, y \geq 0$, and $x^2 + y^2 \leq a^2$, where $a > 0$.

1. $\iint_Q y dA$

2. $\iint_Q e^{x^2+y^2} dA$

Question 3: Evaluate $\iint_S (x + y)dA$, where S is the region in the first quadrant lying inside the disk $x^2 + y^2 \leq a^2$ and under the line $y = \sqrt{3}x$.

Question 4: Evaluate $\iint_T (x^2 + y^2)dA$, where T is the triangle with vertices $(0,0)$, $(1,0)$, and $(1,1)$.

Question 5: For what values of k, and to what value, does the integral $\iint_{x^2+y^2\leq 1} \frac{dA}{(x^2+y^2)^k}$ converge?

Question 6: Evaluate $\iint_D xydA$, where D is the plane region satisfying $x \geq 0, 0 \leq y \leq x$, and $x^2 + y^2 \leq a^2$.

Question 7: Find the volume lying between the paraboloids $z = x^2 + y^2$ and $3z = 4 - x^2 - y^2$.

Question 8: Find the volume lying inside both the sphere $x^2 + y^2 + z^2 = 2a^2$ and the cylinder $x^2 + y^2 = a^2$.

Question 9: Find the volume of the region lying inside all three of the circular cylinders $x^2 + y^2 = a^2, x^2 + z^2 = a^2$, and $y^2 + z^2 = a^2$. Hint: Make a good sketch of the first octant part of the region and use symmetry whenever possible.

The Following Questions Are Obtained From PAST EXAMS

Question 1: a) Evaluate $\int_0^1 \int_y^1 sin(x^2)\, dx\, dy$ PE. SPRING2006

b) After drawing the region for the integral, express $\int_0^1 \int_{\sqrt{1-y}}^1 f(x,y)\,dx\,dy + \int_1^3 \int_{\frac{y-1}{2}}^1 f(x,y)\,dx\,dy$ as one double integral.

Question 2: Let R be the finite region, bounded by the line $y = x$ and the curve $y = x^2$. Find the volume of the solid over R bounded from below by the xy −plane and the graph of $f(x,y) = xy$. PE. SPRING2006

Question 3: Evaluate the double integral $\iint_R \frac{\sqrt{x}}{y} e^{-\frac{\sqrt{x}}{y}} dA$ where R is the region on the $xy-$ plane bounded by the curves $y = 2\sqrt{x}, y = \sqrt{x}, xy = 1, xy = 3$.

PE. SPRING2016

Section 5.4 Triple Integrals

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: In following questions, evaluate the triple integrals over the indicated region. Be alert for simplifications and auspicious orders of iteration.

1. $\iiint_B xyz dV$, over the box B given by $0 \le x \le 1, -2 \le y \le 0, 1 \le z \le 4$

2. $\iiint_R x dV$, over the tetrahedron bounded by the coordinate planes and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

3. $\iiint_R (xy+z^2)dV$, over the set $0 \leq z \leq 1-|x|-|y|$

4. $\iiint_R \sin(\pi y^3)dV$, over the pyramid with vertices $(0,0,0)$, $(0,1,0)$, $(1,1,0)$, $(1,1,1)$, and $(0,1,1)$

5. $\iiint_R ydV$, over that part of the cube $0 \leq x, y, z \leq 1$ lying above the plane $y + z = 1$ and below the plane $x + y + z = 2$.

<u>Question 2:</u> Find the volume of the region lying inside the cylinder $x^2 + 4y^2 = 4$, above the xy-plane, and below the plane $z = 2 + x$.

<u>Question 3:</u> Find $\iiint_T xdV$, where T is the tetrahedron bounded by the planes $x = 1, y = 1, z = 1$, and $x + y + z = 2$.

Section 5.5 Change of Variables in Triple Integrals

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: In following questions, find the volumes of the indicated regions.

1. Above the surface $z = (x^2 + y^2)^{1/4}$ and inside the sphere $x^2 + y^2 + z^2 = 2$

2. Between the paraboloids $z = 10 - x^2 - y^2$ and $z = 2(x^2 + y^2 - 1)$

3. Inside the paraboloid $z = x^2 + y^2$ and inside the sphere $x^2 + y^2 + z^2 = 12$

4. Above the xy-plane, under the paraboloid $z = 1 - x^2 - y^2$, and in the wedge $-x \leq y \leq \sqrt{3}x$

Question 2: Evaluate $\iiint_R (x^2 + y^2 + z^2)dV$, where R is the cylinder $0 \leq x^2 + y^2 \leq a^2, 0 \leq z \leq h$.

Question 3: Find $\iiint_B (x^2 + y^2 + z^2)dV$, where B is the ball of $x^2 + y^2 + z^2 \leq a^2$.

Question 4: Find $\iiint_R x dV$ and $\iiint_R z dV$, over that part of the hemisphere $0 \le z \le \sqrt{a^2 - x^2 - y^2}$ that lies in the first octant.

Section 6.1 Vector Functions of One Variable

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: In following questions, find the velocity, speed, and acceleration at time t of the particle whose position is $\mathbf{r}(t)$. Describe the path of the particle.

1. $\mathbf{r} = a\cos\ \omega t\mathbf{i} + b\mathbf{j} + a\sin\ \omega t\mathbf{k}$

2. $\mathbf{r} = 3\cos\ t\mathbf{i} + 4\sin\ t\mathbf{j} + t\mathbf{k}$

Question 2: A particle moves to the right along the curve $y = 3/x$. If its speed is 10 when it passes through the point $\left(2, \frac{3}{2}\right)$, what is its velocity at that time?

Question 3: An object moves along the curve $y = x^2, z = x^3$, with constant vertical speed $dz/dt = 3$. Find the velocity and acceleration of the object when it is at the point $(2,4,8)$.

Section 6.2 Curves and Parametrizations

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: In following questions, find the required parametrization of the first quadrant part of the circular arc $x^2 + y^2 = a^2$.

1. In terms of the y-coordinate, oriented counterclockwise

2. In terms of the x-coordinate, oriented clockwise

3. In terms of the angle between the tangent line and the positive x-axis, oriented counterclockwise

4. In terms of arc length measured from $(0, a)$, oriented clockwise

5. The plane $x + y + z = 1$ intersects the cylinder $z = x^2$ in a parabola. Parametrize the parabola using $t = x$ as parameter.

Question 2: parametrize the curve of intersection of the given surfaces. Note: the answers are not unique.

Question 3: Find the length of the conical helix $\mathbf{r} = t\cos\ t\mathbf{i} + t\sin\ t\mathbf{j} + t\mathbf{k}$, $(0 \leq t \leq 2\pi)$. Why is the curve called a conical helix?

Question 4: Describe the intersection of the sphere $x^2 + y^2 + z^2 = 1$ and the elliptic cylinder $x^2 + 2z^2 = 1$. Find the total length of this intersection curve.

Question 5: reparametrize the given curve in the same orientation in terms of arc length measured from the point where $t = 0$.

Section 7.1 Line Integrals

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: Let $\mathcal{C}$ be the conical helix with parametric equations $x = t\cos t\,, y = t\,\sin\, t, z = t, (0 \le t \le 2\pi)$.

Find $\int_{\mathcal{C}} z ds$.

Question 2: Find $\int_{\mathcal{C}} \sqrt{1+4x^2z^2}ds$, where $\mathcal{C}$ is the curve of intersection of the surfaces $x^2+z^2=1$ and $y=x^2$.

$\mathcal{C}: \mathbf{r} = e^t\, cos\, t\, \mathbf{i} + e^t\, sin\, t\, \mathbf{j} + t\, \mathbf{k}$, for $0 \le t \le 2\pi$

Question 3: Find $\int_C x ds$ along the first octant part of the curve of intersection of the cylinder $x^2+y^2=a^2$ and the plane $z = x$.

Section 7.2 Vector and Scalar Fields

The Following Questions Are Obtained From **Calculus - A Complete Course 7th ed - R. Adams, C. Essex**

Question 1: In following questions, sketch the given plane vector field and determine its field lines.

1. $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$

2. $\mathbf{F}(x, y) = y\mathbf{i} + x\mathbf{j}$

3. $\mathbf{F}(x, y) = \nabla(x^2 - y)$

Section 7.3 Gradient, Divergence, and Curl

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: In following questions, calculate div $\mathbf{F}$ and curl $\mathbf{F}$ for the given vector fields.

1. $\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$

2. $\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: In following questions, determine whether the given vector field is conservative, and find a potential if it is.

1. $\mathbf{F}(x, y, z) = y\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$

2. $\mathbf{F}(x, y, z) = e^{x^2+y^2+z^2}(xz\mathbf{i} + yz\mathbf{j} + xy\mathbf{k})$

Question 2: Show that the vector field

$$\mathbf{F}(x,y,z) = \frac{2x}{z}\mathbf{i} + \frac{2y}{z}\mathbf{j} - \frac{x^2+y^2}{z^2}\mathbf{k}$$

is conservative and find its potential. Describe the equipotential surfaces. Find the field lines of $\mathbf{F}$.

Section 7.5 Line Integrals of Vector Fields

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: In following questions, evaluate the line integral of the tangential component of the given vector field along the given curve.

1. $\mathbf{F}(x, y, z) = z\mathbf{i} - y\mathbf{j} + 2x\mathbf{k}$ along the curve $x = t, y = t^2, z = t^3$ from $(0,0,0)$ to $(1,1,1)$

2. $\mathbf{F}(x, y, z) = (x - z)\mathbf{i} + (y - z)\mathbf{j} - (x + y)\mathbf{k}$ along the polygonal path from $(0,0,0)$ to $(1,0,0)$ to $(1,1,0)$ to $(1,1,1)$

UG

Question 2: Evaluate $\oint_C x^2y^2dx + x^3ydy$ counterclockwise around the square with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$.

Question 3: Evaluate $\int_C e^{x+y}\sin(y+z)dx + e^{x+y}(\sin(y+z) + \cos(y+z))dy + e^{x+y}\cos(y+z)dz$ along the straight-line segment from $(0,0,0)$ to $\left(1, \frac{\pi}{4}, \frac{\pi}{4}\right)$.

Question 4: If C is the intersection of $z = \ln(1+x)$ and $y = x$ from $(0,0,0)$ to $(1,1,\ln 2)$, evaluate

$$\int_C (2x\sin(\pi y) - e^z)dx + (\pi x^2\cos(\pi y) - 3e^z)dy - xe^z dz$$

Question 5: Evaluate

$$\frac{1}{2\pi}\oint_C \frac{-ydx + xdy}{x^2 + y^2}$$

(a) counterclockwise around the circle $x^2 + y^2 = a^2$,

(b) clockwise around the square with vertices $(-1,-1)$, $(-1,1)$, $(1,1)$, and $(1,-1)$,

(c) counterclockwise around the boundary of the region $1 \le x^2 + y^2 \le 4, y \ge 0$.

a)

b)

The Following Questions Are Obtained From PAST EXAMS

Question 1: It is given that $F(x, y) = (P(x, y), Q(x, y))$ where PE. SPRING2006

a) Prove that $Pdx + Qdy$ is exact (that is (P, Q) is conservative), by finding a potential function $\phi(x, y)$such that $\phi_x = P, \phi_y = Q$.

b) Using ϕ, evaluate $\int_{C_1} Pdx + Qdy$.

c) Evaluate $\int_{C_2} Pdx + Qdy$ where C_2 is the line segment from the point $(1,0)$ to the point $(1,2)$.

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: Evaluate $\oint_{\mathcal{C}}(\sin x + 3y^2)dx + \left(2x - e^{-y^2}\right)dy$, where $\mathcal{C}$ is the boundary of the half-disk $x^2 + y^2 \leq a^2, y \geq 0$, oriented counterclockwise.

Question 2: Evaluate $\oint_{\mathcal{C}}(x^2 - xy)dx + (xy - y^2)dy$ clockwise around the triangle with vertices $(0,0)$, $(1,1)$, and $(2,0)$.

Question 3: Evaluate $\oint_C (x\sin(y^2) - y^2)dx + (x^2 y\cos(y^2) + 3x)dy$, where C is the counterclockwise boundary of the trapezoid with vertices $(0,-2), (1,-1), (1,1)$, and $(0,2)$.

Question 4: Evaluate $\oint_C x^2 y dx - xy^2 dy$, where C is the clockwise boundary of the region $0 \leq y \leq \sqrt{9 - x^2}$.

Question 5: Use a line integral to find the plane area enclosed by the curve $\mathbf{r} = a\cos^3 t\mathbf{i} + b\sin^3 t\mathbf{j}$, $(0 \leq t \leq 2\pi)$.

The Following Questions Are Obtained From PAST EXAMS

Question 1: Evaluate $\oint_C \left(-10y^3x^2 + \cos(x^2)\right)dx + \left(3x^5 + 15y^4x + \sin(y^2)\right)dy$ where C is counterclockwise oriented perimeter of the region bounded by $y = x, x = 0, x^2 + y^2 = 4$ in the first quadrant.

PE. SPRING2006

Question 2: Compute the line integral

PE. SPRING2016

$$\oint_C \left(x^3 \sin\sqrt{x^2+4} - xe^{x+2y}\right)dx + \left(\cos(y^3+y) - 4ye^{x+2y}\right)dy$$

on the curve C which is the boundary of the parallelogram with vertices $(2,0), (0,1), (-2,0), (0,-1)$ with counterclockwise orientation.

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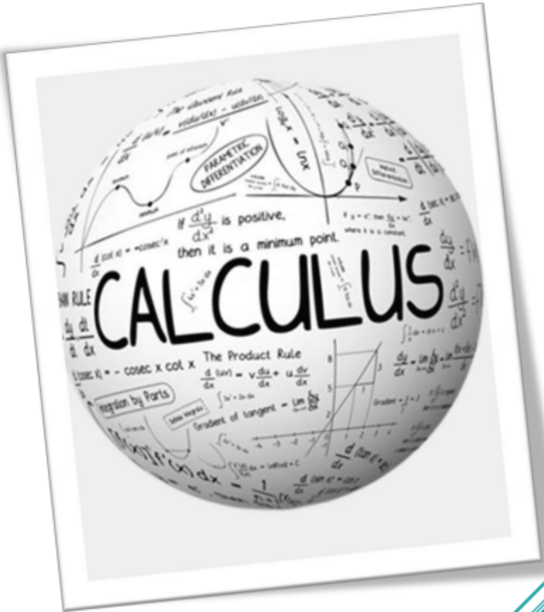
ANSWER KEY (CEVAP ANAHTARI)

[MATH 120 BOOKLET]



RISHA BOOKLET
Road To Success

CALCULUS OF FUNCTIONS OF SEVERAL VARIABLES

Prepared by: Risha

Last Updated: 2021 - 2022

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: Determine whether the given sequence is (a) bounded (above or below), (b) positive or negative (ultimately), (c) increasing, decreasing, or alternating, and (d) convergent, divergent, divergent to ∞ or $-\infty$.

(a) $\left\{\frac{e^n}{\pi^n}\right\} = \left\{\frac{e}{\pi}, \left(\frac{e}{\pi}\right)^2, \left(\frac{e}{\pi}\right)^3, \ldots\right\}$ is bounded, positive, decreasing, and converges to 0, since $e < \pi$.

(b) $\left\{\frac{(-1)^n n}{e^n}\right\} = \left\{\frac{-1}{e}, \frac{2}{e^2}, \frac{-3}{e^3}, \ldots\right\}$ is bounded, alternating, and converges to 0 .

(c) $\frac{(n!)^2}{(2n)!} = \frac{1}{n+1}\frac{2}{n+2}\frac{3}{n+3}\cdots\frac{n}{2n} \leq \left(\frac{1}{2}\right)^n$

Also, $\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{(2n+2)(2n+1)} < \frac{1}{2}$. Thus, the sequence $\left\{\frac{(n!)^2}{(2n)!}\right\}$ is positive, decreasing, bounded, and convergent to 0.

Question 2: Evaluate, wherever possible, the limit of the sequence $\{a_n\}$.

(a) $a_n = (-1)^n \frac{n}{n^3+1} \quad \Rightarrow \quad \lim_{n\to\infty}(-1)^n \frac{n}{n^3+1} = 0.$

(b) $a_n = \frac{n^2 - 2\sqrt{n}+1}{1-n-3n^2} \quad \Rightarrow \quad \lim_{n\to\infty} \frac{n^2-2\sqrt{n}+1}{1-n-3n^2} = \lim_{n\to\infty} \frac{1-\frac{2}{n\sqrt{n}}+\frac{1}{n^2}}{\frac{1}{n^2}-\frac{1}{n}-3} = -\frac{1}{3}$

(c) $a_n = \frac{e^n - e^{-n}}{e^n+e^{-n}} \quad \Rightarrow \quad \lim_{n\to\infty}\frac{e^n-e^{-n}}{e^n+e^{-n}} = \lim_{n\to\infty}\frac{1-e^{-2n}}{1+e^{-2n}} = 1.$

(d) $a_n = n - \sqrt{n^2-4n}$

$$\Rightarrow \quad \lim_{n\to\infty}\left(n-\sqrt{n^2-4n}\right) = \lim_{n\to\infty}\frac{n^2-(n^2-4n)}{n+\sqrt{n^2-4n}} = \lim_{n\to\infty}\frac{4n}{n+\sqrt{n^2-4n}} = \lim_{n\to\infty}\frac{4}{1+\sqrt{1-\frac{4}{n}}} = 2$$

(e) $a_n = \left(\frac{n-1}{n+1}\right)^n$

If $a_n = \left(\frac{n-1}{n+1}\right)^n$, then $a_n = \lim_{n\to\infty}\left(\frac{n-1}{n}\right)^n\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n / \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \frac{e^{-1}}{e} = e^{-2}$

(f) $a_n = \frac{\pi^n}{1+2^{2n}}$

$a_n = \frac{\pi^n}{1+2^{2n}} \Rightarrow 0 < a_n < (\pi/4)^n$. Since $\pi/4 < 1$, therefore $(\pi/4)^n \to 0$ as $n \to \infty$. Thus $\lim_{n\to\infty} a_n = 0$.

Question 3: Let $a_1 = 1$ and $a_{n+1} = \sqrt{1+2a_n}$ $(n = 1,2,3,\dots)$. Show that $\{a_n\}$ is increasing and bounded above. (Hint: Show that 3 is an upper bound.) Hence, conclude that the sequence converges, and find its limit.

Let $a_1 = 1$ and $a_{n+1} = \sqrt{1+2a_n}$ for $n = 1,2,3,\dots$

Then we have $a_2 = \sqrt{3} > a_1$. If $a_{k+1} > a_k$ for some k, then

$$a_{k+2} = \sqrt{1+2a_{k+1}} > \sqrt{1+2a_k} = a_{k+1}$$

Thus, $\{a_n\}$ is increasing by induction. Observe that $a_1 < 3$ and $a_2 < 3$. If $a_k < 3$ then

$$a_{k+1} = \sqrt{1+2a_k} < \sqrt{1+2(3)} = \sqrt{7} < \sqrt{9} = 3.$$

Therefore, $a_n < 3$ for all n, by induction. Since $\{a_n\}$ is increasing and bounded above, it converges. Let $\lim\limits_{n\to\infty} a_n = a$. Then

$$a = \sqrt{1+2a} \Rightarrow a^2 - 2a - 1 = 0 \Rightarrow a = 1 \pm \sqrt{2}$$

Since $a = 1-\sqrt{2} < 0$, it is not appropriate. Hence, we must have $\lim\limits_{n\to\infty} a_n = 1+\sqrt{2}$.

The Following Questions Are Obtained From PAST EXAMS

Question 1: Is the sequence $\left\{\frac{1\cdot 4\cdots(3n+1)}{2\cdot 5\cdots(3n+2)}\right\}_{n=1}^{\infty}$ convergent or divergent? Explain your answer. PE. SPRING2017

$$a_n = \frac{1\cdot 4\cdots(3n+1)}{2\cdot 5\cdots(3n+2)} > 0 \ \ \forall n$$

$$a_{n+1} = a_n \cdot \frac{3n+4}{3n+5} < a_n \qquad \therefore a_n \text{ is decreasing}$$

since $\{a_n\}$ is a deceasing sequence and bounded from below (by 0), the given sequence is convergent.

Question 2: Determine whether the given sequence $\{a_n\}$ is convergent or divergent: PE. SUMMER2016

(a) $a_n = \dfrac{(-4)^n}{2^{2n}+1}$

$$a_n = \frac{(-4)^n}{2^{2n}+1} = \frac{(-1)^n(4)^n}{4^n+1} = \frac{(-1)^n}{1+\frac{1}{4^n}} = \begin{cases} n \text{ is even} \to \lim\limits_{n\to\infty} \dfrac{1}{1+\frac{1}{4^n}} = 1 \\ n \text{ is odd} \to \lim\limits_{n\to\infty} \dfrac{-1}{1+\frac{1}{4^n}} = -1 \end{cases} \qquad \therefore \{a_n\} \text{ is divergent}$$

(b) $a_n = \dfrac{\sin(2n)+\cos(3n)+2}{n^{1/2}}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{\sin(2n)}{n^{1/2}} + \lim_{n\to\infty}\frac{\cos(3n)}{n^{1/2}} + \lim_{n\to\infty}\frac{2}{n^{1/2}} = 0+0+0=0 \qquad \therefore \{a_n\} \text{ is convergent to } 0.$$

By squeeze theorem, $\underbrace{\frac{-1}{n^{1/2}}}_{0\ as\ n\to\infty} \le \underbrace{\frac{\sin(2n)}{n^{1/2}}}_{\therefore 0\ as\ n\to\infty} \le \underbrace{\frac{1}{n^{1/2}}}_{0\ as\ n\to\infty}$

By squeeze theorem, $\underbrace{\frac{-1}{n^{1/2}}}_{0\ as\ n\to\infty} \le \underbrace{\frac{\cos(3n)}{n^{1/2}}}_{\therefore 0\ as\ n\to\infty} \le \underbrace{\frac{1}{n^{1/2}}}_{0\ as\ n\to\infty}$

(c) $a_n = \dfrac{3+\sqrt{n}}{\ln(n+\sqrt{n})}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{3+\sqrt{n}}{\ln(n+\sqrt{n})}\right)\left[\frac{\infty}{\infty}\right] \overset{L`H}{=} \lim_{n\to\infty}\frac{\dfrac{1}{2\sqrt{n}}}{\dfrac{1+\dfrac{1}{2\sqrt{n}}}{n+\sqrt{n}}} = \lim_{n\to\infty}\frac{1}{2\sqrt{n}}\frac{2\sqrt{n}}{2\sqrt{n}+1} = \lim_{n\to\infty}\frac{n+\sqrt{n}}{2\sqrt{n}+1} = \lim_{n\to\infty}\frac{\dfrac{n}{\sqrt{n}}+\dfrac{\sqrt{n}}{\sqrt{n}}}{\dfrac{2\sqrt{n}}{\sqrt{n}}+\dfrac{1}{\sqrt{n}}}$$

$$= \lim_{n\to\infty}\frac{\sqrt{n}+1}{2+\dfrac{1}{\sqrt{n}}} = \infty \qquad \therefore \{a_n\} \text{ is divergent}$$

<u>Question 3:</u> Find the limit of the given sequence if it exists. PE. SUMMER2015

(a) $a_n = \frac{\sin(5n)-3\ln n}{\ln n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{\sin(5n)}{\ln n} - \frac{3\ln n}{\ln n}\right) = \lim_{n\to\infty}\left(\frac{\sin(5n)}{\ln n} - 3\right) = 0-3=-3$$

By squeeze theorem, $\underbrace{\frac{-1}{\ln n}}_{0\ as\ n\to\infty} \le \underbrace{\frac{\sin(5n)}{\ln n}}_{\therefore 0\ as\ n\to\infty} \le \underbrace{\frac{1}{\ln n}}_{0\ as\ n\to\infty}$

(b) $b_n = \dfrac{\sqrt{119}+\sqrt{n}}{\sqrt{\ln(\sqrt{n})+120}}$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{\sqrt{119}+\sqrt{n}}{\sqrt{\ln(\sqrt{n})+120}} = \lim_{x\to\infty}\frac{\sqrt{119}+\sqrt{x}}{\sqrt{\ln(\sqrt{x})+120}}\left[\frac{\infty}{\infty}\right] \overset{L`H}{=} \lim_{n\to\infty}\frac{\dfrac{1}{2\sqrt{x}}}{\dfrac{1}{2\sqrt{\ln(\sqrt{x})+120}}\cdot\dfrac{\frac{1}{2\sqrt{x}}}{\sqrt{x}}}$$

$$= \lim_{n\to\infty} 2\sqrt{x}\cdot\sqrt{\ln(\sqrt{x})+120} = \infty \quad \text{diverges to } \infty$$

(c) $c_n = \dfrac{3n+5}{\sqrt{4n^2+n-2}}$

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{3n+5}{\sqrt{4n^2+n-2}} = \lim_{n\to\infty} \frac{n\left(3+\frac{5}{n}\right)}{n\sqrt{4+\frac{1}{n}-\frac{2}{n^2}}} = \frac{3+0}{\sqrt{4+0-0}} = \frac{3}{2}$$

Question 4: Suppose that $r \in \mathbb{R}$ and $a_n = nr^n$. For what values of r is the sequence $\{a_n\}$ convergent?

PE. SPRING2014

if $r = 0$, then $a_n = 0\ \ \forall n \qquad \lim_{n\to\infty} a_n = 0 \quad \{a_n\} = \{0\}$ is a convergent sequence.

if $|r| \geq 1$, then $\qquad \lim_{n\to\infty} n \cdot r^n$ does not exist since nr^n is unbounded.

if $|r| < 1$ and $r \neq 0$, then consider $f(x) = xr^x$

$$\lim_{x\to\infty} |f(x)| = \lim_{x\to\infty} |xr^x| = \lim_{x\to\infty} \frac{x}{\left|\frac{1}{r^x}\right|} \left[\frac{\infty}{\infty}\right] \overset{L`H}{=} \lim_{x\to\infty} \frac{1}{\left|\frac{1}{r^x}\right| \ln\left(\frac{1}{r^x}\right)} = 0$$

$\therefore \lim_{x\to\infty} f(x) = 0 \qquad\qquad \therefore \lim_{n\to\infty} a_n = 0$

So the given sequence converges $|r| < 1$

Question 5: Find the limit of the given sequence if it exists. PE. SUMMER2014

(a) $a_n = \dfrac{n+119}{\ln(120n)}$

consider $f(x) = \frac{x+119}{\ln(120x)}$, $\qquad$ then $\quad \lim_{x\to\infty} \frac{x+119}{\ln(120x)} \left[\frac{\infty}{\infty}\right] \overset{L`H}{=} \lim_{x\to\infty} \frac{1}{\frac{120}{120x}} = \lim_{x\to\infty} x = \infty$ D.N.E

So the given sequence is divergent.

(b) $b_n = \dfrac{n\sin(n) + \sin(n)}{n^2} = \dfrac{(n+1)\sin(n)}{n^2}$

$$\underbrace{-\frac{(n+1)}{n^2}}_{0\ as\ n\to\infty} \leq \underbrace{\frac{(n+1)\sin(n)}{n^2}}_{\therefore\ 0\ as\ n\to\infty} \leq \underbrace{\frac{n+1}{n^2}}_{0\ as\ n\to\infty} \quad \text{for all } n \geq 1 \text{ since } |sin(x)| \leq 1 \text{ for all } x$$

since $\lim_{n\to\infty} \frac{n+1}{n^2} = \lim_{n\to\infty} \left(\frac{1}{n} + \frac{1}{n^2}\right) = 0 = \lim_{n\to\infty} -\frac{(n+1)}{n^2} \qquad$ Thus, $\lim_{n\to\infty} b_n = 0$ By squeeze theorem

Question 6: Is the sequence $\left\{\frac{(-1)^n n^3}{2^n}\right\}_{n=1}^{\infty}$ convergent or divergent? Explain your answer. PE. SPRING2013

$$\left|\frac{(-1)^n n^3}{2^n}\right| = \frac{n^3}{2^n} \qquad \lim_{n\to\infty}\frac{n^3}{2^n}\left[\frac{\infty}{\infty}\right] \overset{L`H}{=} \lim_{n\to\infty}\frac{3n^2}{2^n \ln 2}\left[\frac{\infty}{\infty}\right] \overset{L`H}{=} \lim_{n\to\infty}\frac{6n}{2^n \ln^2 2}\left[\frac{\infty}{\infty}\right] \overset{L`H}{=} \lim_{n\to\infty}\frac{6}{2^n \ln^3 2} = 0$$

$$Hence\ \lim_{n\to\infty}\frac{(-1)^n n^3}{2^n} = 0$$

Question 7: Let $\{a_n\}_{n=1}^{\infty}$be the sequence defined recursively by PE. SPRING2011

$$a_1 = 3$$

$$a_{n+1} = \sqrt{15 + 2a_n}\,, n = 1,2,3, \ldots.$$

a) Show that $\{a_n\}$ is increasing.

Induction on n:
$$\begin{cases} a_2 = \sqrt{15 + 2a_1} = \sqrt{15 + 6} = \sqrt{21} > 3 = a_1 \\ \text{Assume } a_n > a_{n-1} \\ a_{n+1} = \sqrt{15 + 2a_n} > \sqrt{15 + 2a_{n-1}} = a_n \\ \text{Hence, by induction, } a_n \text{is an increasing sequence} \end{cases}$$

OR $\quad a_{n+1} - a_n = \sqrt{15 + 2a_n} - a_n > 0$ (i.e., $a_n$is increasing) if and only if

$$\sqrt{15 + 2a_n} > a_n \iff 15 + 2a_n > a_n^2 \iff a_n^2 - 2a_n - 15 < 0 \iff (a_n + 3)(a_n - 5) < 0 \iff a_n < 5$$

$0 < a_n < 5$ since a_n terms are all positive. (Check part b to learn how $a_n < 5 \quad \forall n = 1,2,3, \ldots$)

Claim: $a_n < 5 \quad \forall n = 1,2,3, \ldots$

Proof: By induction on n

- $n = 1 \implies a_1 = 3 < 5$
- $n = k$ assume $a_k < 5$
- $n = k + 1 \implies a_{k+1} = \sqrt{15 + 2a_k} < \sqrt{15 + 10} = 5$

b) Show that $\{a_n\}$ is bounded.

$a_1 = 3 > 0$ and since a_n terms are all positive. Thus, $a_n \geq 0 \quad \forall n$. ($a_n$ is bounded from below by zero)

Then as proven in part a) $\quad a_n < 5 \quad \forall n = 1,2,3, \ldots$

Hence, $|a_n| < 5 \quad \forall n = 1,2,3, \ldots$ (a_n is bounded from above by 5)

Thus, $\{a_n\}$ is both increasing and bounded

c) Does $\lim_{n\to\infty} a_n$ exist? Explain. If yes, find it.

since a_n is an increasing and bounded sequence, a_n is convergent. That is, $\lim_{n\to\infty} a_n$ exists.

If $\lim_{n\to\infty} a_n = L$ then also $\lim_{n\to\infty} a_{n+1} = L$

Hence by taking the limit of both sides of the equation $a_{n+1} = \sqrt{15 + 2a_n}$

We obtain $L = \sqrt{15 + 2L} \quad \Rightarrow \quad L^2 - 2L - 15 = (L-5)(L+3) = 0$

Since $a_n < 5 \quad \forall n = 1,2,3,\ldots, L \neq -3$ Thus, $\lim_{n\to\infty} a_n = 5$.

Section 1.2 Infinite Series

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: Determine whether the following series is convergent or divergent. If convergent, find its sum.

(a) $\sum_{n=0}^{\infty} \frac{5}{10^{3n}} = 5\left[1 + \frac{1}{1000} + \left(\frac{1}{1000}\right)^2 + \cdots\right] = \frac{5}{1 - \frac{1}{1000}} = \frac{5000}{999}$

(b) $\sum_{n=0}^{\infty} \frac{1}{e^n} = 1 + \frac{1}{e} + \left(\frac{1}{e}\right)^2 + \cdots = \frac{1}{1 - \frac{1}{e}} = \frac{e}{e-1}$

(c) $\sum_{j=1}^{\infty} \pi^{j/2} \cos(j\pi) = \sum_{j=2}^{\infty} (-1)^j \pi^{j/2}$ diverges because $\lim_{j\to\infty} (-1)^j \pi^{j/2}$ does not exist.

(e) $\sum_{n=0}^{\infty} \frac{3+2^n}{3^{n+2}} = \frac{1}{3}\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n + \frac{1}{9}\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{3} \cdot \frac{1}{1-\frac{1}{3}} + \frac{1}{9} \cdot \frac{1}{1-\frac{2}{3}} = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$.

(f) $\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \cdots$

Since $\frac{1}{(2n-1)(2n+1)} = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$, the partial sum is

$$s_n = \frac{1}{2}\left(1 - \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \frac{1}{2}\left(\frac{1}{2n-3} - \frac{1}{2n-1}\right) + \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) = \frac{1}{2}\left(1 - \frac{1}{2n+1}\right)$$

Hence, $\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \lim_{n\to\infty} s_n = \frac{1}{2}$

(g) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{1\times 2\times 3} + \frac{1}{2\times 3\times 4} + \frac{1}{3\times 4\times 5} + \cdots$

Since $\quad \frac{1}{n(n+1)(n+2)} = \frac{1}{2}\left[\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}\right]$

The partial sum is $s_n = \frac{1}{2}\left(1 - \frac{2}{2} + \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) + \cdots + \frac{1}{2}\left(\frac{1}{n-1} - \frac{2}{n} + \frac{1}{n+1}\right) + \frac{1}{2}\left(\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}\right)$

$$= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{n+1} + \frac{1}{n+2}\right)$$

Hence, $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \lim_{n\to\infty} s_n = \frac{1}{4}$.

Question 2: decide whether the given statement is TRUE or FALSE. If it is true, prove it. If it is false, give a counterexample showing the falsehood.

1. If $a_n = 0$ for every n, then $\sum a_n$ converges.

"If $a_n = 0$ for every n, then $\sum a_n$ converge" is TRUE because $s_n = \sum_{k=0}^{n} 0 = 0$, for every n, and so $\sum a_n = \lim_{n\to\infty} s_n = 0$

2. If $\sum a_n$ converges, then $\sum(1/a_n)$ diverges to infinity.

"If $\sum a_n$ converges, then $\sum 1/a_n$ diverges to infinity" is FALSE. A counterexample is $\sum(-1)^n/2^n$.

3. If $\sum a_n$ and $\sum b_n$ both diverge, then so does $\sum(a_n + b_n)$.

"If $\sum a_n$ and $\sum b_n$ both diverge, then so does $\sum(a_n + b_n)$" is FALSE. Let $a_n = \frac{1}{n}$ and $b_n = -\frac{1}{n}$, then $\sum a_n = \infty$ and $\sum b_n = -\infty$ but $\sum(a_n + b_n) = \sum(0) = 0$.

4. If $a_n \geq c > 0$ for every n, then $\sum a_n$ diverges to infinity.

"If $a_n \geq c > 0$ for all n, then $\sum a_n$ diverges to infinity" is TRUE. We have

$s_n = a_1 + a_2 + a_3 + \cdots + a_n \geq c + c + c + \cdots + c = nc$, and $nc \to \infty$ as $n \to \infty$.

5. If $\sum a_n$ diverges and $\{b_n\}$ is bounded, then $\sum a_n b_n$ diverges.

"If $\sum a_n$ diverges and $\{b_n\}$ is bounded, then $\sum a_n b_n$ diverges" is FALSE. Let $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n+1}$. Then $\sum a_n = \infty$ and $0 \leq b_n \leq 1/2$. But $\sum_3 a_n b_n = \sum \frac{1}{n(n+1)}$ which converges by Example

6. If $a_n > 0$ and $\sum a_n$ converges, then $\sum(a_n)^2$ converges.

"If $a_n > 0$ and $\sum a_n$ converges, then $\sum a_n^2$ converges" is TRUE.

Since $\sum a_n$ converges, therefore $\lim_{n\to\infty} a_n = 0$. Thus, there exists N such that $0 < a_n \leq 1$ for $n \geq N$.

Thus, $0 < a_n^2 \leq a_n$ for $n \geq N$.

If $S_n = \sum_{k=N}^{n} a_k^2$ and $s_n = \sum_{k=N}^{n} a_k$, then $\{S_n\}$ is increasing and bounded above:

$$S_n \leq s_n \leq \sum_{k=1}^{\infty} a_k < \infty$$

Thus $\sum_{k=N}^{\infty} a_k^2$ converges, and so $\sum_{k=1}^{\infty} a_k^2$ converges.

The Following Questions Are Obtained From **PAST EXAMS**

Question 1: Determine whether the following statement is true or false. Explain. **PE. SPRING2017**

If $\lim\limits_{n\to\infty} a_n = 0$, then the series $\sum_{n=1}^{\infty} a_n$ is convergent.

FALSE: Let $a_n = \frac{1}{n}$ Then $\lim\limits_{n\to\infty} \frac{1}{n} = 0$ However, $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent harmonic series.

Question 2: Find the series $\sum_{n=1}^{\infty} a_n$, where $a_1 = \ln 2\,, a_n = \ln\left(1 - \frac{1}{n^2}\right)$ for $n \geq 2$. **PE. SPRING2017**

$$\sum_{n=1}^{\infty} a_n = a_1 + \sum_{n=2}^{\infty} a_n = \ln 2 + \sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right)$$

$\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right) = \sum_{n=2}^{\infty} \ln\left(\frac{n^2-1}{n^2}\right) = \sum_{n=2}^{\infty} \ln\left(\frac{(n-1)(n+1)}{n^2}\right) = s$ So, the nth for the sum is

$$s_n = \ln\left(\frac{1}{2} \cdot \cancel{\frac{3}{2}} \cdot \cancel{\frac{2}{3}} \cdot \cancel{\frac{4}{3}} \cdot \cancel{\frac{5}{4}} \cdots \cancel{\frac{(n-3)(n-1)}{(n-2)^2}} \cdot \cancel{\frac{(n-2)(n)}{(n-1)^2}} \cdot \frac{\cancel{(n-1)}(n+1)}{n^{\cancel{2}}}\right) = \ln\left(\frac{1}{2} \cdot \frac{n+1}{n}\right)$$

Hence, the sum of the series is $s = \lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} \ln\left(\frac{1}{2} \cdot \frac{n+1}{n}\right) \overset{\text{since } \ln x \text{ is cont.}}{=} \ln\left(\lim\limits_{n\to\infty} \frac{1}{2} \cdot \frac{n+1}{n}\right) \ln\left(\frac{1}{2}\right) = -\ln 2$

$$\sum_{n=1}^{\infty} a_n = \ln 2 + \sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right) = \ln 2 + (-\ln 2) = 0$$

Question 3: Let $\sum_{n=1}^{\infty} a_n$ be a series whose $n-$th partial sum is given by **PE. SUMMER2017**

$$S_n = n[\pi - 2\arctan(n)].$$

(a) Is $\sum_{n=1}^{\infty} a_n$ convergent? If yes, find its sum.

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{\pi - 2\arctan(n)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{-2}{1+x^2}}{-\frac{1}{n^2}} = \lim_{n\to\infty} \frac{2x^2}{1+x^2} = 2$$

The series converges to the sum 2.

(b) Is $\{a_n\}$ the sequence convergent?

since $\sum_{n=1}^{\infty} a_n$ is convergent, $\lim_{n\to\infty} a_n = 0$. Hence, the sequence $\{a_n\}$ is convergent.

Question 4: Find the sum of the series $\sum_{n=1}^{\infty} \frac{4}{n^2+2n}$ PE. SPRING2016

$$\sum_{n=1}^{\infty} \frac{4}{n^2+2n} = \sum_{n=1}^{\infty} \frac{4}{n(n+2)} = \sum_{n=1}^{\infty} \left(\frac{A}{n} + \frac{B}{n+2}\right) = \sum_{n=1}^{\infty} \left(\frac{2}{n} - \frac{2}{n+2}\right) = \lim_{n\to\infty} s_n$$

$$s_n = \frac{2}{1} - \cancel{\frac{2}{3}} + \frac{2}{2} - \cancel{\frac{2}{2}} + \cancel{\frac{2}{3}} - \cancel{\frac{2}{5}} + \cancel{\frac{2}{2}} - \cancel{\frac{2}{3}} + \cdots\cdots + \cancel{\frac{2}{n-2}} - \cancel{\frac{2}{n}} + \cancel{\frac{2}{n-1}} - \frac{2}{n+1} + \cancel{\frac{2}{n}} - \frac{2}{n+2} = \frac{2}{1} + \frac{2}{2} - \frac{2}{n+1} - \frac{2}{n+2}$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(\frac{2}{1} + \frac{2}{2} - \frac{2}{n+1} - \frac{2}{n+2}\right) = 3$$

$\therefore \sum_{n=1}^{\infty} \frac{4}{n^2+2n}$ is convergent to 3.

Question 5: Determine whether the following series is convergent or divergent. If convergent, find the sum $\sum_{k=1}^{\infty} \ln\left(\frac{\arctan(k+1)}{\arctan(k)}\right)$ PE. SPRING2006

$$\sum_{k=1}^{\infty} \ln\left(\frac{\arctan(k+1)}{\arctan(k)}\right) = \sum_{k=1}^{\infty} (\ln(\arctan(k+1)) - \ln(\arctan(k)))$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} (\ln(\arctan(k+1)) - \ln(\arctan(k)))$$

$$= \lim_{n\to\infty} [(\cancel{\ln(\arctan 2)} - \ln(\arctan 1)) + (\cancel{\ln(\arctan 3)} - \cancel{\ln(\arctan 2)}) + (\cancel{\ln(\arctan 4)} - \cancel{\ln(\arctan 3)}) + \cdots$$

$$+ (\cancel{\ln(\arctan(n))} - \cancel{\ln(\arctan(n-1))}) + (\ln(\arctan(n+1)) - \cancel{\ln(\arctan(n))})]$$

$$\lim_{n\to\infty} (\ln(\arctan(n+1)) - \ln(\arctan 1)) = \lim_{n\to\infty} \ln\left(\frac{\arctan(n+1)}{\arctan 1}\right) = \lim_{n\to\infty} \ln\left(\frac{\arctan(n+1)}{\frac{\pi}{4}}\right) = \ln\left(\frac{\frac{\pi}{2}}{\frac{\pi}{4}}\right)$$

$$= \ln 2 \quad \text{convergent}$$

Section 1.3 Convergence Tests for Positive Series

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: Determine whether the followings are convergent or divergent. Show your work.

1. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+n+1}$ converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ since

2. $\sum_{n=8}^{\infty} \frac{1}{\pi^n+5}$ converges by comparison with the geometric series $\sum_{n=8}^{\infty} \left(\frac{1}{\pi}\right)^n$ since $0 < \frac{1}{\pi^n+5} < \frac{1}{\pi^n}$.

3. $\sum_{n=1}^{\infty} \frac{n^2}{1+n\sqrt{n}}$ diverges to infinity since $\lim_{n\to\infty} \frac{n^2}{1+n\sqrt{n}} = \infty$.

4. $$\sum_{n=1}^{\infty} \frac{1+(-1)^n}{\sqrt{n}} = 0 + \frac{2}{\sqrt{2}} + 0 + \frac{2}{\sqrt{4}} + 0 + \frac{2}{\sqrt{6}} + \cdots = 2\sum_{k=1}^{\infty} \frac{1}{\sqrt{2k}} = \sqrt{2}\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$$ diverges to infinity.

5. $\sum_{n=1}^{\infty} \frac{n^4}{n!}$ converges by the ratio test since

$$\lim_{n\to\infty} \frac{\frac{(n+1)^4}{(n+1)!}}{\frac{n^4}{n!}} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^4 \frac{1}{n+1} = 0 < 1$$

6. $\sum_{n=1}^{\infty} \frac{(2n)!6^n}{(3n)!}$ converges by the ratio test since

$$\lim_{n\to\infty} \frac{(2n+2)!\,6^{n+1}}{(3n+3)!} / \frac{(2n)!\,6^n}{(3n)!} = \lim_{n\to\infty} \frac{(2n+2)(2n+1)6}{(3n+3)(3n+2)(3n+1)} = 0 < 1$$

7. $\sum_{n=1}^{\infty} \frac{1+n!}{(1+n)!}$ diverges by comparison with the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n+1}$ since $\frac{1+n!}{(1+n)!} > \frac{n!}{(1+n)!} = \frac{1}{n+1}$.

8. $\sum_{n=1}^{\infty} \frac{n^n}{\pi^n n!}$ converges by the ratio test since $\lim \frac{(n+1)^{n+1}}{\pi^{(n+1)}(n+1)!} / \frac{n^n}{\pi^n n!} = \frac{1}{\pi}\lim\left(1+\frac{1}{n}\right)^n = \frac{e}{\pi} < 1$.

Question 2: Use the root test to show that $\sum_{n=1}^{\infty} \frac{2^{n+1}}{n^n}$ converges.

Let $a_n = 2^{n+1}/n^n$. Then $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{2\times 2^{1/n}}{n} = 0$

Since this limit is less than 1, $\sum_{n=1}^{\infty} a_n$ converges by the root test.

Question 3: Determine whether the series $\sum_{n=1}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2}$ converges. Show that $a_n \geq 1/(2n)$.

We have $a_n = \frac{(2n)!}{2^{2n}(n!)^2} = \frac{1\times2\times3\times4\times\cdots\times2n}{(2\times4\times6\times8\times\cdots\times2n)^2} = \frac{1\times3\times5\times\cdots\times(2n-1)}{2\times4\times6\times\cdots\times(2n-2)\times2n} = 1 \times \frac{3}{2} \times \frac{5}{4} \times \frac{7}{6} \times \cdots \times \frac{2n-1}{2n-2} \times \frac{1}{2n} > \frac{1}{2n}$

Therefore $\sum_{n=1}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2}$ diverges to infinity by comparison with the harmonic series $\sum_{n=1}^{\infty} \frac{1}{2n}$.

The Following Questions Are Obtained From PAST EXAMS

Question 1: Determine whether the followings are convergent or divergent. Show your work. PE. SPRING2005

a) $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}$

We apply ratio test, with $a_n = \frac{(n!)^2}{(2n)!}$:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n!)^2(n+1)^2}{(2n)!\,(2n+1)(2n+2)} \cdot \frac{(2n)!}{(n!)^2} = \lim_{n\to\infty} \frac{n+1}{2(2n+1)} = \frac{1}{4} < 1.$$

So, by ratio test, the series converges.

b) $\sum_{n=0}^{\infty} \frac{1}{2+3^n}$

We have $\quad 0 \le \frac{1}{2+3^n} \le \frac{1}{3^n} = \left(\frac{1}{3}\right)^n$

$\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n$ is a geometric series with $r = 1/3$, so it is convergent.

Therefore, by comparison test, $\sum_{n=0}^{\infty} \frac{1}{2+3^n}$ is convergent.

Question 2:

a) Prove that for $a_n \ge 0$, if $\sum_{n=0}^{\infty} a_n$ is convergent then $\sum_{n=0}^{\infty} a_n^2$ is also convergent. PE. SPRING2005

Since $\sum_{n=0}^{\infty} a_n$ is convergent, we have $\lim_{n\to\infty} a_n = 0$. This implies that there exists $N \in \mathbf{N}$ such that for $n \ge N$, we have $0 \le a_n \le 1$. So for all $n \ge N, 0 \le a_n^2 \le a_n$.

Since $\sum_{n=0}^{\infty} a_n$ is convergent, $\sum_{n=0}^{\infty} a_n^2$ is also convergent by comparison test.

b) The terms of a sequence are defined recursively by the equations $a_1 = 2,\ a_{n+1} = \left(\frac{4n+1}{5n+2}\right) a_n$.

Is $(a_n)_{n=1}^{\infty}$ convergent? (Explain, and if so, what is the limit?) PE. SPRING2005

Consider $\sum_{n=1}^{\infty} a_n$. We have $a_n \ge 0$. Apply ratio test to this series.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{4n+1}{5n+2} = \frac{4}{5} < 1$$

So, by ratio test, the series converges. This implies $\lim_{n\to\infty} a_n = 0$.

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: Determine whether the following series converge absolutely, converge conditionally, or diverge.

1. $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+\ln n}$ converges absolutely since $\left|\frac{(-1)^n}{n^2+\ln n}\right| \le \frac{1}{n^2}$ and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges.

2. $\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{2^n} = \sum_{n=1}^{\infty}\frac{1}{2^n}$ is a positive, convergent geometric series so must converge absolutely.

3. $\sum_{n=0}^{\infty}\frac{-n}{n^2+1}$ diverges to $-\infty$ since all terms are negative and $\sum_{n=0}^{\infty}\frac{n}{n^2+1}$ diverges to infinity by comparison with $\sum_{n=0}^{\infty}\frac{1}{n}$.

4. $\sum_{n=1}^{\infty}\frac{100\cos(n\pi)}{2n+3} = \sum_{n=1}^{\infty}\frac{100(-1)^n}{2n+3}$ converges by the alternating series test but only conditionally since

$$\left|\frac{100(-1)^n}{2n+3}\right| = \frac{100}{2n+3}$$

Question 2: For the following series, find the smallest integer n that ensures that the partial sum s_n approximates the sum s of the series with error less than 0.001 in absolute value.

$$\sum_{n=0}^{\infty}(-1)^n\frac{3^n}{n!}$$

Since the terms of the series $s = \sum_{n=0}^{\infty}(-1)^n\frac{3^n}{n!}$ are alternating in sign and ultimately decreasing in size (they decrease after the third term), the size of the error in the approximation $s \approx s_n$ does not exceed that of the first omitted term (provided $n \ge 3$):

$|s - s_n| \le \frac{3^{n+1}}{(n+1)!} < 0.001$ if $n = 12$. Thus, twelve terms will suffice to approximate s with eiror less than 0.001 in absolute value.

Question 3: Determine the values of x for which the following series converge absolutely, converge conditionally, or diverge.

1. $\sum_{n=1}^{\infty} \frac{1}{2n-1}\left(\frac{3x+2}{-5}\right)^n$

Let $a_n = \frac{1}{2n-1}\left(\frac{3x+2}{-5}\right)^n$. Apply the ratio test

$$\rho = l \left|\frac{1}{2n+1}\left(\frac{3x+2}{-5}\right)^{n+1} \times \frac{2n-1}{1}\left(\frac{3x+2}{-5}\right)^{-n}\right| = \left|\frac{3x+2}{5}\right| < 1$$

if and only if $\left|x+\frac{2}{3}\right| < \frac{5}{3}$, that is $-\frac{7}{3} < x < 1$.

If $x = -\frac{7}{3}$, then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{2n-1}$, which diverges.

If $x = 1$, then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$, which converges conditionally.

Thus, the series converges absolutely if $-\frac{7}{3} < x < 1$, converges conditionally if $x = 1$ and diverges elsewhere.

2. $\sum_{n=1}^{\infty} \frac{1}{n}\left(1+\frac{1}{x}\right)^n$

Let $a_n = \frac{1}{n}\left(1+\frac{1}{x}\right)^n$. Apply the ratio test $\rho = \lim \left|\frac{1}{n+1}\left(1+\frac{1}{x}\right)^{n+1} \times \frac{n}{1}\left(1+\frac{1}{x}\right)^{-n}\right| = \left|1+\frac{1}{x}\right| < 1$

if and only if $|x+1| < |x|$, that is, $-2 < \frac{1}{x} < 0 \Rightarrow x < -\frac{1}{2}$. If $x = -\frac{1}{2}$, then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, which converges conditionally.

Thus, the series converges absolutely if $x < -\frac{1}{2}$, converges conditionally if $x = -\frac{1}{2}$ and diverges elsewhere. It is undefined at $x = 0$.

Question 4: Which of the following statements are TRUE and which are FALSE? Justify your assertion of truth or give a counterexample to show falsehood.

(a) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.

" $\sum a_n$ converges implies $\sum(-1)^n a_n$ converges" is FALSE. $a_n = \frac{(-1)^n}{n}$ is a counterexample.

(b) If $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} (-1)^n a_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

" $\sum a_n$ converges and $\sum(-1)^n a_n$ converges implies $\sum a_n$ converges absolutely" is FALSE. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n+1}$ is a counterexample.

(c) If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges absolutely.

" $\sum a_n$ converges absolutely implies $\sum(-1)^n a_n$ converges absolutely" is TRUE, because $|(-1)^n a_n| = |a_n|$.

The Following Questions Are Obtained From PAST EXAMS

Question 1: Approximate $\frac{1}{e}$ with an error less than $\frac{1}{119}$. PE. SPRING2005

We have $e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$

Then, $\frac{1}{e} = e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots$.

Since the series is alternating.

$$E_n(1) = \frac{1}{(n+1)!} < \frac{1}{119}$$

which gives $n + 1 = 5$. Therefore:

$$\frac{1}{e} \approx 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} = \frac{3}{8}$$

Question 2: For each series below determine whether it is: absolutely convergent, conditionally convergent, or divergent. You must **verify** your answers, **state** any theorem you use. PE. SPRING2006

a) $\sum_{n=1}^{\infty} e^{a_n}$, where $\sum_{n=120}^{\infty} a_n$ is convergent

since $\sum a_n$ is convergent by n^{th} term test $\quad \therefore \lim_{n\to\infty} a_n = 0$

$\Rightarrow \lim_{n\to\infty} e^{a_n} = e^{\lim_{n\to\infty} a_n} = e^0 = 1 \neq 0 \Rightarrow$ by n^{th} term test $\sum_{n=1}^{\infty} e^{a_n}$ is divergent.

b) $\sum_{n=1}^{\infty} \frac{\cos(n)+e^{-n}}{n^3-2n+4}$

$$0 \leq \left| \frac{\overbrace{\cos(n)}^{\leq 1} + \overbrace{e^{-n}}^{\leq 1}}{n^3 - 2n + 4} \right| \leq \frac{2}{|n^3 - 2n + 4|} = b_n$$

Since $\lim_{n\to\infty} \frac{b_n}{\frac{1}{n^3}} = 2$, we have $\sum_{n=1}^{\infty} b_n$ is convergent by limit comparison test as $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent.

Therefore, $\sum_{n=1}^{\infty} \frac{\cos(n)+e^{-n}}{n^3-2n+4}$ is absolutely convergent by comparison test.

b) $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$

Let $f(x) = \frac{1}{x\ln x}$ and

we have $\begin{cases} f(x) \text{ is continuous on } [2,\infty) \\ f'(x) = \frac{-(\ln(x)+1)}{(x\ln x)^2} < 0 \text{ on } [2,\infty) \to f(x) \text{ is decreasing on } [2,\infty) \\ \lim_{x\to\infty} \frac{1}{x\ln x} = 0 \end{cases}$

Thus, by the integral test, $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ is convergent if and only if $\int_2^{\infty} \frac{1}{x\ln x} dx$ is convergent.

$$\int_2^{\infty} \frac{1}{x\ln x} dx = \lim_{c\to\infty} \int_2^{c} \frac{1}{x\ln x} dx \to \begin{cases} u = \ln x \\ du = \frac{1}{x} dx \end{cases} = \lim_{c\to\infty} \int \frac{1}{u} du = \lim_{c\to\infty} \ln|u|$$

$$\lim_{c\to\infty} \ln(\ln x)\Big|_2^c = \lim_{c\to\infty} \overbrace{\ln(\ln c)}^{\infty} - \ln(\ln 2) = \infty$$

Hence $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ is divergent to ∞

Section 1.5 Power Series

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: Determine the center, radius, and interval of convergence of each of the following power series.

1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} x^n$

We have $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} x^n$. The centre of convergence is $x = 0$. The radius of convergence is

$$R = \lim_{n\to\infty} \left| \frac{(-1)^n}{n^4 2^{2n}} \cdot \frac{(n+1)^4 2^{2n+2}}{(-1)^{n+1}} \right| = \lim_{n\to\infty} \left| \left(\frac{n+1}{n}\right)^4 \cdot 4 \right| = 4$$

At $x = 4$, the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$, which converges. At $x = -4$, the series is $\sum_{n=1}^{\infty} \frac{1}{n^4}$, which also converges.

Hence, the interval of convergence is $[-4,4]$.

2. $\sum_{n=1}^{\infty} \frac{(4x-1)^n}{n^n}$

We have $\sum_{n=1}^{\infty} \frac{(4x-1)^n}{n^n} = \sum_{n=1}^{\infty} \left(\frac{4}{n}\right)^n \left(x - \frac{1}{4}\right)^n$. The centre of convergence is $x = \frac{1}{4}$. The radius of convergence is

$$R = \lim_{n\to\infty} \frac{4^n}{n^n} \cdot \frac{(n+1)^{n+1}}{4^{n+1}} = \frac{1}{4} \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n (n+1) = \infty.$$

Hence, the interval of convergence is $(-\infty, \infty)$.

Question 2: Determine the Cauchy product of the series $1 + x + x^2 + x^3 + \cdots$ and $1 - x + x^2 - x^3 + \cdots$. On what interval and to what function does the product series converge?

We have $1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ and $1 - x + x^2 - x^3 + \cdots = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$

holds for $-1 < x < 1$. Since $a_n = 1$ and $b_n = (-1)^n$ for $n = 0,1,2, \dots$, we have

$$C_n = \sum_{j=0}^{n} (-1)^{n-j} = \begin{cases} 0, & \text{if } n \text{ is odd} \\ 1, & \text{if } n \text{ is even} \end{cases}$$

Then the Cauchy product is $1 + x^2 + x^4 + \cdots = \sum_{n=0}^{\infty} x^{2n} = \frac{1}{1-x} \cdot \frac{1}{1+x} = \frac{1}{1-x^2}$ for $-1 < x < 1$.

Question 3: Determine following power series representations for the indicated functions. On what interval is each representation valid?

(a) $\frac{1}{(2-x)^2}$ in powers of x

$\frac{1}{2-x} = \frac{1}{2} \cdot \frac{1}{1-\frac{x}{2}} = \frac{1}{2} + \frac{x}{2^2} + \frac{x^2}{2^3} + \frac{x^3}{2^4} \cdots$ for $-2 < x < 2$. Now differentiate to get

$$\frac{1}{(2-x)^2} = \frac{1}{2^2} + \frac{2x}{2^3} + \frac{3x^2}{2^4} + \cdots = \sum_{n=0}^{\infty} \frac{(n+1)x^n}{2^{n+2}}, \ (-2 < x < 2)$$

(b) $\frac{1}{1+2x}$ in powers of x

$\frac{1}{1+2x} = \sum_{n=0}^{\infty} (-2x)^n = 1 - 2x + 2^2x^2 - 2^3x^3 + \cdots \ \left(-\frac{1}{2} < x < \frac{1}{2}\right).$

(c) $\frac{1}{x^2}$ in powers of $x + 2$

Let $x + 2 = t$, so $x = t - 2$. Then

$$\frac{1}{x^2} = \frac{1}{(2-t)^2} = \sum_{n=0}^{\infty} \frac{(n+1)t^n}{2^{n+2}} = \sum_{n=0}^{\infty} \frac{(n+1)(x+2)^n}{2^{n+2}}, \ (-4 < x < 0)$$

(d) $\frac{1-x}{1+x}$ in powers of x

$$\frac{1-x}{1+x}=\frac{2}{1+x}-1=2(1-x+x^2-x^3+\cdots)-1=1+2\sum_{n=1}^{\infty}(-x)^n\ (-1<x<1)$$

Question 4: Determine the interval of convergence and the sum of each of the following series.

a) $3+4x+5x^2+6x^3+\cdots=\sum_{n=0}^{\infty}(n+3)x^n$

We differentiate the series $\sum_{n=0}^{\infty} x^n=1+x+x^2+x^3+\cdots=\frac{1}{1-x}$

and multiply by x to get $\sum_{n=0}^{\infty} nx^n=x+2x^2+3x^3+\cdots=\frac{x}{(1-x)^2}$ for $-1<x<1$. Therefore,

$$\sum_{n=0}^{\infty}(n+3)x^n=\sum_{n=0}^{\infty}nx^n+3\sum_{n=0}^{\infty}x^n=\frac{x}{(1-x)^2}+\frac{3}{1-x}=\frac{3-2x}{(1-x)^2}\qquad \text{for } (-1<x<1).$$

b) $1-\frac{x^2}{2}+\frac{x^4}{3}-\frac{x^6}{4}+\frac{x^8}{5}-\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n+1}$

Since $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots=\ln(1+x)$ for $-1<x\le 1$, therefore

$$x^2-\frac{x^4}{2}+\frac{x^6}{3}-\frac{x^8}{4}+\cdots=\ln(1+x^2)$$

for $-1\le x\le 1$, and, dividing by x^2,

$$1-\frac{x^2}{2}+\frac{x^4}{3}-\frac{x^6}{4}+\cdots=\begin{cases}\dfrac{\ln(1+x^2)}{x^2} & \text{if } -1\le x\le 1, x\ne 0\\ 1 & \text{if } x=0\end{cases}$$

Question 5: Find the sums of the following series.

1. $\sum_{n=0}^{\infty}\frac{n+1}{2^n}$

With $x=1/2$, $\sum_{n=0}^{\infty}\frac{n+1}{2^n}=\sum_{k=1}^{\infty}k\left(\frac{1}{2}\right)^{k-1}=\frac{1}{\left(1-\frac{1}{2}\right)^2}=4$

2. $\sum_{n=1}^{\infty}\frac{(-1)^n n(n+1)}{2^n}$

$$\sum_{n=1}^{\infty}nx^{n-1}=\frac{1}{(1-x)^2},\ (-1<x<1)$$

Differentiate with respect to x and then replace n by $n+1$:

$\sum_{n=2}^{\infty} n(n-1)x^{n-2} = \frac{2}{(1-x)^3}$, $(-1 < x < 1)$ $\qquad$ $\sum_{n=1}^{\infty} (n+1)nx^{n-1} = \frac{2}{(1-x)^3}$, $(-1 < x < 1)$

Now let $x = -1/2$: $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n(n+1)}{2^{n-1}} = \frac{16}{27}$

Finally, multiply by $-1/2$: $\sum_{n=1}^{\infty} (-1)^{n} \frac{n(n+1)}{2^{n}} = -\frac{8}{27}$

The Following Questions Are Obtained From PAST EXAMS

<u>Question 1:</u> Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$

PE. SPRING2005

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{3^{n+1}|x|^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{3^n|x|^n} = 3|x| \lim_{n\to\infty} \sqrt{\frac{n+1}{n+2}} = 3|x|.$$

So, if $3|x| < 1$, by ratio test the given power series is convergent.
This gives $|x| < 1/3$, so $R = 1/3$.
For the interval of convergence, we must check the end points.
If $x = 1/3$:

$$\sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{\sqrt{n+1} \cdot 3^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$$

$a_n = \frac{(-1)^n}{\sqrt{n+1}}$ is alternating,

- $\frac{1}{\sqrt{n+1}} > 0,$
- $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n}{\sqrt{n+1}} = 0,$
- $\frac{|a_{n+1}|}{|a_n|} = \frac{\sqrt{n+1}}{\sqrt{n+2}} = \sqrt{\frac{n+1}{n+2}} < 1$, so $|a_n|$ is decreasing.

  So by AST, the series is convergent.
  If $x = -1/3$:

$$\sum_{n=0}^{\infty} \frac{(-3)^n}{\sqrt{n+1}(-3)^n} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$$

$\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ is divergent by p-test, so the above series is divergent by limit comparison.
So the interval of convergence is $I = (-1/3, 1/3]$.

Question 2: Give an example of a power series which is convergent in $(1,5)$, and divergent on $(-\infty,1)\cup(5,\infty)$.

(No condition at $x=1$ and $x=5$) PE. SPRING2005

Consider, $\sum_{n=0}^{\infty} a_n(x-3)^n$; a power series around 3. We want the radius of convergence to be 2. So, we want $\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\frac{1}{2}$.

Let $a_n=\frac{1}{2^n}$, then we get the desired limit and interval of convergence. So

$$\sum_{n=0}^{\infty}\frac{(x-3)^n}{2^n}$$

is a power series satisfying the conditions we want.

Question 3: For the series $\sum_{n=1}^{\infty}\frac{(3x+2)^n}{\sqrt{n}}$ PE. SPRING2006

a) Find the radius R of convergence

$\sum_{n=1}^{\infty}\frac{(3x+2)^n}{\sqrt{n}}=\sum_{n=1}^{\infty}\frac{\left(3\left(x+\frac{2}{3}\right)\right)^n}{\sqrt{n}}=\sum_{n=1}^{\infty}\frac{3^n}{\sqrt{n}}\left(x-\left(-\frac{2}{3}\right)\right)^n$ is a power series around $-2/3$

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to\infty}\frac{\cancel{3^{n+1}}}{\sqrt{n+1}}\left|x+\frac{2}{3}\right|^{\cancel{n+1}}\cdot\frac{\sqrt{n}}{\cancel{3^n}}\frac{1}{\cancel{\left|x+\frac{2}{3}\right|^n}}=3\left|x+\frac{2}{3}\right|\lim_{n\to\infty}\sqrt{\frac{n}{n+1}}=3\left|x+\frac{2}{3}\right|$$

Given power series is convergent if $\left|x+\frac{2}{3}\right|<1/3$

so $R=1/3$

b) Find the interval I of convergence

$$x=-1\Rightarrow\sum_{n=1}^{\infty}\frac{(3x+2)^n}{\sqrt{n}}=\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}\quad\begin{cases}\text{alternating series}\\ a_n=\dfrac{1}{\sqrt{n}}\\ \lim\limits_{n\to\infty}\dfrac{1}{\sqrt{n}}=0\end{cases}\quad\text{convergent series by (A.S.T)}$$

$x=\frac{-1}{3}\Rightarrow\sum_{n=1}^{\infty}\frac{(3x+2)^n}{\sqrt{n}}=\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ divergent (by p-test)

$\mathrm{I}=[-1,-1/3)$

c) If $f(x)=\sum_{n=1}^{\infty}\frac{(3x+2)^n}{\sqrt{n}}$, $x\in I$, find 2006$^{\text{th}}$ derivative $f^{(2006)}\left(\frac{-2}{3}\right)$

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(-2/3)}{n!}\left(x+\frac{2}{3}\right)^n=\underbrace{f\left(-\frac{2}{3}\right)}_{=0}+\sum_{n=1}^{\infty}\frac{f^{(n)}(-2/3)}{n!}\left(x+\frac{2}{3}\right)^n=\sum_{n=1}^{n}\frac{3^n n!}{\sqrt{n}n!}\left(x+\frac{2}{3}\right)^n$$

$$\Rightarrow f^{(2006)}(-2/3)=\frac{3^{2006}(2006)!}{\sqrt{2006}}$$

Question 4: Find the radius **R** and interval **I** of convergence of $\sum_{k=1}^{\infty}\frac{1}{k}(x-1)^k$ PE. SPRING2006

Let's apply Ratio Test

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|=\lim_{k\to\infty}\frac{\frac{1}{k+1}|x-1|^{k+1}}{\frac{1}{k}|x-1|^k}=|x-1|\lim_{k\to\infty}\frac{k}{k+1}=|x-1|$$

Hence given power series around $x_0=1$ is convergent (absolutely) if $|x-1|<1$. That is convegent in $(0,2)$.

Thus, $\mathbf{R}=1$.

Check the end points:

$x=0\Rightarrow\sum_{k=1}^{\infty}\frac{1}{k}(-1)^k$ is convergent alternating series $\left(a_k=\frac{1}{k}>0,\ a_k \text{ is decreasing}, \lim_{k\to\infty}a_k=0\right)$

$x=2\Rightarrow\sum_{k=1}^{\infty}\frac{1}{k}$ is divergent (by p −test)

Thus, $\mathbf{I}=[0,2)$

Section 1.6 Taylor and Maclaurin Series

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: Find Maclaurin series representations for the following functions. For what values of x is each representation valid?

a. $\cos^2(x/2)$

$$\cos^2\left(\frac{x}{2}\right)=\frac{1}{2}(1+\cos x)=\frac{1}{2}\left(1+1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\right)=1+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}\ (\text{for all } x)$$

b. $\tan^{-1}(5x^2)$

$$\tan^{-1}(5x^2)=(5x^2)-\frac{(5x^2)^3}{3}+\frac{(5x^2)^5}{5}-\frac{(5x^2)^7}{7}+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)}(5x^2)^{2n+1}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n5^{2n+1}}{(2n+1)}x^{4n+2}\ \left(\text{for } -\frac{1}{\sqrt{5}}\le x\le\frac{1}{\sqrt{5}}\right)$$

c. $\left(e^{2x^2}-1\right)/x^2$

$$\frac{e^{2x^2}-1}{x^2}=\frac{1}{x^2}\left(e^{2x^2}-1\right)=\frac{1}{x^2}\left(1+2x^2+\frac{(2x^2)^2}{2!}+\frac{(2x^2)^3}{3!}+\cdots-1\right)=2+\frac{2^2x^2}{2!}+\frac{2^3x^4}{3!}+\frac{2^4x^6}{4!}+\cdots$$

$$=\sum_{n=0}^{\infty}\frac{2^{n+1}}{(n+1)!}x^{2n} \text{ (for all } x\neq 0)$$

Question 2: Find the required Taylor series representations of the following functions. Where is each series representation valid?

1. $f(x)=\ln\ x$ in powers of $x-3$

Let $y=x-3$; then $x=y+3$. Hence,

$$\ln\ x=\ln\ (y+3)=\ln\ 3+\ln\ \left(1+\frac{y}{3}\right)=\ln\ 3+\frac{y}{3}-\frac{1}{2}\left(\frac{y}{3}\right)^2+\frac{1}{3}\left(\frac{y}{3}\right)^3-\frac{1}{4}\left(\frac{y}{3}\right)^4+\cdots$$

$$=\ln\ 3+\frac{(x-3)}{3}-\frac{(x-3)^2}{2\cdot 3^2}+\frac{(x-3)^3}{3\cdot 3^3}-\frac{(x-3)^4}{4\cdot 3^4}+\cdots=\ln\ 3+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n\cdot 3^n}(x-3)^n\ (0<x\leq 6).$$

2. $f(x)=\cos^2\ x$ about $\frac{\pi}{8}$

Let $y=x-\frac{\pi}{8}$; then $x=y+\frac{\pi}{8}$. Thus,

$$\cos^2 x=\cos^2\left(y+\frac{\pi}{8}\right)=\frac{1}{2}\left[1+\cos\left(2y+\frac{\pi}{4}\right)\right]=\frac{1}{2}\left[1+\frac{1}{\sqrt{2}}\cos(2y)-\frac{1}{\sqrt{2}}\sin(2y)\right]$$

$$=\frac{1}{2}+\frac{1}{2\sqrt{2}}\left[1-\frac{(2y)^2}{2!}+\frac{(2y)^4}{4!}-\cdots\right]-\frac{1}{2\sqrt{2}}\left[2y-\frac{(2y)^3}{3!}+\frac{(2y)^5}{5!}-\cdots\right]$$

$$=\frac{1}{2}+\frac{1}{2\sqrt{2}}\left[1-2y-\frac{(2y)^2}{2!}+\frac{(2y)^3}{3!}+\frac{(2y)^4}{4!}-\frac{(2y)^5}{5!}-\cdots\right]$$

$$=\frac{1}{2}+\frac{1}{2\sqrt{2}}\left[1-2\left(x-\frac{\pi}{8}\right)-\frac{2^2}{2!}\left(x-\frac{\pi}{8}\right)^2+\frac{2^3}{3!}\left(x-\frac{\pi}{8}\right)^3+\frac{2^4}{4!}\left(x-\frac{\pi}{8}\right)^4-\frac{2^5}{5!}\left(x-\frac{\pi}{8}\right)^5-\cdots\right]$$

$$=\frac{1}{2}+\frac{1}{2\sqrt{2}}+\frac{1}{2\sqrt{2}}\sum_{n=1}^{\infty}(-1)^n\left[\frac{2^{2n-1}}{(2n-1)!}\left(x-\frac{\pi}{8}\right)^{2n-1}+\frac{2^{2n}}{(2n)!}\left(x-\frac{\pi}{8}\right)^{2n}\right]\qquad \text{(for all } x).$$

3. $f(x)=xe^x$ in powers of $x+2$

Let $u=x+2$. Then $x=u-2$, and $xe^x=(u-2)e^{u-2}=(u-2)e^{-2}\sum_{n=0}^{\infty}\frac{u^n}{n!}$ (for all u)

$$= \sum_{n=0}^{\infty} \frac{e^{-2}u^{n+1}}{n!} - \sum_{n=0}^{\infty} \frac{2e^{-2}u^{n}}{n!}$$

In the first sum replace n by $n-1$.

$$xe^{x} = \sum_{n=1}^{\infty} \frac{e^{-2}u^{n}}{(n-1)!} - \sum_{n=0}^{\infty} \frac{2e^{-2}u^{n}}{n!} = -\frac{2}{e^2} + \sum_{n=1}^{\infty} \frac{1}{e^2}\left(\frac{1}{(n-1)!} - \frac{2}{n!}\right)u^n$$

$$= -\frac{2}{e^2} + \sum_{n=1}^{\infty} \frac{1}{e^2}\left(\frac{1}{(n-1)!} - \frac{2}{n!}\right)(x+2)^n \text{ (for all } x)$$

Question 3: Find the sums of the following series.

(a) $x^3 - \frac{x^9}{3! \times 4} + \frac{x^{15}}{5! \times 16} - \frac{x^{21}}{7! \times 64} + \frac{x^{27}}{9! \times 256} - \cdots = 2\left[\frac{x^3}{2} - \frac{1}{3!}\left(\frac{x^3}{2}\right)^3 + \frac{1}{5!}\left(\frac{x^3}{2}\right)^5 - \cdots\right]$

$= 2\sin\left(\frac{x^3}{2}\right)$ (for all x)

(b) $1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \frac{x^6}{7!} + \frac{x^8}{9!} + \cdots = \frac{1}{x}\sinh x = \frac{e^x - e^{-x}}{2x}$ if $x \neq 0$. The sum is 1 if $x = 0$.

The Following Questions Are Obtained From PAST EXAMS

Question 1: Consider $f(x) = \frac{1}{1+x^k}$; $k \geq 1$ an integer. PE. SPRING2005

a) Find the power series expansion of $f(x)$ in powers of x.

Since $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$ for all $|x| < 1$

we have $\frac{1}{1+x^k} = \sum_{n=0}^{\infty} (-1)^n x^{nk}$

b) Use the series in part (a), to expand $g(x) = x \arctan(x)$ in powers of x.

$g(x) = x \arctan x = x\int_0^x \frac{1}{1+u^2} du = x\int_0^x \sum_{n=0}^{\infty} (-1)^n u^{2n} du = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{2n+1}$ for all $|x| < 1$.

c) Find the expansion of $h(x) = \sqrt{x+1} \arctan(\sqrt{x+1})$ around $a = -1$. Indicate the interval of convergence.

$h(x) = \sqrt{x+1} \arctan \sqrt{x+1} = \sum_{n=0}^{\infty} (-1)^n \frac{\left[(x+1)^{1/2}\right]^{2n+2}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n (x+1)^{n+1}}{2n+1}$ for all $|x+1| < 1$.

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Question 2: Find an approximation of $\int_0^2 \frac{\sin x}{x} dx$ with an error less than $\frac{1}{1000}$ PE. SPRING2006

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \forall x \in \mathbb{R} \qquad \Rightarrow \frac{\sin x}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} \quad \forall x \in \mathbb{R}$$

$$\Rightarrow \int_0^2 \frac{\sin x}{x} dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}\Bigg|_0^2 = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)(2n+1)!}$$

$$\text{error} \approx 2 - \frac{2^3}{3.3!} + \frac{2^5}{5.5!} - \frac{2^7}{7.7!} \qquad \text{error} < \frac{2^9}{9.9!} < 10^{-3}$$

Section 1.7 Applications of Taylor and Maclaurin Series

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: Use Maclaurin or Taylor following series to calculate the function values, with error less than 5×10^{-5} in absolute value.

a. $\sin(0.1)$

We have $\sin(0.1) = 0.1 - \frac{(0.1)^3}{3!} + \frac{(0.1)^5}{5!} - \frac{(0.1)^7}{7!} + \cdots$.

Since $\frac{(0.1)^5}{5!} = 8.33 \times 10^{-8} < 5 \times 10^{-5}$, therefore, $\sin(0.1) = 0.1 - \frac{(0.1)^3}{3!} \approx 0.09983$

with error less than 5×10^{-5} in absolute value.

b. $\cos 5°$

$$\cos 5° = \cos\frac{5\pi}{180} = \cos\frac{\pi}{36} \approx 1 - \frac{1}{2!}\left(\frac{\pi}{36}\right)^2 + \frac{1}{4!}\left(\frac{\pi}{36}\right)^4 - \cdots + \frac{(-1)^n}{(2n)!}\left(\frac{\pi}{36}\right)^{2n}$$

$|\text{ Error }| < \frac{1}{(2n+2)!}\left(\frac{\pi}{36}\right)^{2n+2} < \frac{1}{(2n+2)!9^{2n+2}} < 0.00005$ if $n = 1$.

$\cos 5° \approx 1 - \frac{1}{2!}\left(\frac{\pi}{36}\right)^2 \approx 0.996192$ with error less than 0.00005.

c. $\tan^{-1} 0.2$

We have $\tan^{-1}(0.2) = 0.2 - \frac{(0.2)^3}{3} + \frac{(0.2)^5}{5} - \frac{(0.2)^7}{7} + \cdots$.

Since $\frac{(0.2)^7}{7} < 5 \times 10^{-5}$, therefore $\tan^{-1}(0.2) \approx 0.2 - \frac{(0.2)^3}{3} + \frac{(0.2)^5}{5} \approx 0.19740$

with error less than 5×10^{-5} in absolute value.

Question 2: Find Maclaurin series for the following functions.

(a) $J(x)=\int_0^x \frac{e^t-1}{t}dt$

$$=\int_0^x\left(1+\frac{t}{2!}+\frac{t^2}{3!}+\frac{t^3}{4!}+\cdots\right)dt=x+\frac{x^2}{2!\cdot 2}+\frac{x^3}{3!\cdot 3}+\frac{x^4}{4!\cdot 4}+\cdots=\sum_{n=1}^{\infty}\frac{x^n}{n!\cdot n}$$

(b) $L(x)=\int_0^x \cos(t^2)dt$

$$L(x)=\int_0^x \cos(t^2)dt=\int_0^x\left(1-\frac{t^4}{2!}+\frac{t^8}{4!}-\frac{t^{12}}{6!}+\cdots\right)dt=x-\frac{x^5}{2!\cdot 5}+\frac{x^9}{4!\cdot 9}-\frac{x^{13}}{6!\cdot 13}+\cdots$$

$$=\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+1}}{(2n)!\cdot(4n+1)}$$

Question 3: Evaluate the limit (Do NOT use L'Hôpital's rule)

24. $\lim\limits_{x\to\infty}\frac{(e^x-1-x)^2}{x^2-\ln(1+x^2)}$

We have $$\lim_{x\to 0}\frac{(e^x-1-x)^2}{x^2-\ln(1+x^2)}=\lim_{x\to 0}\frac{\left(\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\right)^2}{\frac{x^4}{2}-\frac{x^6}{3}+\frac{x^8}{4}-\cdots}=\lim_{x\to 0}\frac{\frac{x^4}{4}\left(1+\frac{x}{3}+\frac{x^2}{12}+\cdots\right)^2}{\frac{x^4}{2}-\frac{x^6}{3}+\frac{x^8}{4}-\cdots}=\frac{\left(\frac{1}{4}\right)}{\left(\frac{1}{2}\right)}=\frac{1}{2}$$

Section 2.1 Analytic Geometry in Three Dimensions

The Following Questions Are Obtained From **Calculus - A Complete Course 7th ed - R. Adams, C. Essex**

Question 1: Show that the triangle with vertices (1,2,3), (4,0,5), and (3,6,4) has a right angle.

If $A=(1,2,3), B=(4,0,5)$, and $C=(3,6,4)$, then

$|AB|=\sqrt{3^2+(-2)^2+2^2}=\sqrt{17}$ $\quad |AC|=\sqrt{2^2+4^2+1^2}=\sqrt{21}$ $\quad |BC|=\sqrt{(-1)^2+6^2+(-1)^2}=\sqrt{38}$

Since $|AB|^2+|AC|^2=17+21=38=|BC|^2$, the triangle ABC has a right angle at A.

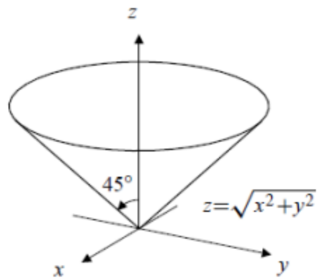
Question 2: In the following questions, describe (and sketch if possible) the set of points in $\mathbb{R}^3$ that satisfy the given equation or inequality.

1. $y^2+z^2\le 4$

$y^2+z^2\le 4$ represents all points inside and on the circular cylinder of radius 2 with central axis along the x-axis (a solid cylinder).

2. $z\ge\sqrt{x^2+y^2}$

$z \geq \sqrt{x^2 + y^2}$ represents every point whose distance above the xy-plane is not less than its horizontal distance from the z-axis. It therefore consists of all points inside and on a circular cone with axis along the positive z-axis, vertex at the origin, and semi-vertical angle 45°.



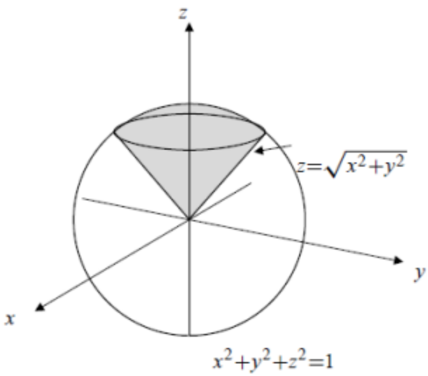
Question 3: In the following questions, describe (and sketch if possible) the set of points in $\mathbb{R}^3$ that satisfy the given pair of equations or inequalities.

1. $\begin{cases} x^2 + y^2 + z^2 = 4 \\ x^2 + y^2 + z^2 = 4x \end{cases}$

$\begin{cases} x^2 + y^2 + z^2 = 4 \\ x^2 + y^2 + z^2 = 4z \end{cases}$ is the circle in which the sphere of radius 2 centred at the origin intersects the sphere of radius 2 centred at $(0,0,2)$. (The second equation can be rewritten $x^2 + y^2 + (z-2)^2 = 4$ for easier recognition.) Subtracting the equations of the two spheres we get $z = 1$, so the circle must lie in the plane $z = 1$ as well.

2. $\begin{cases} x^2 + y^2 + z^2 \leq 1 \\ \sqrt{x^2 + y^2} \leq z \end{cases}$

$\begin{cases} x^2 + y^2 + z^2 \leq 1 \\ \sqrt{x^2 + y^2} \leq z \end{cases}$ represents all points which are inside or on the sphere of radius 1 centred at the origin and which are also inside or on the upper half of the circular cone with axis along the z-axis, vertex at the origin, and semi-vertical angle 45°.



Question 4: In the following questions, specify the boundary and the interior of the plane sets S whose points (x, y) satisfy the given conditions. Is S open, closed, or neither?

1. $|x| + |y| \leq 1$

$$S = \{(x, y): |x| + |y| \leq 1\}$$

The boundary of S consists of all points on the edges of the square with vertices $(\pm 1, 0)$ and $(0, \pm 1)$. The interior of S consists of all points inside that square. S is closed since it contains all its boundary points. It is bounded since all points in it are at distance not greater than 1 from the origin.

2. $x^2 + y^2 < 1, y + z > 2$

$$S = \{(x, y, z): x^2 + y^2 < 1, y + z > 2\}$$

Boundary: the part of the cylinder $x^2 + y^2 = 1$ that lies on or above the plane $y + z = 2$ together with the part of that plane that lies inside the cylinder.

Interior: all points that are inside the cylinder $x^2 + y^2 = 1$ and above the plane $y + z = 2$. S is open.

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: Use vectors to show that the triangle with vertices $(-1,1)$, $(2,5)$, and $(10,-1)$ is right-angled.

If $a = (-1,1), B = (2,5)$ and $C = (10,-1)$, then $\overrightarrow{AB} = 3\mathbf{i} + 4\mathbf{j}$ and $\overrightarrow{BC} = 8\mathbf{i} - 6\mathbf{j}$. Since $\overrightarrow{AB} \cdot \overrightarrow{BC} = 0$, therefore, $\overrightarrow{AB} \perp \overrightarrow{BC}$. Hence, $\triangle ABC$ has a right angle at B.

Question 2: For what value of t is the vector $2t\mathbf{i} + 4\mathbf{j} - (10 + t)\mathbf{k}$ perpendicular to the vector $\mathbf{i} + t\mathbf{j} + \mathbf{k}$?

The two vectors are perpendicular if their dot product is zero:

$$(2t\mathbf{i} + 4\mathbf{j} - (10 + t)\mathbf{k}) \cdot (\mathbf{i} + t\mathbf{j} + \mathbf{k}) = 0 \qquad 2t + 4t - 10 - t = 0 \Rightarrow t = 2$$

The vectors are perpendicular if $t = 2$.

Question 3: If a vector $\mathbf{u}$ in $\mathbb{R}^3$ makes angles α, β, and γ with the coordinate axes, show that

$$\hat{\mathbf{u}} = \cos\ \alpha\mathbf{i} + \cos\ \beta\mathbf{j} + \cos\ \gamma\mathbf{k}$$

is a unit vector in the direction of $\mathbf{u}$, so $\cos^2\ \alpha + \cos^2\ \beta + \cos^2\ \gamma = 1$. The numbers $\cos\ \alpha, \cos\ \beta$, and $\cos\ \gamma$ are called the direction cosines of $\mathbf{u}$.

If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, then $\cos\ \alpha \frac{\mathbf{u}\cdot\mathbf{i}}{|\mathbf{u}|} = \frac{u_1}{|\mathbf{u}|}$. Similarly, $\cos\ \beta = \frac{u_2}{|\mathbf{u}|}$ and $\cos\ \gamma = \frac{u_3}{|\mathbf{u}|}$. Thus, the unit vector in the direction of $\boldsymbol{u}$ is

$$\hat{\mathbf{u}} = \frac{\mathbf{u}}{|\mathbf{u}|} = \cos\ \alpha\mathbf{i} + \cos\ \beta\mathbf{j} + \cos\ \gamma\mathbf{k}$$

and so $\cos^2\ \alpha + \cos^2\ \beta + \cos^2\ \gamma = |\hat{\mathbf{u}}|^2 = 1$.

Question 4: Find the three angles of the triangle with vertices $(1,0,0)$, $(0,2,0)$, and $(0,0,3)$.

If $A = (1,0,0), B = (0,2,0)$, and $C = (0,0,3)$, then

$$\angle ABC = \cos^{-1} \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|BA||BC|} = \cos^{-1} \frac{4}{\sqrt{5}\sqrt{13}} \approx 60.26^\circ$$

$$\angle BCA = \cos^{-1} \frac{\overrightarrow{CB} \cdot \overrightarrow{CA}}{|CB||CA|} = \cos^{-1} \frac{9}{\sqrt{10}\sqrt{13}} \approx 37.87^\circ$$

$$\angle CAB = \cos^{-1}\frac{\overrightarrow{AC}\cdot\overrightarrow{AB}}{|AC||AB|} = \cos^{-1}\frac{1}{\sqrt{10}\sqrt{5}} \approx 81.87^\circ$$

<u>Question 5:</u> Find two-unit vectors each of which is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

Vector $\mathbf{x} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ if

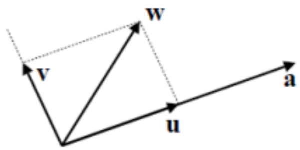
$$\begin{aligned}\mathbf{u}\cdot\mathbf{x} = 0 \quad &\Leftrightarrow \quad 2x + y - 2z = 0\\ \mathbf{v}\cdot\mathbf{x} = 0 \quad &\Leftrightarrow \quad x + 2y - 2z = 0\end{aligned}$$

Subtracting these equations, we get $x - y = 0$, so $x = y$. The first equation now gives $3x = 2z$. Now $\mathbf{x}$ is a unit vector if $x^2 + y^2 + z^2 = 1$, that is, if $x^2 + x^2 + \frac{9}{4}x^2 = 1$, or $x = \pm 2/\sqrt{17}$. The two-unit vectors are

$$\mathbf{x} = \pm\left(\frac{2}{\sqrt{17}}\mathbf{i} + \frac{2}{\sqrt{17}}\mathbf{j} + \frac{3}{\sqrt{17}}\mathbf{k}\right)$$

<u>Question 6:</u> If a is a nonzero vector and $\mathbf{w}$ is any vector, find vectors $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{w} = \mathbf{u} + \mathbf{v}$, $\mathbf{u}$ is parallel to $\mathbf{a}$, and $\mathbf{v}$ is perpendicular to $\mathbf{a}$.

Let $\mathbf{u} = \frac{\mathbf{w}\cdot\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a}$, (the vector projection of $\mathbf{w}$ along $\mathbf{a}$). Let $\mathbf{v} = \mathbf{w} - \mathbf{u}$. Then $\mathbf{w} = \mathbf{u} + \mathbf{v}$. Clearly $\mathbf{u}$ is parallel to $\mathbf{a}$, and $v\cdot a = w\cdot a - \frac{w\cdot a}{|a|^2}a\cdot a = w\cdot a - w\cdot a = 0$ so $\mathbf{v}$ is perpendicular to $\mathbf{a}$.



Section 2.3 The Cross Product in 3-Space

The Following Questions Are Obtained From **Calculus - A Complete Course 7th ed - R. Adams, C. Essex**

<u>Question 1:</u> Find the area of the triangle with vertices $(1,2,0)$, $(1,0,2)$, and $(0,3,1)$.

If $A = (1,2,0)$, $B = (1,0,2)$, and $C = (0,3,1)$, then $\overrightarrow{AB} = -2\mathbf{j} + 2\mathbf{k}$, $\overrightarrow{AC} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$, and the area of triangle ABC is $\frac{|\overrightarrow{AB}\times\overrightarrow{AC}|}{2} = \frac{|-4\mathbf{i}-2\mathbf{j}-2\mathbf{k}|}{2} = \sqrt{6}$sq. units.

<u>Question 2:</u> Find a unit vector perpendicular to the vectors $\mathbf{i} + \mathbf{j}$ and $\mathbf{j} + 2\mathbf{k}$

A vector perpendicular to $\mathbf{i} + \mathbf{j}$ and $\mathbf{j} + 2\mathbf{k}$ is $\pm(\mathbf{i} + \mathbf{j}) \times (\mathbf{j} + 2\mathbf{k}) = \pm(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$

which has length 3. A unit vector in that direction is $\pm\left(\frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}\right)$

Question 3: Find the volume of the tetrahedron with vertices $(1,0,0)$, $(1,2,0)$, $(2,2,2)$, and $(0,3,2)$.

The tetrahedron with vertices $(1,0,0)$, $(1,2,0)$, $(2,2,2)$, and $(0,3,2)$ is spanned by $\mathbf{u} = 2\mathbf{j}$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, and $\mathbf{w} = -\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$. By Exercise 14, its volume is

$V = \frac{1}{6}\left\|\begin{matrix} 0 & 2 & 0 \\ 1 & 2 & 2 \\ -1 & 3 & 2 \end{matrix}\right\| = \frac{4}{3}$ cu. units.

Question 4: For what value of k do the four points $(1,1,-1)$, $(0,3,-2)$, $(-2,1,0)$, and $(k,0,2)$ all lie in a plane?

The points $A = (1,1,-1)$, $B = (0,3,-2)$, $C = (-2,1,0)$, and $D = (k,0,2)$ are coplanar if $(\overrightarrow{AB} \times \overrightarrow{AC}) \cdot \overrightarrow{AD} = 0$. Now

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & -1 \\ -3 & 0 & 1 \end{vmatrix} = 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$$

Thus the four points are coplanar if

$$2(k-1) + 4(0-1) + 6(2+1) = 0,$$

that is, if $k = -6$.

Question 5: The product $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ is called a vector triple product. Since it is perpendicular to $\mathbf{v} \times \mathbf{w}$, it must lie in the plane of $\mathbf{v}$ and $\mathbf{w}$. Show that

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$$

As suggested in the hint, let the x-axis lie in the direction of $\mathbf{v}$, and let the y-axis be such that $\mathbf{w}$ lies in the xy plane. Thus $\mathbf{v} = v_1\mathbf{i}$, $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j}$

Thus $\mathbf{v} \times \mathbf{w} = v_1w_2\mathbf{i} \times \mathbf{j} = v_1w_2\mathbf{k}$, and $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1w_2\mathbf{k})$

$= u_1v_1w_2\mathbf{i} \times \mathbf{k} + u_2v_1w_2\mathbf{j} \times \mathbf{k} = -u_1v_1w_2\mathbf{i} - u_1v_1w_2\mathbf{j}$

But $(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} = (u_1w_1 + u_2w_2)v_1\mathbf{i} - u_1v_1(w_1\mathbf{i} + w_2\mathbf{j}) = u_2v_1w_2\mathbf{i} - u_1v_1w_2\mathbf{j}$.

Thus, $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$.

The Following Questions Are Obtained From PAST EXAMS

Question 1: a) Find all vectors $\vec{v}$ that satisfy the equation $(-\vec{\imath}+2\vec{\jmath}+3\vec{k})\times\vec{v}=\vec{\imath}+5\vec{\jmath}-3\vec{k}$ PE. SPRING2005

Let $\vec{v}=(a,b,c)$. Then we should have:

$$(-1,2,3)\times(a,b,c)=\begin{vmatrix} i & j & k \\ -1 & 2 & 3 \\ a & b & c \end{vmatrix}=(2c-3b,3a+c,-2a-b)=(1,5,-3)$$

This happens if and only if $-3b+2c=1,3a+c=5$, and $-2a-b=-3$ which gives

$$\vec{v}=t\vec{\imath}+(3-2t)\vec{\jmath}+(5-3t)\vec{k}\ \text{ for all } t\in\mathbb{R}.$$

b) Show that the four points $A(1,2,-1),B(0,1,5),C(-1,2,1)$ and $D(2,1,3)$ are coplanar (i.e., are on the same plane). PE. SPRING2005

Let the three vectors $\vec{u},\vec{v}$ and $\vec{w}$ represent the three segments $\overrightarrow{AB},\overrightarrow{AC}$ and $\overrightarrow{AD}$, respectively. Then

$\vec{u}=(-1,-1,6),\vec{v}=(-2,0,2)$ and $\vec{w}=(1,-1,4)$.

The four points are coplanar if the volume of the parallelepiped spanned by the three vectors is 0:

$$\begin{vmatrix} -1 & -1 & 6 \\ -2 & 0 & 2 \\ 1 & -1 & 4 \end{vmatrix}=-\begin{vmatrix} 0 & 2 \\ 1 & 4 \end{vmatrix}+\begin{vmatrix} -2 & 2 \\ 1 & -1 \end{vmatrix}+6\begin{vmatrix} -2 & 0 \\ 1 & -1 \end{vmatrix}=-2-10+12=0$$

Section 2.4 Planes and Lines

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: In following questions, find equations of the planes satisfying the given conditions.

1. Passing through the origin and having normal $\mathbf{i}-\mathbf{j}+2\mathbf{k}$

The plane through the origin having normal $\mathbf{i}-\mathbf{j}+2\mathbf{k}$ has equation $x-y+2z=0$.

2. Passing through the three points $(-2,0,0)$, $(0,3,0)$, and $(0,0,4)$

The plane passing through $(-2,0,0)$, $(0,3,0)$, and $(0,0,4)$ has equation

$\frac{x}{-2}+\frac{y}{3}+\frac{z}{4}=1$, or $6x-4y-3z=-12$.

3. Passing through the line $x+y=2, y-z=3$, and perpendicular to the plane $2x+3y+4z=5$

A plane through the line $x+y=2, y-z=3$ has equation of the form $x+y-2+\lambda(y-z-3)=0$.

This plane will be perpendicular to $2x+3y+4z=5$ if $(2)(1)+(1+\lambda)(3)-(\lambda)(4)=0$,

that is, if $\lambda=5$. The equation of the required plane is $x+6y-5z=17$

Question 2: find equations of the line specified in vector and scalar parametric forms and in standard form. Through $(2,-1,-1)$ and parallel to each of the two planes $x+y=0$ and $x-y+2z=0$

A line parallel to $x+y=0$ and to $x-y+2z=0$ is parallel to the cross product of the normal vectors to these two planes, that is, to the vector $(\mathbf{i}+\mathbf{j})\times(\mathbf{i}-\mathbf{j}+2\mathbf{k})=2(\mathbf{i}-\mathbf{j}-\mathbf{k})$

Since the line passes through $(2,-1,-1)$, its equations are, in vector parametric form

$$\mathbf{r}=(2+t)\mathbf{i}-(1+t)\mathbf{j}-(1+t)\mathbf{k}$$

or in scalar parametric form $x=2+t,\ y=-(1+t),\ z=-(1+t)$

or in standard form $x-2=-(y+1)=-(z+1)$

Question 3: If $P_1=(x_1,y_1,z_1)$ and $P_2=(x_2,y_2,z_2)$, show that the equations

$$\begin{cases} x=x_1+t(x_2-x_1) \\ y=y_1+t(y_2-y_1) \\ z=z_1+t(z_2-z_1) \end{cases}$$

represent a line through P_1 and P_2.

The equations

$$\begin{cases} x=x_1+t(x_2-x_1) \\ y=y_1+t(y_2-y_1) \\ z=z_1+t(z_2-z_1) \end{cases}$$

certainly, represent a straight line. Since $(x,y,z)=(x_1,y_1,z_1)$ if $t=0$, and

$(x,y,z)=(x_2,y_2,z_2)$ if $t=1$, the line must pass through P_1 and P_2.

Question 4: Find the required distances in the following questions.

1. From the origin to the plane $x + 2y + 3z = 4$

The distance from $(0,0,0)$ to $x + 2y + 3z = 4$ is $\frac{4}{\sqrt{1^2+2^2+3^2}} = \frac{4}{\sqrt{14}}$ units.

2. From the origin to the line $x + y + z = 0,2x - y - 5z = 1$

A vector parallel to the line $x + y + z = 0,2x - y - 5z = 1$ is

$$\mathbf{a} = (\mathbf{i} + \mathbf{j} + \mathbf{k}) \times (2\mathbf{i} - \mathbf{j} - 5\mathbf{k}) = -4\mathbf{i} + 7\mathbf{j} - 3\mathbf{k}$$

We need a point on this line: if $z = 0$ then $x + y = 0$ and $2x - y = 1$, so $x = 1/3$ and $y = -1/3$. The position vector of this point is $\mathbf{r}_1 = \frac{1}{3}\mathbf{i} - \frac{1}{3}\mathbf{j}$

The distance from the origin to the line is $s = \frac{|\mathbf{r}_1 \times \mathbf{a}|}{|\mathbf{a}|} = \frac{|\mathbf{i}+\mathbf{j}+\mathbf{k}|}{\sqrt{74}} = \sqrt{\frac{3}{74}}$ units.

3. Between the lines

$$\begin{cases} x + 2y = 3 \\ y + 2z = 3 \end{cases} \text{and} \begin{cases} x + y + z = 6 \\ x - 2z = -5 \end{cases}$$

The line $\begin{cases} x + 2y = 3 \\ y + 2z = 3 \end{cases}$ contains the points $(1,1,1)$ and $(3,0,3/2)$, so is parallel to the vector $2\mathbf{i} - \mathbf{j} + \frac{1}{2}\mathbf{k}$, or to $4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

The line $\begin{cases} x + y + z = 6 \\ x - 2z = -5 \end{cases}$ contains the points $(-5,11,0)$ and $(-1,5,2)$, and so is parallel to the vector $4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$, or to $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$.

Using the values $\mathbf{r}_1 = \mathbf{i} + \mathbf{j} + \mathbf{k}$ $\quad \mathbf{a}_1 = 4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ $\quad \mathbf{r}_2 = -\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$ $\quad \mathbf{a}_2 = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$

we calculate the distance between the two lines by the following formula as

$$s = \frac{|(\mathbf{r}_1 - \mathbf{r}_2) \cdot (\mathbf{a}_1 \times \mathbf{a}_2)|}{|\mathbf{a}_1 \times \mathbf{a}_2|} = \frac{|(2\mathbf{i} - 4\mathbf{j} - \mathbf{k}) \cdot (\mathbf{i} - 2\mathbf{j} - 8\mathbf{k})|}{|\mathbf{i} - 2\mathbf{j} - 8\mathbf{k}|} = \frac{18}{\sqrt{69}} \text{ units.}$$

The Following Questions Are Obtained From PAST EXAMS

Question 1: Find the distance between the lines

$\mathcal{L}_1$: intersection of the planes ($x + 2y = 3$ and $y + 2z = 3$) and

$\mathcal{L}_2$: intersection of the planes ($x + y + z = 6$ and $x - 2z = -5$) PE. SPRING2005

$\mathcal{L}_1$ contains the points $(1,1,1), (3,0,3/2)$, so it is parallel to the vector $(3,0,3/2) - (1,1,1) = (2,-1,1/2)$, so it is parallel to the vector $\vec{u}_1 = (4,-2,1)$

$\mathcal{L}_2$ contains the points $(-5,11,0), (-1,5,2)$, so it is parallel to the vector $(-1,5,2) - (-5,11,0) = (4,-6,2)$, so it is parallel to the vector $\vec{u}_2 = (2,-3,1)$

Using the vectors $\vec{r}_1 = (1,1,1), \vec{r}_2 = (-1,5,2), \vec{u}_1 = (4,-2,1)$ and $\vec{u}_2 = (2,-3,1)$ we find the distance:

$$d = \frac{|(\vec{r}_2 - \vec{r}_1) \cdot (\vec{u}_1 \times \vec{u}_2)|}{|\vec{u}_1 \times \vec{u}_2|} = \frac{|(2,-4,-1) \cdot (1,-2,-8)|}{|(1,-2,-8)|} = \frac{18}{\sqrt{69}}$$

Question 2: Let $L_1 = \begin{cases} x = 1 + t \\ y = -9 + 2t \quad t \in R \\ z = 5 - t \end{cases}$ and $L_2 = \begin{cases} x = 2 - 4s \\ y = -s \qquad s \in R \\ z = 1 + s \end{cases}$ PE. SPRING2006

a) Find an equation of the line L which is perpendicular to both L_1 and L_2 and passes through the point of intersection of L_1 and L_2.

$$\vec{v} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ 1 & 2 & -1 \\ -4 & -1 & 1 \end{vmatrix} = \mathrm{i}(2-1) - \mathrm{j}(1-4) + \mathrm{k}(-1+8) = \langle 1,3,7 \rangle$$

$$P = L_1 \cap L_2 \qquad \left.\begin{matrix} 1 + t = 2 - 4s \\ -9 + 2t = -s \end{matrix}\right\} \qquad \left.\begin{matrix} s = -1 \\ t = 5 \end{matrix}\right\} \Rightarrow P = (6,1,0)$$

L: $x = 6 + 1t$

$y = 1 + 3t \quad t \in \mathbb{R}$

$z = 0 + 7t$

b) Find the distance between the line L_1 and the plane containing the point $P_0(1,1,1)$ and the line L_2

$$\vec{n} // \left(\vec{V}_1 \times \overrightarrow{P_2P_0}\right) = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ -4 & -1 & 1 \\ -1 & 1 & 0 \end{vmatrix} = \mathrm{i}(-1) - \mathrm{j}(1) + \mathrm{k}(-5) = \langle -1,-1,-5 \rangle$$

Note that

$$\vec{n} \cdot \vec{V}_1 = \langle -1,-1,-5 \rangle \cdot \langle 1,2,-1 \rangle = -1 - 2 + 5 = 2 \neq 0$$

so $\vec{n} \not\perp \vec{v}_1 \Rightarrow L_1 \not\parallel P \Rightarrow L_1 \cap P \neq \emptyset$

$\Rightarrow$ distance is zero

[Or simply, as part (a) indicates, L_1 & L_2 interests at $P(6,1,0)$. Hence distance is zero]

Section 2.5 Quadric Surfaces

The Following Questions Are Obtained From **Calculus - A Complete Course 7th ed - R. Adams, C. Essex**

Question 1: Identify the surfaces represented by the equations in following questions and sketch their graphs.

a. $2x^2 + 2y^2 + 2z^2 - 4x + 8y - 12z + 27 = 0$

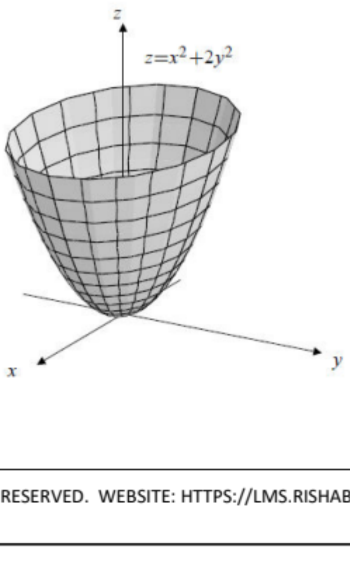
$$2x^2 + 2y^2 + 2z^2 - 4x + 8y - 12z + 27 = 0$$

$$2(x^2 - 2x + 1) + 2(y^2 + 4y + 4) + 2(z^2 - 6z + 9) = -27 + 2 + 8 + 18(x-1)^2 + (y+2)^2 + (z-3)^2 = \frac{1}{2}$$

This is a sphere with radius $1/\sqrt{2}$ and center $(1,-2,3)$.

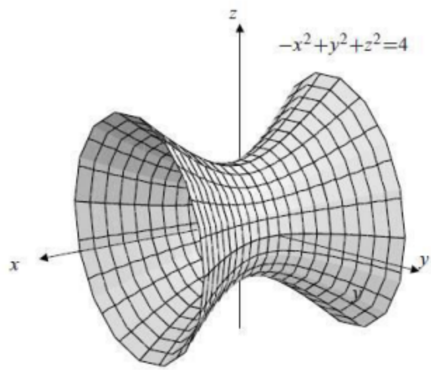
b. $z = x^2 + 2y^2$

$z = x^2 + 2y^2$ represents an elliptic paraboloid with vertex at the origin and axis along the positive z-axis. Cross sections in planes $z = k > 0$ are ellipses with semi-axes $\sqrt{k}$ and $\sqrt{k/2}$.



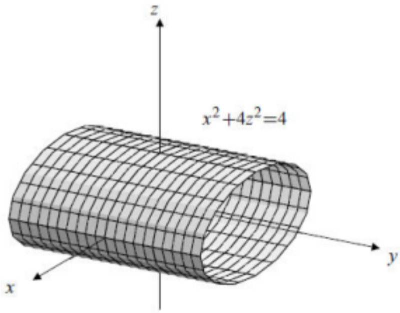
c. $-x^2 + y^2 + z^2 = 4$

$-x^2 + y^2 + z^2 = 4$ represents a hyperboloid of one sheet, with circular cross-sections in all planes perpendicular to the x-axis.



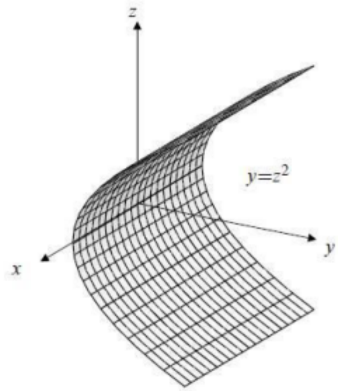
d. $x^2 + 4z^2 = 4$

$x^2 + 4z^2 = 4$ represents an elliptic cylinder with axis along the y-axis.



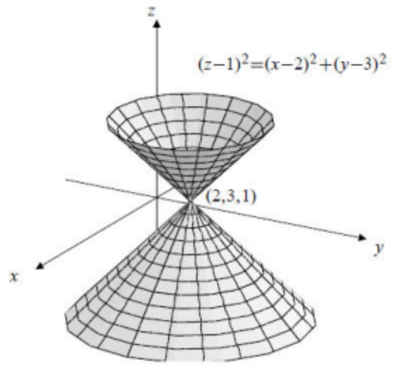
e. $y = z^2$

$y = z^2$ represents a parabolic cylinder with vertex line along the x-axis.



f. $(z-1)^2 = (x-2)^2 + (y-3)^2$

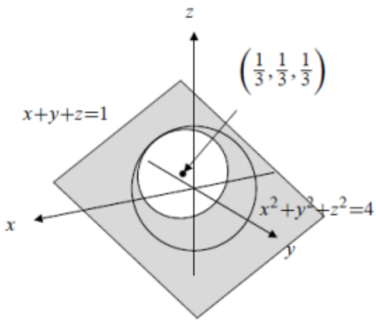
$(z-1)^2 = (x-2)^2 + (y-3)^2$ represents a circular cone with axis along the line $x = 2, y = 3$, and vertex at $(2,3,1)$



<u>Question 2:</u> Describe and sketch the geometric objects represented by the systems of equations in following questions

a. $\begin{cases} x^2 + y^2 + z^2 = 4 \\ x + y + z = 1 \end{cases}$

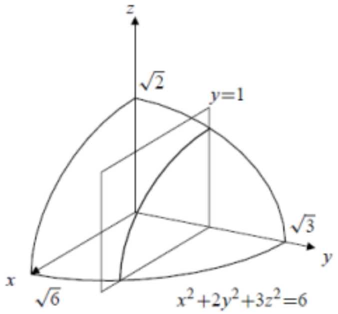
$\begin{cases} x^2 + y^2 + z^2 = 4 \\ x + y + z = 1 \end{cases}$ represents the circle of intersection of a sphere and a plane. The circle lies in the plane $x + y + z = 1$, and has centre $(1/3,1/3,1/3)$ and radius $\sqrt{4-(3/9)} = \sqrt{11/3}$.



b. $\begin{cases} x^2 + 2y^2 + 3z^2 = 6 \\ y = 1 \end{cases}$

$\begin{cases} x^2 + 2y^2 + 3z^2 = 6 \\ y = 1 \end{cases}$ is an ellipse in the plane

$y = 1$. Its projection onto the xz-plane is the ellipse $x^2 + 3z^2 = 4$. One quarter of the ellipse is shown in the figure.



Section 3.1 Functions of Several Variables

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: Specify the domains of the following functions.

1. $f(x,y) = \frac{xy}{x^2-y^2}$

The domain consists of all points not on the lines $x = \pm y$.

2. $f(x,y) = \sqrt{4x^2 + 9y^2 - 36}$

The domain consists of all points (x, y) lying on or outside the ellipse $4x^2 + 9y^2 = 36$

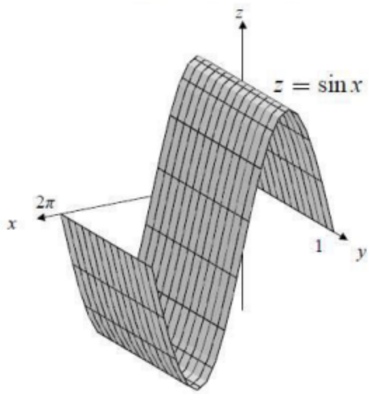
3. $f(x,y) = \sin^{-1}(x + y)$

The domain consists of all points in the strip $-1 \leq x + y \leq 1$.

Question 2: Sketch the graphs of the following functions.

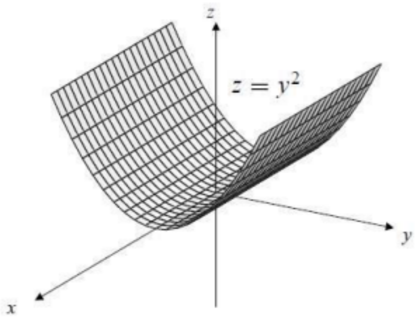
1. $f(x,y) = \sin\ x,\ (0 \leq x \leq 2\pi,\ 0 \leq y \leq 1)$

$f(x,y) = \sin\ x, 0 \leq x \leq 2\pi, 0 \leq y \leq 1$

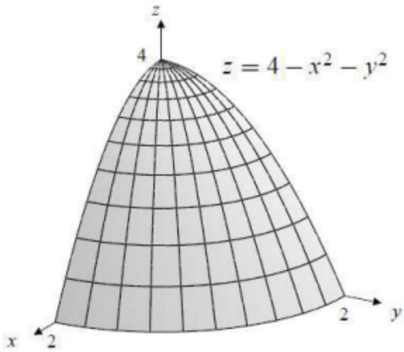


2. $f(x,y) = y^2,\ (-1 \leq x \leq 1,\ -1 \leq y \leq 1)$

$z = f(x,y) = y^2$



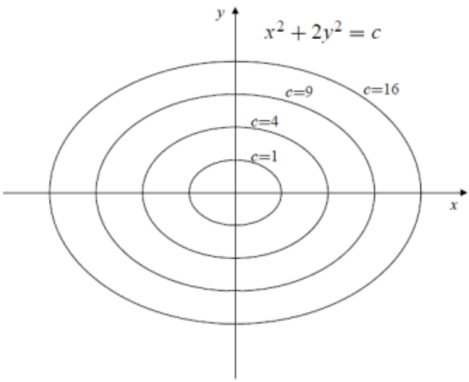
3. $f(x,y) = 4 - x^2 - y^2$, $(x^2 + y^2 \leq 4, x \geq 0, y \geq 0)$



Question 3: Sketch some of the level curves of the following functions.

1. $f(x,y) = x^2 + 2y^2$

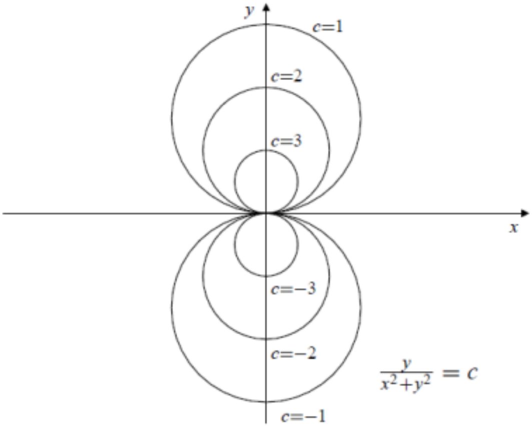
$f(x,y) = x^2 + 2y^2 = C$, a family of similar ellipses centered at the origin.



2. $f(x,y) = \frac{y}{x^2+y^2}$

$f(x,y) = \dfrac{y}{x^2 + y^2} = C.$

This is the family $x^2 + \left(y - \frac{1}{2C}\right)^2 = \frac{1}{4C^2}$ of circles passing through the origin and having centres on the y-axis. The origin itself is, however, not on any of the level curves.



Section 3.2 Limits and Continuity

The Following Questions Are Obtained From **Calculus - A Complete Course 7th ed - R. Adams, C. Essex**

Question 1: In following questions, evaluate the indicated limit or explain why it does not exist.

1. $\lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2}$ $\qquad \lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2} = 0$

2. $\lim_{(x,y)\to(0,1)} \frac{x^2(y-1)^2}{x^2+(y-1)^2}$

$\lim_{(x,y)\to(0,1)} \frac{x^2(y-1)^2}{x^2+(y-1)^2} = 0,$ because $0 \le \left|\frac{x^2(y-1)^2}{x^2+(y-1)^2}\right| \le x^2$ and $x^2 \to 0$ as $(x,y) \to (0,1)$.

3. $\lim_{(x,y)\to(0,0)} \frac{\sin(x-y)}{\cos(x+y)}$

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x-y)}{\cos(x+y)} = \frac{\sin\ 0}{\cos\ 0} = 0.$$

4. $\lim_{(x,y)\to(1,2)} \frac{2x^2-xy}{4x^2-y^2}$

The fraction is not defined at points of the line $y = 2x$ and so cannot have a limit at $(1,2)$. However, by cancelling the common factor $2x - y$, we get

$$\lim_{(x,y)\to(1,2)} \frac{2x^2-xy}{4x^2-y^2} = \lim_{(x,y)\to(1,2)} \frac{x}{2x+y} = \frac{1}{4}.$$

5. $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{2x^4+y^4}$

If $x = 0$ and $y \neq 0$, then $\frac{x^2y^2}{2x^4+y^4} = 0$. If $x = y \neq 0$, then $\frac{x^2y^2}{2x^4+y^4} = \frac{x^4}{2x^4+x^4} = \frac{1}{3}$.

Therefore $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{2x^4+y^4}$ does not exist.

Question 2: How can the function $f(x,y) = \frac{x^3-y^3}{x-y}$, $(x \neq y)$,

be defined along the line $x = y$ so that the resulting function is continuous on the whole xy-plane?

For $x \neq y$, we have $f(x,y) = \frac{x^3-y^3}{x-y} = x^2 + xy + y^2$

The latter expression has the value $3x^2$ at points of the line $x = y$. Therefore, if we extend the definition of $f(x,y)$ so that $f(x,x) = 3x^2$, then the resulting function will be equal to $x^2 + xy + y^2$ everywhere, and so continuous everywhere.

Question 3: What condition must the nonnegative integers m, n, and p satisfy to guarantee that $\lim_{(x,y)\to(0,0)} x^my^n/(x^2+y^2)^p$ exists? Prove your answer.

Since $|x| \leq \sqrt{x^2+y^2}$ and $|y| \leq \sqrt{x^2+y^2}$, we have $\left|\frac{x^my^n}{(x^2+y^2)^p}\right| \leq \frac{(x^2+y^2)^{(m+n)/2}}{(x^2+y^2)^p} = (x^2+y^2)^{-p+(m+n)/2}$.

The expression on the right $\to 0$ as $(x,y) \to (0,0)$, provided $m + n > 2p$. In this case

$$\lim_{(x,y)\to(0,0)} \frac{x^my^n}{(x^2+y^2)^p} = 0.$$

The Following Questions Are Obtained From PAST EXAMS

Question 1: Consider the function $f(x,y) = \frac{x^3}{x^2+y^2}$ if $(x,y) \neq (0,0)$, $f(0,0) = 0$. Show that: PE. SPRING2005

a) f is continuous everywhere

Solution: We need to check only continuity at the point $(0,0)$.

$$0 \leq \left|\frac{x^3}{x^2+y^2}\right| = \left|\frac{x^2 \cdot x}{x^2+y^2}\right| \leq |x|$$

so, by the squeeze theorem, we have

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = 0 = f(0,0)$$

meaning that f is continuous at $(0,0)$.

b) f is **not** differentiable $(0,0)$

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k)-f(0,0)-hf_x(0,0)-kf_y(0,0)}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{f(h,k)-h}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{\frac{h^3}{h^2+k^2}-h}{\sqrt{h^2+k^2}}$$

along the line $k = h$ we have

$$\lim_{h\to 0} \frac{-h/2}{\sqrt{2}|h|}$$

which does not exist. Hence, f is not differentiable at $(0,0)$.

Question 2: Determine whether the following limits exist: PE. SPRING2006

a) $\lim\limits_{(x,y)\to(1,2)} \frac{(x-1)(y-2)}{(x-1)^2+(y-2)^2} = \lim\limits_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

along $y = mx$, $\lim\limits_{x\to 0} \frac{mx^2}{x^2+m^2x^2} = \frac{m}{1+m^2}$ depends on m so $\lim\limits_{(x,y)\to(1,2)} \frac{(x-1)(y-2)}{(x-1)^2+(y-2)^2}$ does not exist

b) $\lim\limits_{(x,y)\to(0,0)} \frac{xy^5}{x^2+y^2}$

$$0 \le \left|\frac{xy^5}{x^2+y^2}\right| = |xy^3| \frac{y^2}{x^2+y^2} \le \underbrace{|xy^3|}_{0 \text{ as } (x,y)\to(0,0)}$$

so, by the squeeze theorem, we have

$$\lim_{(x,y)\to(0,0)} |xy^3| = \lim_{(x,y)\to(0,0)} \frac{xy^5}{x^2+y^2} = 0$$

Question 3: Let $f(x,y) = \begin{cases} \frac{x-2y^2}{x-y} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$ PE. SPRING2006

a) Compute $f_x(0,0) = \frac{\partial f}{\partial x}(0,0)$

$\lim\limits_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = \lim\limits_{t\to 0} \frac{\frac{t^2}{t}}{t} = \lim\limits_{t\to 0} 1 = 1$ Thus, $f_x(0,0) = 1$

b) Is $f_x(x,y)$ continuous at $(0,0)$?

if $x \neq y$ then $f_x(x,y) = \frac{2x(x-y)-(x^2-2y^2)}{(x-y)^2} = \frac{x^2+2y^2-2xy}{(x-y)^2}$

Thus, along $y = mx$ with $m \neq 1$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{x^2+2m^2x^2-2xmx}{(x-mx)^2} = \lim_{x\to 0} \frac{\cancel{x^2}(1+3m^2-2m)}{\cancel{x^2}(1-m)^2} = \frac{1+2m^2-2m}{(1-m)^2}$$

The limit result depends on m so limit does NOT exist. Hence $f_x(x,y)$ is NOT continuous at $(0,0)$

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: In following questions, find all the first partial derivatives of the function specified, and evaluate them at the given point.

1. $g(x,y,z) = \frac{xz}{y+z}$, $(1,1,1)$

$g(x,y,z) = \frac{xz}{y+z}$ $\quad g_1(x,y,z) = \frac{z}{y+z}, g_1(1,1,1) = \frac{1}{2}$ $\quad g_2(x,y,z) = \frac{-xz}{(y+z)^2}, g_2(1,1,1) = -\frac{1}{4}$

$$g_3(x,y,z) = \frac{xy}{(y+z)^2}, g_3(1,1,1) = \frac{1}{4}$$

2. $z = \tan^{-1}\left(\frac{y}{x}\right)$, $(-1,1)$

$z = \tan^{-1}\left(\frac{y}{x}\right)$ $\quad \frac{\partial z}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}$ $\quad \frac{\partial z}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$

$$\left.\frac{\partial z}{\partial x}\right|_{(-1,1)} = -\frac{1}{2}, \left.\frac{\partial z}{\partial y}\right|_{(-1,1)} = -\frac{1}{2}$$

3. $w = \ln\left(1+e^{xyz}\right)$, $(2,0,-1)$

$w = \ln(1+e^{xyz}), \frac{\partial w}{\partial x} = \frac{yze^{xyz}}{1+e^{xyz}}$ $\quad \frac{\partial w}{\partial y} = \frac{xze^{xyz}}{1+e^{xyz}}, \frac{\partial w}{\partial z} = \frac{xye^{xyz}}{1+e^{xyz}}$

At $(2,0,-1)$: $\frac{\partial w}{\partial x} = 0, \frac{\partial w}{\partial y} = -1, \frac{\partial w}{\partial z} = 0$

Question 2: In following questions, calculate the first partial derivatives of the given functions at $(0,0)$. You will have to use limit Definition.

1. $f(x,y) = \begin{cases} \frac{2x^3-y^3}{x^2+3y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$

$f_1(0,0) = \lim_{h\to 0} \frac{2h^3-0}{h(h^2+0)} = 2$ $\quad f_2(0,0) = \lim_{k\to 0} \frac{-k^3-0}{k(0+3k^2)} = -\frac{1}{3}.$

2. $f(x,y) = \begin{cases} \frac{x^2-2y^2}{x-y}, & \text{if } x \neq y \\ 0, & \text{if } x = y. \end{cases}$

$$f_1(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{h-0}{h} = 1, \qquad f_2(0,0) = \lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0}\frac{2k}{k} = 2$$

Question 3: In following questions, find equations of the tangent plane and normal line to the graph of the given function at the point with specified values of x and y.

1. $f(x,y) = e^{xy}$ at $(2,0)$

$$f(x,y) = e^{xy}, f_1(x,y) = ye^{xy}, f_2(x,y) = xe^{xy} \qquad f(2,0) = 1,\quad f_1(2,0) = 0, \qquad f_2(2,0) = 2$$

Tangent plane to $z = e^{xy}$ at $(2,0)$ has equation $z = 1 + 2y$.

Normal line: $x = 2, y = 2 - 2z$.

2. $f(x,y) = \frac{x}{x^2+y^2}$ at $(1,2)$

$$f_1(x,y) = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \qquad f_2(x,y) = -\frac{2xy}{(x^2+y^2)^2}$$

$$f(1,2) = \frac{1}{5},\ f_1(1,2) = \frac{3}{25},\ f_2(1,2) = -\frac{4}{25}.$$

The tangent plane at $x = 1, y = 2$ is

$$z = \frac{1}{5} + \frac{3}{25}(x-1) - \frac{4}{25}(y-2) \text{ or } 3x - 4y - 25z = -10.$$

Normal line: $\frac{x-1}{3} = \frac{y-2}{-4} = \frac{5z-1}{-125}$.

3. $f(x,y) = \tan^{-1}(y/x)$ at $(1,-1)$

$$f(x,y) = \tan^{-1}\left(\frac{y}{x}\right), f(1,-1) = -\frac{\pi}{4}, \qquad f_1(x,y) = \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2},$$

$$f_2(x,y) = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}, \qquad f_1(1,-1) = f_2(1,-1) = \frac{1}{2}.$$

The tangent plane is $z = -\frac{\pi}{4} + \frac{1}{2}(x-1) + \frac{1}{2}(y+1)$, or $z = -\frac{\pi}{4} + \frac{1}{2}(x+y)$.

Normal line: $2(x-1) = 2(y+1) = -z - \frac{\pi}{4}$.

Question 4: Find all horizontal planes that are tangent to the surface with equation $z = xye^{-(x^2+y^2)/2}$. At what points are they tangent?

$z = xye^{-(x^2+y^2)/2}$

$$\frac{\partial z}{\partial x} = ye^{-(x^2+y^2)/2} - x^2ye^{-(x^2+y^2)/2} = y(1-x^2)e^{-(x^2+y^2)/2}$$

$$\frac{\partial z}{\partial y} = x(1-y^2)e^{-(x^2+y^2)/2} \text{ (by symmetry)}$$

The tangent planes are horizontal at points where both of these first partials are zero, that is, points satisfying

$$y(1-x^2) = 0 \text{ and } x(1-y^2) = 0$$

These points are $(0,0), (1,1), (-1,-1), (1,-1)$ and $(-1,1)$.

At $(0,0)$ the tangent plane is $z = 0$.

At $(1,1)$ and $(-1,-1)$ the tangent plane is $z = 1/e$.

At $(1,-1)$ and $(-1,1)$ the tangent plane is $z = -1/e$.

Question 5: In following questions, show that the given function satisfies the given partial differential equation.

1. $w = x^2 + yz,\ x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = 2w$

$w = x^2 + yz, \frac{\partial w}{\partial x} = 2x, \frac{\partial w}{\partial y} == z, \frac{\partial w}{\partial z} = y$. Therefore, $x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = 2x^2 + yz + yz = 2(x^2 + yz) = 2w$.

2. $z = f(x^2 - y^2)$, where f is any differentiable function of one variable,

$y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0. \qquad z = f(x^2 - y^2),$

$$\frac{\partial z}{\partial x} = f'(x^2 - y^2)(2x), \qquad \frac{\partial z}{\partial y} = f'(x^2 - y^2)(-2y).$$

Thus $y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = (2xy - 2xy)f'(x^2 - y^2) = 0.$

Question 6: $f(x,y)=\begin{cases}\frac{2xy}{x^2+y^2}, & if\ (x,y)\neq(0,0)\\ 0, & if\ (x,y)=(0,0)\end{cases}$

Note that f is not continuous at $(0,0)$. Therefore, its graph is not smooth there. Show, however, that $f_1(0,0)$ and $f_2(0,0)$ both exist. Hence, the existence of partial derivatives does not imply that a function of several variables is continuous. This is in contrast to the single-variable case.

$$f(x,y)=\frac{2xy}{x^2+y^2}\ \text{ if } (x,y)\neq(0,0),\ f(0,0)=0$$

$$f_1(0,0)=\lim_{h\to0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to0}\frac{0-0}{h}=0 \qquad f_2(0,0)=\lim_{k\to0}\frac{f(0,k)-f(0,0)}{h}=\lim_{k\to0}\frac{0-0}{k}=0$$

Thus $f_1(0,0)$ and $f_2(0,0)$ both exist even though f is not continuous at $(0,0)$.

Question 7: Let $f(x,y)=\begin{cases}\frac{x^3-y^3}{x^2+y^2}, & \text{if } (x,y)\neq(0,0)\\ 0, & \text{if } (x,y)=(0,0)\end{cases}.$

Calculate $f_1(x,y)$ and $f_2(x,y)$ at all points (x,y) in the plane. Is f continuous at $(0,0)$? Are f_1 and f_2 continuous at $(0,0)$?

If $(x,y)\neq(0,0)$, then $f_1(x,y)\ =\frac{(x^2+y^2)3x^2-(x^3-y^3)2x}{(x^2+y^2)^2}=\frac{x^4+3x^2y^2+2xy^3}{(x^2+y^2)^2}$

$$f_2(x,y)=\frac{(x^2+y^2)(-3y^2)-(x^3-y^3)2y}{(x^2+y^2)^2}=-\frac{y^4+3x^2y^2+2x^3y}{(x^2+y^2)^2}.$$

Also, at $(0,0)$,

$$f_1(0,0)=\lim_{h\to0}\frac{h^3}{h\cdot h^2}=1,\ f_2(0,0)=\lim_{k\to0}\frac{-k^3}{k\cdot k^2}=-1.$$

Neither f_1 nor f_2 has a limit at $(0,0)$ (the limits along $x=0$ and $y=0$ are different in each case), so neither function is continuous at $(0,0)$. However, f is continuous at $(0,0)$ because

$$|f(x,y)|\leq\left|\frac{x^3}{x^2+y^2}\right|+\left|\frac{y^3}{x^2+y^2}\right|\leq|x|+|y|,$$

which $\to 0$ as $(x,y)\to(0,0)$

The Following Questions Are Obtained From PAST EXAMS

Question 1: Let $f(u)$ be a differentiable function with $f(1) = 1, f'(1) = 2$ and $g(x, y) = xf\left(\frac{y}{x^2}\right)$.

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$$g_x(x, y) = f\left(\frac{y}{x^2}\right) + xf'\left(\frac{y}{x^2}\right)y(-2)\frac{1}{x^3}$$

$$g_x(1,1) = f(1) + 1 \cdot f'(1) \cdot 1 \cdot (-2) \cdot \frac{1}{1} = 1 + 2(-2) = -3$$

Section 3.4 Higher-Order Derivatives

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: find all the second partial derivatives of the given function.

$z = \sqrt{3x^2 + y^2}$

$$\frac{\partial z}{\partial x} = \frac{3x}{\sqrt{3x^2 + y^2}}, \ \frac{\partial z}{\partial y} = \frac{y}{\sqrt{3x^2 + y^2}}, \quad \rightarrow \quad \frac{\partial^2 z}{\partial x^2} = \frac{\sqrt{3x^2 + y^2}(3) - 3x\dfrac{3x}{\sqrt{3x^2 + y^2}}}{3x^2 + y^2} = \frac{3y^2}{(3x^2 + y^2)^{3/2}},$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\sqrt{3x^2 + y^2} - y\dfrac{y}{\sqrt{3x^2 + y^2}}}{3x^2 + y^2} = \frac{3x^2}{(3x^2 + y^2)^{3/2}} \quad \rightarrow \quad \frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial^2 z}{\partial y\, \partial x} = -\frac{3xy}{(3x^2 + y^2)^{\frac{3}{2}}}.$$

Question 2: Show that the function is harmonic in the plane regions indicated.

$f(x, y) = \frac{x}{x^2+y^2}$ everywhere except at the origin.

$$f_1(x, y) = \frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$f_2(x, y) = -\frac{2xy}{(x^2 + y^2)^2}$$

$$f_{11}(x, y) = \frac{(x^2 + y^2)^2(-2x) - (y^2 - x^2)2(x^2 + y^2)(2x)}{(x^2 + y^2)^4} = \frac{2x^3 - 6xy^2}{(x^2 + y^2)^3}$$

$$f_{22}(x, y) = -\frac{(x^2 + y^2)^2(2x) - 2xy2(x^2 + y^2)(2y)}{(x^2 + y^2)^4} = \frac{-2x^3 + 6xy^2}{(x^2 + y^2)^3}$$

Evidently, $f_{11}(x,y)+f_{22}(x,y)=0$ for $(x,y)\neq(0,0)$. Hence f is harmonic except at the origin.

Question 3: Let $F(x,y)=\begin{cases}\frac{2xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$

Calculate $F_1(x,y), F_2(x,y), F_{12}(x,y)$, and $F_{21}(x,y)$ at points $(x,y)\neq(0,0)$. Also calculate these derivatives at $(0,0)$. Observe that $F_{21}(0,0)=2$ and $F_{12}(0,0)=-2$. Does this result show that the partials F_{12} and F_{21} are continuous at $(0,0)$.? Explain why.

Let $f(x,y)=\begin{cases}\dfrac{2xy}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$

For $(x,y)\neq(0,0)$, we have

$$f_1(x,y)=\frac{(x^2+y^2)2y-2xy(2x)}{(x^2+y^2)^2}=\frac{2y(y^2-x^2)}{(x^2+y^2)^2} \qquad f_2(x,y)=\frac{2x(x^2-y^2)}{(x^2+y^2)^2} \text{ (by symmetry)}$$

Let $F(x,y)=(x^2-y^2)f(x,y)$. Then we calculate

$$F_1(x,y)=2xf(x,y)+(x^2-y^2)f_1(x,y)=2xf(x,y)-\frac{2y(y^2-x^2)^2}{(x^2+y^2)^2}$$

$$F_2(x,y)=-2yf(x,y)+(x^2-y^2)f_2(x,y)=-2yf(x,y)+\frac{2x(x^2-y^2)^2}{(x^2+y^2)^2}$$

$$F_{12}(x,y)=\frac{2(x^6+9x^4y^2-9x^2y^4-y^6)}{(x^2+y^2)^3}=F_{21}(x,y)$$

For the values at $(0,0)$ we revert to the definition of derivative to calculate the partials:

$$F_1(0,0)=\lim_{h\to0}\frac{F(h,0)-F(0,0)}{h}=0=F_2(0,0)$$

$$F_{12}(0,0)=\lim_{k\to0}\frac{F_1(0,k)-F_1(0,0)}{k}=\lim_{k\to0}\frac{-2k(k^4)}{k(k^4)}=-2$$

$$F_{21}(0,0)=\lim_{h\to0}\frac{F_2(h,0)-F_2(0,0)}{h}=\lim_{h\to0}\frac{2h(h^4)}{h(h^4)}=2$$

Thus, the partials F_{12} and F_{21} are not continuous at $(0,0)$. (Observe, for instance, that $F_{12}(x,x)=0$, while $F_{12}(x,0)=2$ for $x\neq0$.)

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: write appropriate versions of the Chain Rule for the indicated derivatives.

dw/dt if $w = f(x,y), x = g(r,s), y = h(r,t), r = k(s,t)$, and $s = m(t)$

If $w = f(x,y)$ where $x = g(r,s), y = h(r,t), r = k(s,t)$ and $s = m(t)$, then

$$\frac{dw}{dt} = f_1(x,y)[g_1(r,s)(k_1(s,t)m'(t)+k_2(s,t)) + g_2(r,s)m'(t)] + f_2(x,y)[h_1(r,t)(k_1(s,t)m'(t)$$

$$+k_2(s,t)) + h_2(r,t)]$$

Question 2: Use two methods to calculate dz/dt given that $z = txy^2$, $x = t + \ln(y + t^2)$, and $y = e^t$.

If $z = txy^2$, where $x = t + \ln(y + t^2)$ and $y = e^t$, then

Method I.

$$\frac{dz}{dt} = \frac{\partial z}{\partial t} + \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial t} + \frac{\partial x}{\partial y}\frac{\partial y}{\partial t}\right) + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = xy^2 + ty^2\left(1 + \frac{y+2t}{y+t^2}\right) + 2txy^2$$

Method II.

$$z = t\big(t + \ln(e^t + t^2)\big)e^{2t}$$

$$\frac{\partial z}{\partial t} = \big(t + \ln(e^t + t^2)\big)e^{2t} + te^{2t}\left(1 + \frac{e^t + 2t}{e^t + t^2}\right) + 2te^{2t}\big(t + \ln(e^t + t^2)\big) = xy^2 + ty^2\left(1 + \frac{y+2t}{y+t^2}\right) + 2txy^2$$

Question 3: In following questions, assume that f has continuous partial derivatives of all orders.

1. If $f(x,y)$ is harmonic, show that $f\left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$ is also harmonic, that is, $f_{11}(u,v) + f_{22}(u,v) = 0$.

Let $u = \frac{x}{x^2+y^2}, v = -\frac{y}{x^2+y^2}$. Then $\frac{\partial u}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2}$ $\quad \frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2}$ $\quad \frac{\partial u}{\partial y} = -\frac{2xy}{(x^2+y^2)^2}$ $\quad \frac{\partial v}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2}$

We have

$$\frac{\partial}{\partial x}f(u,v) = f_1(u,v)\frac{\partial u}{\partial x} + f_2(u,v)\frac{\partial v}{\partial x} \qquad \frac{\partial}{\partial y}f(u,v) = f_1(u,v)\frac{\partial u}{\partial y} + f_2(u,v)\frac{\partial v}{\partial y}$$

$$\frac{\partial^2}{\partial x^2}f(u,v) = f_{11}\left(\frac{\partial u}{\partial x}\right)^2 + f_{12}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + f_1\frac{\partial^2 u}{\partial x^2} + f_{21}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + f_{22}\left(\frac{\partial v}{\partial x}\right)^2 + f_2\frac{\partial^2 v}{\partial x^2}$$

$$\frac{\partial^2}{\partial y^2}f(u,v) = f_{11}\left(\frac{\partial u}{\partial y}\right)^2 + f_{12}\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + f_1\frac{\partial^2 u}{\partial y^2} + f_{21}\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + f_{22}\left(\frac{\partial v}{\partial y}\right)^2 + f_2\frac{\partial^2 v}{\partial y^2}.$$

Noting that $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \frac{1}{(x^2+y^2)^2} = \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 \qquad \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = 0$

we have $\frac{\partial^2}{\partial x^2}f(u,v) + \frac{\partial^2}{\partial y^2}f(u,v) = f_{11}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] + f_{22}\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] + 2f_{12}\left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right]$

$$+f_1\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] + f_2\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right] = f_1\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] + f_2\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right]$$

because we are given that f is harmonic, that is, $f_{11}(u,v) + f_{22}(u,v) = 0$.

Thus, $\frac{\partial^2}{\partial x^2}f(u,v) + \frac{\partial^2}{\partial y^2}f(u,v) = 0 \qquad \therefore\ f\left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$ is harmonic for $(x,y) \neq (0,0)$

2. Find $\frac{\partial^3}{\partial x\,\partial y^2}f(2x+3y, xy)$ in terms of partial derivatives of the function f.

$$\frac{\partial^3}{\partial x\,\partial y^2}f(2x+3y,xy) = \frac{\partial^2}{\partial x\,\partial y}(3f_1 + xf_2) = \frac{\partial}{\partial x}(9f_{11} + 3xf_{12} + 3xf_{21} + x^2 f_{22})$$

$$= \frac{\partial}{\partial x}(9f_{11} + 6xf_{12} + x^2 f_{22}) = 18f_{111} + 9yf_{112} + 6f_{12} + 12xf_{121} + 6xyf_{122} + 2xf_{22} + 2x^2 f_{221} + x^2 y f_{222}$$

$$= 18f_{111} + (12x + 9y)f_{112} + (6xy + 2x^2)f_{122} + x^2 y f_{222} + 6f_{12} + 2xf_{22}$$

where all partials are evaluated at $(2x+3y, xy)$.

<u>Question 4:</u> (a) Show that $F(x,y) = -F(y,x)$ for all (x,y).

(b) Show that $F_1(x,y) = -F_2(y,x)$ and $F_{12}(x,y) = -F_{21}(y,x)$ for $(x,y) \neq (0,0)$.

(c) Show that $F_1(0,y) = -2y$ for all y and, hence, that $F_{12}(0,0) = -2$.

(d) Deduce that $F_2(x,0) = 2x$ and $F_{21}(0,0) = 2$.

$$F(x,y) = \begin{cases} \dfrac{2xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

a) For $(x,y) \neq (0,0)$,

$$F(x,y) = \frac{2xy(x^2-y^2)}{x^2+y^2} = -\frac{2xy(y^2-x^2)}{x^2+y^2} = -F(y,x)$$

Since $0 = -0$, this holds for $(x, y) = (0,0)$ also.

b) For $(x, y) \neq (0,0)$,

$$F_1(x,y) = \frac{\partial}{\partial x}F(x,y) = -\frac{\partial}{\partial x}F(y,x) = -F_2(y,x)$$

$$F_{12}(x,y) = \frac{\partial}{\partial y}F_1(x,y) = -\frac{\partial}{\partial y}F_2(y,x) = -F_{21}(y,x)$$

c) If $(x, y) \neq (0,0)$,

then $F_1(x,y) = \frac{2y(x^2-y^2)}{x^2+y^2} + 2xy\frac{\partial}{\partial x}\frac{x^2-y^2}{x^2+y^2}$ Thus $F_1(0,y) = -2y + 0 = -2y$ for $y \neq 0$.

This result holds for $y = 0$ also, since $F_1(0,0) = \lim_{h\to 0}(0-0)/h = 0$..

d) By (b) and (c), $F_2(x,0) = -F_1(0,x) = 2x$, and $F_{21}(0,0) = 2$.

<u>Question 5:</u> (a) Use question 4 (b) to find $F_{12}(x,x)$ for $x \neq 0$.

(b) Is $F_{12}(x,y)$ continuous at $(0,0)$? Why?

a) Since $F_{12}(x,y) = -F_{21}(y,x)$ for $(x,y) \neq (0,0)$, we have $F_{12}(x,x) = -F_{21}(x,x)$ for $x \neq 0$.

However, all partial derivatives of the rational function F are continuous except possibly at the origin. Thus $F_{12}(x,x) = F_{21}(x,x)$ for $x \neq 0$. Therefore, $F_{12}(x,x) = 0$ for $x \neq 0$.

b) F_{12} cannot be continuous at $(0,0)$ because its value there (which is -2) differs from the value of $F_{21}(0,0)$ (which is 2). Alternatively, $F_{12}(0,0)$ is not the limit of $F_{12}(x,x)$ as $x \to 0$.

The Following Questions Are Obtained From PAST EXAMS

<u>Question 1:</u> Given that the function $z = f\left(\frac{x}{y}, \frac{y}{x}\right)$ has continuous partial derivatives in the neighborhood of the point $(x,y) = (1,-1)$ with

$$\begin{aligned} &f_1(-1,-1) = 2 = f_2(1,-1), \quad f_1(1,-1) = -2 = f_2(-1,-1) \\ &f_{11}(-1,-1) = 6 = f_{12}(1,-1), \quad f_{11}(1,-1) = 8 = f_{21}(1,-1) \\ &f_{12}(-1,-1) = -5 = f_{21}(-1,-1), \quad f_{22}(1,-1) = 1 = f_{22}(-1,-1) \end{aligned}$$

Find $z_{12} = \frac{\partial^2 z}{\partial y\, \partial x}$ at the point $(x,y) = (1,-1)$. PE. SPRING2005

Solution: Let $s = \frac{x}{y},\ t = \frac{y}{x}$ then $(x,y) = (1,-1)$ gives $(s,t) = (-1,-1)$ and $z = f(x/y, y/x) = f(s,t)$. Now,

$$z_1 = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial z}{\partial t}\frac{\partial t}{\partial x} = \frac{1}{y}f_1(s,t) - \frac{y}{x^2}f_2(s,t).$$

Similarly,

$$\begin{aligned} z_{12} = &\ \frac{\partial}{\partial y}\left[\frac{1}{y}f_1(s,t) - \frac{y}{x^2}f_2(s,t)\right] \\ = &\ \frac{-1}{y^2}f_1(s,t) + \frac{1}{y}\left[f_{11}(s,t)\frac{\partial s}{\partial y} + f_{12}(s,t)\frac{\partial t}{\partial y}\right] - \frac{1}{x^2}f_2(s,t) - \frac{y}{x^2}\left[f_{21}(s,t)\frac{\partial s}{\partial y} + f_{22}(s,t)\frac{\partial t}{\partial y}\right] \\ = &\ \frac{-1}{y^2}f_1(s,t) + \frac{1}{y}\left[f_{11}(s,t)\frac{-x}{y^2} + f_{12}(s,t)\frac{1}{x}\right] - \frac{1}{x^2}f_2(s,t) - \frac{y}{x^2}\left[f_{21}(s,t)\frac{-x}{y^2} + f_{22}(s,t)\frac{1}{x}\right] \end{aligned}$$

Put $x = 1, y = -1$ to get

$$z_{12}(1,-1) = -f_1(-1,-1) - [-f_{11}(-1,-1) + f_{12}(-1,-1)] - f_2(-1,-1) + [-f_{21}(-1,-1) + f_{22}(-1,-1)] = 17$$

Section 3.6 Linear Approximations

The Following Questions Are Obtained From **Calculus - A Complete Course 7th ed - R. Adams, C. Essex**

Question 1: In following questions, use suitable linearization to find approximate values for the given functions at the points indicated.

1. $f(x,y) = \frac{24}{x^2+xy+y^2}$ at $(2.1, 1.8)$

$$f_1(x,y) = \frac{-24(2x+y)}{(x^2+xy+y^2)^2}, \qquad f_2(x,y) = \frac{-24(x+2y)}{(x^2+xy+y^2)^2}$$

$$f(2,2) = 2,\ f_1(2,2) = -1,\ f_2(2,2) = -1$$

$$f(2.1,1.8) \approx f(2,2) + 0.1f_1(2,2) - 0.2f_2(2,2) = 2 - 0.1 + 0.2 = 2.1$$

2. $f(x,y) = xe^{y+x^2}$ at $(2.05, -3.92)$

$f(x,y) = xe^{y+x^2}$ $\qquad f(2,-4) = 2$ $\qquad f_1(x,y) = e^{y+x^2}(1+2x^2)$

$f_1(2,-4) = 9$ $\qquad f_2(x,y) = xe^{y+x^2}$ $\qquad f_2(2,-4) = 2$

$$f(2.05,-3.92) \approx f(2,-4) + 0.05f_1(2,-4) + 0.08f_2(2,-4) = 2 + 0.45 + 0.16 = 2.61$$

Question 2: write the differential of the given function and use it to estimate the value of the function at the given point by starting with a known value at a nearby point.

$$u = x\sin(x+y), \ \text{ at } \ x = \frac{\pi}{2} + \frac{1}{20}, y = \frac{\pi}{2} - \frac{1}{30}$$

We have du $= (\sin(x+y) + x\cos(x+y))dx + x\cos(x+y)dy$.

If $x = y = \pi/2, dx = 1/20$, and $dy = 1/30$, then $u = 0$ and $du = \left(0 - \frac{\pi}{2}\right)\left(\frac{1}{20}\right) + \left(-\frac{\pi}{2}\right)\left(-\frac{1}{30}\right) = -\frac{\pi}{120}$.

Therefore, at $x = \frac{\pi}{2} + \frac{1}{20}, y = \frac{\pi}{2} - \frac{1}{30}$, we have $u \approx -\frac{\pi}{120}$.

Question 3: By approximately what percentage will the value of $w = \frac{x^2y^3}{z^4}$ " increases by 2%, and z increases by 3%?

$$w = \frac{x^2y^3}{z^4} \quad \frac{\partial w}{\partial x} = \frac{2xy^3}{z^4} = \frac{2w}{x} \qquad \frac{\partial w}{\partial y} = \frac{3x^2y^2}{z^4} = \frac{3w}{y} \quad \frac{\partial w}{\partial z} = -\frac{4x^2y^3}{z^5} = -\frac{4w}{x}.$$

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz \qquad \frac{dw}{w} = 2\frac{dx}{x} + 3\frac{dy}{y} - 4\frac{dz}{z}.$$

Since x increases by 1%, then $\frac{dx}{x} = \frac{1}{100}$. Similarly, $\frac{dy}{y} = \frac{2}{100}$ and $\frac{dz}{z} = \frac{3}{100}$. Therefore

$$\frac{\Delta w}{w} \approx \frac{dw}{w} = \frac{2 + 6 - 12}{100} = -\frac{4}{100}$$

and w decreases by about 4%.

The Following Questions Are Obtained From PAST EXAMS

Question 1: Let $z = f(x,y) = \sqrt{20 - x^2 - 7y^2}$. Use Tangent Plane Approximation to approximate $f(1.95, 1.08)$.

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Solution: We have

$$f_x(x,y) = \frac{-x}{\sqrt{20 - x^2 - 7y^2}}, \ f_y(x,y) = \frac{-7y}{\sqrt{20 - x^2 - 7y^2}}$$

So $f_x(2,1) = -2/3$ and $f_y(2,1) = -7/3$. Therefore

$$f(1.95, 1.08) \approx f(2,1) + f_z(2,1)(1.95 - 2) + f_y(2,1)(1.08 - 1) = 3 + (-2/3)(-0.05) + (-7/3)(0.08)$$

$$= 3 + \frac{0.1 - 0.56}{3}$$

Question 2: Find $sin\left(0.002\left(\frac{\pi}{2} - 0.01\right)^2\right)$ appraximately using the tangent plane (differential) approximation at a suitable point.

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Hint: Consider the function $f(x, y) = \sin\left(x\left(\frac{\pi}{2} - y\right)^2\right)$.

Let $f(x, y) = \sin\left(x\left(\frac{\pi}{2} - y\right)^2\right)$. Then $sin\left(0.002\left(\frac{\pi}{2} - 0.01\right)^2\right) = f(0.002, 0.01)$

$$f_x = \left(\frac{\pi}{2} - y\right)^2 \cos\left(x\left(\frac{\pi}{2} \cdot y\right)^2\right) \Rightarrow f_x(0,0) = \left(\frac{\pi}{2}\right)^2$$

$$f_y = -2x\left(\frac{\pi}{2} \cdot y\right)\cos\left(x\left(\frac{\pi}{2} \cdot y\right)^2\right) \Rightarrow f_y(0,0) = 0$$

$$f(0.002, 0.01) \approx \overbrace{f(0,0)}^{=0} + f_x(0,0)(0.002) + f_y(0,0)(0.01) = 0 + \left(\frac{\pi}{2}\right)^2 (0.002) + 0 = \left(\frac{\pi}{2}\right)^2 (0.002)$$

Section 3.7 Gradients and Directional Derivatives

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: find:

(a) the gradient of the given function at the point indicated,

(b) an equation of the plane tangent to the graph of the given function at the point whose x and y coordinates are given, and

(c) an equation of the straight-line tangent, at the given point, to the level curve of the given function passing through that point.

$$f(x, y) = e^{xy} \text{ at } (2,0)$$

$f(x, y) = e^{xy}$, $\nabla f = ye^{xy}\mathbf{i} + xe^{xy}\mathbf{j}$, $\nabla f(2,0) = 2\mathbf{j}$. Tangent plane to $z = f(x, y)$ at $(2,0,1)$ has equation $2y = z - 1$, or $2y - z = -1$.

Tangent line to $f(x, y) = 1$ at $(2,0)$ has equation $y = 0$.

Question 2: find an equation of the tangent plane to the level surface of the given function that passes through the given point.

$f(x,y,z) = \cos(x + 2y + 3z)$ at $\left(\frac{\pi}{2}, \pi, \pi\right)$

$f\left(\frac{\pi}{2}, \pi, \pi\right) = \cos\frac{11\pi}{2} = 0.$ $\qquad \nabla f(x,y,z) = -\sin(x + 2y + 3z)(1 + 2j + 3\mathbf{k}),$

$\nabla f\left(\frac{\pi}{2}, \pi, \pi\right) = -\sin\frac{11\pi}{2}(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

Tangent plane to $f(x,y,z) = 0$ at $\left(\frac{\pi}{2}, \pi, \pi\right)$ has equation $x - \frac{\pi}{2} + 2(y - \pi) + 3(z - \pi) = 0$, at the given point in the specified direction.

Question 3: find the rate of change of the given function at the given point in the specified direction.

$f(x,y) = 3x - 4y$ at $(0,2)$ in the direction of the vector $-2\mathbf{i}$

$f(x,y) = 3x - 4y,$ $\qquad \nabla f(0,2) = \nabla f(x,y) = 3\mathbf{i} - 4\mathbf{j}$

Question 4: In what directions at the point $(2,0)$ does the function $f(x,y) = xy$ have rate of change -1? Are there directions in which the rate is -3? How about -2?

$f(x,y) = xy$, $\nabla f(x,y) = y\mathbf{i} + x\mathbf{j}$, $\nabla f(2,0) = 2\mathbf{j}$

Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ be a unit vector. Thus $u_1^2 + u_2^2 = 1$. We have $-1 = D_{\mathbf{u}}f(2,0)\mathbf{u} \cdot \nabla f(2,0) = 2u_2$

if $u_2 = -\frac{1}{2}$, and therefore $u_1 = \pm\frac{\sqrt{3}}{2}$. At $(2,0)$, f has rate of change -1 in the directions $\pm\frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$.

If $-3 = D_{\mathbf{u}}f(2,0) = 2u_2$, then $u_2 = -\frac{3}{2}$. This is not possible for a unit vector $\mathbf{u}$, so there is no direction at $(2,0)$ in which f changes at rate -3.

If $-2 = D_{\mathbf{u}}f(2,0) = 2u_2$, then $u_2 = -1$ and $u_1 = 0$. At $(2,0)$, f has rate of change -2 in the direction $-$j.

Question 5: In what directions at the point (a,b,c) does the function $f(x,y,z) = x^2 + y^2 - z^2$ increase at half of its maximal rate at that point?

$f(x,y,z) = x^2 + y^2 - z^2$ $\qquad \nabla f(a,b,c) = 2a\mathbf{i} + 2b\mathbf{j} - 2c\mathbf{k}.$

The maximum rate of change of f at (a,b,c) is in the direction of rate of change of f at (a,b,c) is in the direction of $\nabla f(a,b,c)$, and is equal to $|\nabla f(a,b,c)|$.

Let u be a unit vector making an angle θ with $\nabla f(a,b,c)$. The rate of change of f at (a,b,c) in the direction of u will be half of the maximum rate of change of f at that point provided

$$\frac{1}{2}|\nabla f(a,b,c)| = \mathbf{u}\cdot\nabla f(a,b,c) = |\nabla f(a,b,c)|\cos\,\theta,$$

that is, if $\cos\,\theta = \frac{1}{2}$, which means $\theta = 60^\circ$. At (a,b,c), f increases at half its maximal rate in all directions making 60° angles with the direction $a\mathbf{i} + b\mathbf{j} - c\mathbf{k}$.

Question 6: Find $\nabla f(a,b)$ for the differentiable function $f(x,y)$ given the directional derivatives

$$D_{(\mathrm{i+j})/\sqrt{2}} f(a,b) = 3\sqrt{2} \text{ and } D_{(3\mathrm{i}-4\mathrm{j})/5} f(a,b) = 5.$$

Let $\boldsymbol{\nabla} f(a,b) = u\mathbf{i} + v\mathbf{j}$. Then

$$3\sqrt{2} = D_{(\mathrm{i+j})/\sqrt{2}} f(a,b) = \frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}\cdot(u\mathbf{i}+v\mathbf{j}) = \frac{u+v}{\sqrt{2}}$$

$$5 = D_{(3\mathrm{i}-4\mathrm{j})/5} f(a,b) = \frac{3\mathbf{i}-4\mathbf{j}}{5}\cdot(u\mathbf{i}+v\mathbf{j}) = \frac{3u-4v}{5}$$

Thus, $u + v = 6$ and $3u - 4v = 25$. This system has solution $u = 7, v = -1$. Thus, $\nabla f(a,b) = 7\mathbf{i} - \mathbf{j}$.

Question 7: Find an equation of the curve in the xy-plane that passes through the point (1,1) and intersects all level curves of the function $f(x,y) = x^4 + y^2$ at right angles.

Let the curve be $y = g(x)$. At (x,y) this curve has normal $\nabla(g(x) - y) = g'(x)\mathbf{i} - \mathbf{j}$

A curve of the family $x^4 + y^2 = C$ has normal $\nabla(x^4 + y^2) = 4x^3\mathbf{i} + 2y\mathbf{j}$

These curves will intersect at right angles if their normals are perpendicular: Thus, we require that

$$0 = 4x^3 g'(x) - 2y = 4x^3 g'(x) - 2g(x) \text{ or, equivalently, } \frac{g'(x)}{g(x)} = \frac{1}{2x^3}$$

Integration gives $\ln\,|g(x)| = -\frac{1}{4x^2} + \ln\,|C|$, or $g(x) = Ce^{-(1/4x^2)}$

Since the curve passes through (1,1), we must have $1 = g(1) = Ce^{-1/4}$, so $C = e^{1/4}$.

The required curve is $y = e^{(1/4)-(1/4x^2)}$.

Question 8: Find a vector tangent to the curve of intersection of the two cylinders $x^2 + y^2 = 2$ and $y^2 + z^2 = 2$ at the point $(1, -1, 1)$.

At $(1, -1, 1)$ the surface $x^2 + y^2 = 2$ has normal

$$\mathbf{n}_1 = \nabla(x^2 + y^2)|_{(1,-1,1)} = 2\mathbf{i} - 2\mathbf{j}$$

and $y^2 + z^2 = 2$ has normal

$$\mathbf{n}_2 = \nabla(y^2 + z^2)|_{(1,-1,1)} = -2\mathrm{j} + 2\mathrm{k}$$

A vector tangent to the curve of intersection of the two surfaces at $(1, -1, 1)$ must be perpendicular to both these normals. Since

$$(\mathbf{i} - \mathbf{j}) \times (-\mathbf{j} + \mathbf{k}) = -(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

the vector $\mathbf{i} + \mathbf{j} + \mathbf{k}$, or any scalar multiple of this vector, is tangent to the curve at the given point.

Question 9: Let $f(x, y) = \begin{cases} \frac{\sin(xy)}{\sqrt{x^2+y^2}}, & \text{if } (x, y) \neq (0,0) \\ 0, & \text{if } (x, y) = (0,0) \end{cases}$.

(a) Calculate $\nabla f(0,0)$.

(b) Use the definition of directional derivative to calculate $D_{\mathbf{u}} f(0,0)$, where $\mathbf{u} = (\mathbf{i} + \mathbf{j})/\sqrt{2}$.

(c) Is $f(x, y)$ differentiable at $(0,0)$? Why?

a) $f_1(0,0) = \lim_{h \to 0} \frac{0-0}{h} = 0 = f_2(0,0)$. Thus $\nabla f(0,0) = 0$.

b) If $\mathbf{u} = (\mathbf{i} + \mathbf{j})/\sqrt{2}$, then $D_{\mathbf{u}} f(0,0) = \lim_{h \to 0^+} \frac{1}{h} \frac{\sin(h^2/2)}{\sqrt{h^2}} = \frac{1}{2}$

c) f cannot be differentiable at $(0,0)$; if it were, then the directional derivative obtained in part (b) would have been $\mathrm{u} \cdot \nabla f(0,0) = 0$.

The Following Questions Are Obtained From PAST EXAMS

Question 1: Show that for any tangent plane to the surface $\sqrt{x}+\sqrt{y}+\sqrt{z}=\sqrt{a}, (a>0)$ the sum of x, y, and z axis intercepts is constant.

(x_0 is called x intercept if the graph intersects x axis at the point $(x_0, 0,0)$.) PE. SPRING2005

Solution: Let $f(x,y,z)=\sqrt{x}+\sqrt{y}+\sqrt{z}-\sqrt{a}$ then the surface, S, is given by $f(x,y,z)=0$.

Let the point $P_0(x_0,y_0,z_0)\in S$ be the point of tangency. Then

$$\vec{\nabla}f(x,y,z)=\frac{1}{2\sqrt{x}}\vec{\imath}+\frac{1}{2\sqrt{y}}\vec{\jmath}+\frac{1}{2\sqrt{z}}\vec{k}$$

and hence

$$\vec{n}=\vec{\nabla}f(x_0,y_0,z_0)=\frac{1}{2\sqrt{x_0}}\vec{\imath}+\frac{1}{2\sqrt{y_0}}\vec{\jmath}+\frac{1}{2\sqrt{z_0}}\vec{k}$$

is a normal vector to the tangent plane. Therefore, the equation of the tangent plane is obtained as

$$\frac{1}{2\sqrt{x_0}}(x-x_0)+\frac{1}{2\sqrt{y_0}}(y-y_0)+\frac{1}{2\sqrt{z_0}}(z-z_0)=0$$

Now,

x-intercept is obtained, for $y=z=0$, as: $\left(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0}\right)\sqrt{x_0}$

y-intercept is obtained, for $x=z=0$, as: $\left(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0}\right)\sqrt{y_0}$

z-intercept is obtained, for $x=y=0$, as: $\left(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0}\right)\sqrt{z_0}$. Therefore, the sum of intercepts is:

$$\left(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0}\right)^2=(\sqrt{a})^2=a$$

which is a constant.

Question 2: Consider the function $f(x,y)=\frac{x^3}{x^2+y^2}$ if $(x,y)\neq(0,0)$, $f(0,0)=0$. Show that; PE. SPRING2005

a) all directional derivatives of f **exist** at $(0,0)$, using limit definition of directionsl derivative
Solution: Let $\vec{u}=a\vec{\imath}+b\vec{\jmath}$ with $a^2+b^2=1$. Then

$$D_{\vec{u}}f(0,0)=\lim_{h\to 0}\frac{f(ah,bh)-f(0,0)}{h}=\lim_{h\to 0}\frac{a^3h^3/[h^2(a^2+b^2)]-0}{h}=a^3$$

which completes the proof.

b) find the set of all vectors $\vec{v}$ for which the equation $D_{\vec{v}}f(0,0) = \nabla f(0,0) \cdot \vec{v}$ is satisfied.

Solution: Let $\vec{v} = v_1\vec{\imath} + v_2\vec{\jmath}$. Then, by (b) we have $D_{\vec{v}}f(0,0) = v_1^3$ and $\vec{\nabla}f(0,0) = \vec{\imath}$.
Therefore, $\vec{\nabla}f(0,0) \cdot \vec{v} = v_1$. Now, the required condition is $v_1 = v_1^3$ which gives us $v_1 = 0, v_1 = \pm 1$. For, $v_1 = 0$ we have $v_2 = \pm 1$ and for $v_1 = \pm 1$ we have $v_2 = 0$. Thus, the directions are $\pm\vec{\imath}$ and $\pm\vec{\jmath}$.

Question 3: Find a vector $\vec{v}$ in the xy −plane such that $h(x,y) = 2x^2 - 6xy + 3y^2$ increases most rapidly when (x,y) moves along $\vec{v}$ starting at $(1,1)$. PE. SPRING2006

$\text{Max}_{\vec{v}}\,(D_{\vec{v}}h(x,y)) = \text{Max}_{\vec{v}}\,(\nabla h \cdot \vec{v}) = \text{Max}_{\vec{v}}\,(\|\nabla h\| \cos\,\theta) = \|\nabla h\|\,\text{Max}_{\hat{v}}\,(\cos\,\theta) = \|\nabla h\|$
Hence to get Maximum value of $D_{\vec{v}}h(x,y)$ (Which is $\|\nabla h\|$), we must take $\hat{v}$ so that $\cos\,\theta = 1$ i.e, $\theta = 0$ i.e.,
$\vec{v} = \frac{\nabla h}{\|\nabla h\|}$

$h_x = 4x - 6y \quad \rightarrow h_x(1,1) = -2$ and $h_y = -6x + 6y \quad \rightarrow h_y(1,1) = 0$

Thus, $\vec{v} = \frac{\nabla h(1,1)}{\|\nabla h(1,1)\|} = \frac{\langle -2,0\rangle}{2} = \langle -1,0\rangle$

Section 3.8 Implicit Functions

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: In following questions, calculate the indicated derivative from the given equation(s). What condition on the variables will guarantee the existence of a solution that has the indicated derivative? Assume that any general functions F, G, and H have continuous first partial derivatives.

1. $\frac{\partial x}{\partial y}$ if $xy^3 = y - z$

$$\frac{\partial x}{\partial y} = \frac{1 - 3xy^2}{y^3}. \qquad xy^3 = y - z:\ x = x(y,z) \qquad y^3\frac{\partial x}{\partial y} + 3xy^2 = 1$$

The given equation has a solution $x = x(y,z)$ with this partial derivative near any point where $y \neq 0$.

2. $\frac{\partial x}{\partial w}$ if $x^2y^2 + y^2z^2 + z^2t^2 + t^2w^2 - xw = 0$

$x^2y^2 + y^2z^2 + z^2t^2 + t^2w^2 - xw = 0$: $\quad x = x(y, z, t, w)$

$2xy^2\frac{\partial x}{\partial w} + 2t^2w - w\frac{\partial x}{\partial w} - x = 0$ $\quad \frac{\partial x}{\partial w} = \frac{x - 2t^2w}{2xy^2 - w}$.

The given equation has a solution with this derivative wherever $w \neq 2xy^2$.

3. $\frac{dy}{dx}$ if $F(x, y, x^2 - y^2) = 0$

$F(x, y, x^2 - y^2) = 0$: $y = y(x)$ $\quad F_1 + F_2\frac{dy}{dx} + F_3\left(2x - 2y\frac{dy}{dx}\right) = 0$

$$\frac{dy}{dx} = \frac{F_1(x, y, x^2 - y^2) + 2xF_3(x, y, x^2 - y^2)}{2yF_3(x, y, x^2 - y^2) - F_2(x, y, x^2 - y^2)}$$

The given equation has a solution with this derivative near any point where

$$F_2(x, y, x^2 - y^2) \neq 2yF_3(x, y, x^2 - y^2)$$

4. $\left(\frac{\partial x}{\partial y}\right)_z$ if $x^2 + y^2 + z^2 + w^2 = 1$, and $x + 2y + 3z + 4w = 2$

$\begin{cases} x^2 + y^2 + z^2 + w^2 = 1 \\ x + 2y + 3z + 4w = 2 \end{cases} \Rightarrow \begin{cases} x = x(y, z) \\ w = w(y, z) \end{cases}$

$2x\frac{\partial x}{\partial y} + 2y + 2w\frac{\partial w}{\partial y} = 0 \times 2$ $\quad \frac{\partial x}{\partial y} + 2 + 4\frac{\partial w}{\partial y} = 0 \times w$

$(4x - w)\frac{\partial x}{\partial y} + 4y - 2w = 0$ $\quad \left(\frac{\partial x}{\partial y}\right)_z = \frac{210 - 4y}{4x - 10}$

The given equations have a solution of the indicated form with this derivative near any point where $w \neq 4x$.

Section 4.1 Extreme Values

The Following Questions Are Obtained From **Calculus - A Complete Course 7th ed - R. Adams, C. Essex**

Question 1: In following questions, find and classify the critical points of the given functions.

(a) $f(x, y) = x^2 + 2y^2 - 4x + 4y$

$f_1(x, y) = 2x - 4 = 0$ if $x = 2$ $\quad f_2(x, y) = 4y + 4 = 0$ if $y = -1$.

Critical point is $(2, -1)$. Since $f(x, y) \to \infty$ as $x^2 + y^2 \to \infty$, f has a local (and absolute) minimum value at that critical point.

(b) $f(x,y) = x^3 + y^3 - 3xy$

$f_1(x,y) = 3(x^2 - y),\ f_2(x,y) = 3(y^2 - x)$

For critical points: $x^2 = y$ and $y^2 = x$. Thus $x^4 - x = 0$, that is, $x(x-1)(x^2+x+1) = 0$. Thus $x = 0$ or $x = 1$. The critical points are (0,0) and (1,1). We have

$$A = f_{11}(x,y) = 6x \qquad B = f_{12}(x,y) = -3 \qquad C = f_{22}(x,y) = 6y$$

At (0,0): $A = C = 0, B = -3$. Thus $AC < B^2$, and $(0,0)$ is a saddle point of f.

At (1,1): $A = C = 6, B = -3$, so $AC > B^2$. Thus f has a local minimum value at (1,1).

(c) $f(x,y) = \cos(x+y)$

$f(x,y) = \cos(x+y),\ f_1 = -\sin(x+y) = f_2$.

All points on the lines $x + y = n\pi$ (n is an integer) are critical points. If n is even, $f = 1$ at such points; if n is odd, $f = -1$ there. Since $-1 \leq f(x,y) \leq 1$ at all points in $\mathbb{R}^2$, f must have local and absolute maximum values at points $x + y = n\pi$ with n even, and local and absolute minimum values at such points with n odd.

(d) $f(x,y) = x\sin\ y$

For critical points we have

$$f_1 = \sin\ y = 0,\ f_2 = x\cos\ y = 0$$

Since $\sin\ y$ and $\cos\ y$ cannot vanish at the same point, the only critical points correspond to $x = 0$ and $\sin\ y = 0$. They are $(0, n\pi)$, for all integers n. All are saddle points.

(e) $f(x,y) = x^2ye^{-(x^2+y^2)}$

$$f_1(x,y) = 2xy(1-x^2)e^{-(x^2+y^2)} \qquad f_2(x,y) = x^2(1-2y^2)e^{-(x^2+y^2)}$$

$$A = f_{11}(x,y) = 2y(1-5x^2+2x^4)e^{-(x^2+y^2)} \qquad B = f_{12}(x,y) = 2x(1-x^2)(1-2y^2)e^{-(x^2+y^2)}$$

$$C = f_{22}(x,y) = 2x^2y(2y^2-3)e^{-(x^2+y^2)}$$

For critical points: $xy(1-x^2) = 0 \qquad x^2(1-2y^2) = 0$

The critical points are $(0,y)$ for all y, $(\pm1, 1/\sqrt{2})$, and $(\pm1, -1/\sqrt{2})$.

Evidently, $f(0,y) = 0$. Also $f(x,y) > 0$ if $y > 0$ and $x \neq 0$, and $f(x,y) < 0$ if $y < 0$ and $x \neq 0$. Thus f has a local minimum at $(0,y)$ if $y > 0$, and a local maximum if $y < 0$. The origin is a saddle point.

At $(\pm 1, 1/\sqrt{2})$: $A = C = -2\sqrt{2}e^{-3/2}, B = 0$, and so $AC > B^2$. Thus f has local maximum values at these two points.

At $(\pm 1, -1/\sqrt{2})$: $A = C = 2\sqrt{2}e^{-3/2}, B = 0$, and so $AC > B^2$. Thus f has local minimum values at these two points.

Since $f(x,y) \to 0$ as $x^2 + y^2 \to \infty$, the value $f(\pm 1, 1/\sqrt{2}) = e^{-3/2}/\sqrt{2}$ is the absolute maximum value for f, and the value $f(\pm 1, -1/\sqrt{2}) = -e^{-3/2}/\sqrt{2}$ is the absolute minimum value.

(f) $f(x,y) = xe^{-x^3+y^3}$

$f_1(x,y) = (1-3x^3)e^{-x^3+y^3}$ $\qquad$ $f_2(x,y) = 3xy^2e^{-x^3+y^3}$

$A = f_{11}(x,y) = 3x^2(3x^3-4)e^{-x^3+y^3}$ $\qquad$ $B = f_{12}(x,y) = -3y^2(3x^3-1)e^{-x^3+y^3}$

$C = f_{22}(x,y) = 3xy(3y^3+2)e^{-x^3+y^3}$

For critical points: $3x^3 = 1$ and $3xy^2 = 0$. The only critical point is $(3^{-1/3}, 0)$. At that point we have $B = C = 0$ so the second derivative test is inconclusive.

However, note that $f(x,y) = f(x,0)e^{y^3}$, and e^{y^3} has an inflection point at $y = 0$. Therefore $f(x,y)$ has neither a maximum nor a minimum value at $(3^{-1/3}, 0)$, so has a saddle point there.

(g) $f(x,y,z) = xy + x^2z - x^2 - y - z^2$

$f_1(x,y,z) = y + 2x(z-1)$ $\qquad$ $f_2(x,y,z) = x - 1$ $\qquad$ $f_3(x,y,z) = x^2 - 2z.$

The only critical point is $\left(1,1,\frac{1}{2}\right)$. We have

$$D = f\left(1+h, 1+k, \frac{1}{2}+m\right) - f\left(1,1,\frac{1}{2}\right) = 1 + h + k + hk + \frac{1+2h+h^2}{2} + (1+2h+h^2)m$$

$$= -1 - 2h - h^2 - 1 - k - \frac{1}{4} - m - m^2 - \left(-\frac{3}{4}\right) = \frac{h^2(2m-1) + 2h(k+2m) - 2m^2}{2}$$

If $m = h$ and $k = 0$, then $D = \dfrac{h^2(1+2h)}{2} > 0$ for small $|h|$.

If $h = k = 0$, then $D = -m^2 < 0$ for $m \neq 0$. Thus f has a saddle point at $\left(1,1,\frac{1}{2}\right)$.

Question 2: Find the maximum and minimum values of $f(x,y) = xye^{-x^2-y^4}$

$$f_1(x,y) = y(1-2x^2)e^{-(x^2+y^4)} \qquad\qquad f_2(x,y) = x(1-4y^4)e^{-(x^2+y^4)}$$

For critical points $y(1-2x^2) = 0$ and $x(1-4y^4) = 0$. The critical points are

$$(0,0),\ \left(\pm\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right),\ \left(\pm\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$$

We have $f(0,0) = 0$ $\qquad f\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right) = \frac{1}{2}e^{-\frac{3}{4}} > 0$

$$f\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right) = f\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}e^{-3/4} < 0$$

Since $f(x,y) \to 0$ as $x^2+y^2 \to \infty$, the maximum and minimum values of f are $\frac{1}{2}e^{-3/4}$ and $-\frac{1}{2}e^{-3/4}$ respectively.

Question 3: The material used to make the bottom of a rectangular box is twice as expensive per unit area as the material used to make the top or side walls. Find the dimensions of the box of given volume V for which the cost of materials is minimum.

Let the length, width, and height of the box be x, y, and z, respectively. Then $V = xyz$. If the top and side walls cost $\$k$ per unit area, then the total cost of materials for the box is

$$C = 2kxy + kxy + 2kxz + 2kyz = k\left[3xy + 2(x+y)\frac{V}{xy}\right] = k\left[3xy + \frac{2V}{x} + \frac{2V}{y}\right]$$

where $x > 0$ and $y > 0$. Since $C \to \infty$ as $x \to 0+$ or $y \to 0+$ or $x^2+y^2 \to \infty$, C must have a minimum value at a critical point in the first quadrant. For CP:

$$0 = \frac{\partial C}{\partial x} = k\left(3y - \frac{2V}{x^2}\right) \qquad\qquad 0 = \frac{\partial C}{\partial y} = k\left(3x - \frac{2V}{y^2}\right)$$

Thus $3x^2y = 2V = 3xy^2$, so that $x = y = (2V/3)^{1/3}$ and $z = V/(2V/3)^{2/3} = (9V/4)^{1/3}$

Question 4: Find the three positive numbers a, b, and c, whose sum is 30 and for which the expression ab^2c^3 is maximum.

Given that $a > 0, b > 0, c > 0$, and $a + b + c = 30$, we want to maximize

$$P = ab^2c^3 = (30 - b - c)b^2c^3 = 30b^2c^3 - b^3c^3 - b^2c^4$$

Since $P = 0$ if $b = 0$ or $c = 0$ or $b + c = 30$ (i.e., $a = 0$), the maximum value of P will occur at a critical point (b, c) satisfying $b > 0, c > 0$, and $b + c < 30$. For CP:

$$0 = \frac{\partial P}{\partial b} = 60bc^3 - 3b^2c^3 - 2bc^4 = bc^3(60 - 3b - 2c)$$

$$0 = \frac{\partial P}{\partial c} = 90b^2c^2 - 3b^3c^2 - 4b^2c^3 = b^2c^2(90 - 3b - 4c)$$

Hence $9b + 6c = 180 = 6b + 8c$, from which we obtain $3b = 2c = 30$. The three numbers are $b = 10, c = 15$, and $a = 30 - 10 - 15 = 5$.

The Following Questions Are Obtained From PAST EXAMS

Question 1: Find and classify all the critical points of the function $f(x, y) = \frac{1}{3}x^3 + xy^2 - x$.

PE. SPRING2005

Solution: We first find the critical points:

$$f_1(x, y) = x^2 + y^2 - 1 = 0 \qquad (1)$$
$$f_2(x, y) = 2xy = 0 \qquad (2)$$

The equation in (2) gives us $x = 0$ or $y = 0$. If $x = 0$, (1) implies that $y = \pm 1$ and if $y = 0$, (1) implies that $x = \pm 1$. Therefore, we have $P_1(0,1), P_2(0,-1), P_3(1,0), P_4(-1,0)$ as critical points.

$$f_{11}(x, y) = 2x,\ f_{12}(x, y) = f_{21}(x, y) = 2y,\ f_{22}(x, y) = 2x$$
$$\Delta = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ 2y & 2x \end{vmatrix} = 4(x^2 - y^2)$$

and

$\Delta(P_1) = -4 < 0$ so P_1 is a SADDLE point,

$\Delta(P_2) = -4 < 0$ so P_2 is a SADDLE point,

$\Delta(P_3) = 4 > 0$ and $f_{11}(P_3) = 2 > 0$ so P_3 is a LOCAL MIN point,

$\Delta(P_4) = 4 > 0$ and $f_{11}(P_4) = -2 < 0$ so P_4 is a LOCAL MAX point.

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Question 2: a) Find all critical points of $f(x,y)=2x^3-6xy+3y^2$.

$$\begin{cases} f_x = 6x^2-6y=0 \\ f_y=-6x+6y=0 \end{cases} \rightarrow \quad \begin{matrix} x^2=y \\ x=y \end{matrix} \quad \Rightarrow x=x^2 \quad \Rightarrow x=0 \ \text{ or } \ x=1$$

$$\begin{cases} x=0 \Rightarrow y=0 \\ x=1 \Rightarrow y=1 \end{cases} \qquad \rightarrow (0{,}0)\ \&\ (1{,}1) \text{ are the critical points}$$

b) Classify all critical points of $f(x,y)=2x^3-6xy+3y^2$ as local maximum, local minimum or a saddle point using socond derivative test.

$f_{xx}=12x \qquad f_{yy}=6 \qquad f_{xy}=-6$

at $(0{,}0)$; $f_{xx}f_{yy}-f_{xy}^2=-36<0 \qquad \Rightarrow (0{,}0)$ saddle point

at $(1{,}1)$, $f_{xx}f_{yy}-f_{xy}^2=(12)(6)-(-6)^2=72-36>0 \ \ \& \ \ f_{xx}(1{,}1)=12>0$ so $(1{,}1)$ is a local minimum.

Section 4.2 Extreme Values of Functions Defined on Restricted Domains

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: Find the maximum and minimum values of $f(x,y)=xy-y^2$ on the disk $x^2+y^2\le 1$.

$f(x,y)=xy-y^2$ on $D=\{(x,y): x^2+y^2\le 1\}$.

For critical points:

$$0=f_1(x,y)=y, \qquad 0=f_2(x,y)=x-2y.$$

The only CP is $(0{,}0)$, which lies inside D. We have $f(0{,}0)=0$.

The boundary of D is the circle $x=\cos\, t, y=\sin\, t, -\pi\le t\le\pi$. On this circle we have

$$g(t)=f(\cos t,\sin t)=\cos t\sin t-\sin^2 t=\frac{1}{2}[\sin 2t+\cos 2t-1],\ (-\pi\le t\le\pi).$$

$$g(0)=g(2\pi)=0 \qquad g'(t)=\cos 2t-\sin 2t.$$

The critical points of g satisfy $\cos\, 2t=\sin\, 2t$, that is, $\tan\, 2t=1$, so $2t=\pm\frac{\pi}{4}$ or $\pm\frac{5\pi}{4}$, and $t=\pm\frac{\pi}{8}$ or $\pm\frac{5\pi}{8}$. We have

$$g\left(\frac{\pi}{8}\right) = \frac{1}{2\sqrt{2}} - \frac{1}{2} + \frac{1}{2\sqrt{2}} = \frac{1}{\sqrt{2}} - \frac{1}{2} > 0 \qquad g\left(-\frac{\pi}{8}\right) = -\frac{1}{2\sqrt{2}} - \frac{1}{2} + \frac{1}{2\sqrt{2}} = -\frac{1}{2}$$

$$g\left(\frac{5\pi}{8}\right) = -\frac{1}{2\sqrt{2}} - \frac{1}{2} - \frac{1}{2\sqrt{2}} = -\frac{1}{\sqrt{2}} - \frac{1}{2} \qquad g\left(-\frac{5\pi}{8}\right) = \frac{1}{2\sqrt{2}} - \frac{1}{2} - \frac{1}{2\sqrt{2}} = -\frac{1}{2}.$$

Thus, the maximum and minimum values of f on the disk D are $\frac{1}{\sqrt{2}} - \frac{1}{2}$ and $-\frac{1}{\sqrt{2}} - \frac{1}{2}$ respectively.

Question 2: Find the maximum value of $f(x,y) = xy - x^3y^2$ over the square $0 \le x \le 1, 0 \le y \le 1$.

$f_1 = y - 3x^2y^2 = y(1 - 3x^2y)$, $\qquad f_2 = x - 2x^3y = x(1 - 2x^2y)$.

$(0,0)$ is a critical point. Any other critical points must satisfy $3x^2y = 1$ and $2x^2y = 1$, that is, $x^2y = 0$. Therefore, $(0,0)$ is the only critical point, and it is on the boundary of S. We need therefore only consider the values of f on the boundary of S.

On the sides $x = 0$ and $y = 0$ of $S, f(x,y) = 0$.

On the side $x = 1$ we have $f(1,y) = y - y^2 = g(y)$, $(0 \le y \le 1)$. g has maximum value $\frac{1}{4}$ at its critical point $y = \frac{1}{2}$.

On the side $y = 1$ we have $f(x,1) = x - x^3 = h(x)$,$(0 \le x \le 1)$.

h has critical point given by $1 - 3x^2 = 0$; only $x = 1/\sqrt{3}$ is on the side of S.

$h\left(\frac{1}{\sqrt{3}}\right) = \frac{2}{3\sqrt{3}} > \frac{1}{4}$.

On the square $S, f(x,y)$ has minimum value 0 (on the sides $x = 0$ and $y = 0$ and at the corner $(1,1)$ of the square), and maximum value $2/(3\sqrt{3})$ at the point $(1/\sqrt{3}, 1)$. There is a smaller local maximum value at $(1,1/2)$.

Question 3: Find the maximum and minimum values of $f(x,y) = \sin\ x\cos\ y$ on the closed triangular region bounded by the coordinate axes and the line $x + y = 2\pi$.

Since $-1 \le f(x,y) = \sin\ x\cos\ y \le 1$ everywhere, and since $f(\pi/2,0) = 1, f(3\pi/2,0) = -1$, and both $(\pi/2,0)$ and $(3\pi/2,0)$ belong to the triangle bounded by $x = 0, y = 0$ and $x + y = 2\pi$, therefore the maximum and minimum values of f over that triangle must be 1 and - 1 respectively.

Question 4: Find the maximum value of

$$f(x,y) = \sin\ x\sin\ y\sin\ (x + y)$$

over the triangle bounded by the coordinate axes and the line $x + y = \pi$.

$f(x,y) = \sin\ x\sin\ y\sin\ (x + y)$ on the triangle T shown in the figure. Evidently $f(x,y) = 0$ on the boundary of

T, and $f(x,y) > 0$ at all points inside T. Thus, the minimum value of f on T is zero, and the maximum value must occur at an interior critical point. For critical points inside T we must have

$$0 = f_1(x,y) = \cos\ x \sin\ y \sin\,(x+y) + \sin\ x \sin\ y \cos\,(x+y)$$

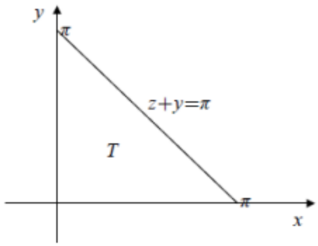
$$0 = f_2(x,y) = \sin\ x \cos\ y \sin\,(x+y) + \sin\ x \sin\ y \cos\,(x+y)$$

Therefore $\cos\ x \sin\ y = \cos y \sin\ x$, which implies $x = y$ for points inside T, and

$\cos x \sin x \sin\ 2x + \sin^2 x \cos 2x = 0$

$2\sin^2 x \cos^2 x + 2\sin^2 x \cos^2 x - \sin^2 x = 0 \qquad 4\cos^2 x = 1$

Thus $\cos\ x = \pm 1/2$, and $x = \pm\pi/3$. The interior critical point is $(\pi/3, \pi/3)$, where f has the value $3\sqrt{3}/8$. This is the maximum value of f on T.



Question 5: The temperature at all points in the disk $x^2 + y^2 \le 1$ is given by

$$T = (x+y)e^{-x^2-y^2}$$

Find the maximum and minimum temperatures at points of the disk.

$T = (x+y)e^{-x^2-y^2}$ on $D = \{(x,y): x^2 + y^2 \le 1\}$.

For critical points:

$$0 = \frac{\partial T}{\partial x} = \big(1 - 2x(x+y)\big)e^{-x^2-y^2} \qquad\qquad 0 = \frac{\partial T}{\partial y} = (1 - 2y(x+y))e^{-x^2-y^2}$$

The critical points are given by

$2x(x+y) = 1 = 2y(x+y)$, which forces $x = y$ and $4x^2 = 1$, so $x = y = \pm\frac{1}{2}$

The two critical points are $\left(\frac{1}{2},\frac{1}{2}\right)$ and $\left(-\frac{1}{2},-\frac{1}{2}\right)$,

both of which lie inside D. T takes the values $\pm e^{-1/2}$ at these points.

On the boundary of D, $x = \cos\ t, y = \sin\ t, 0 \le t \le 2\pi$, so that

$$T = (\cos\ t + \sin\ t)e^{-1} = g(t),\ (0 \le t \le 2\pi)$$

We have $g(0) = g(2\pi) = e^{-1}$. For critical points of g :

$$0 = g'(t) = (\cos\ t - \sin\ t)e^{-1}$$

so $\tan\ t = 1$ and $t = \pi/4$ or $t = 5\pi/4$. Observe that $g(\pi/4) = \sqrt{2}e^{-1}$, and $g(5\pi/4) = -\sqrt{2}e^{-1}$

Since $e^{-1/2} > \sqrt{2}e^{-1}$ (because $e > 2$), the maximum and minimum values of T on the disk are $\pm e^{-1/2}$, the values at the interior critical points.

<u>Question 6:</u> Find the maximum and minimum values of $xy^2 + yz^2$ over the ball $x^2 + y^2 + z^2 \le 1$.

Let $f(x, y, z) = xy^2 + yz^2$ on the ball $B: x^2 + y^2 + z^2 \le 1$. First look for interior critical points:

$$0 = f_1 = y^2,\ 0 = f_2 = 2xy + z^2,\ 0 = f_3 = 2yz$$

All points on the x-axis are CPs, and $f = 0$ at all such points.

Now consider the boundary sphere $z^2 = 1 - x^2 - y^2$. On it

$$f(x, y, z) = xy^2 + y(1 - x^2 - y^2) = xy^2 + y - x^2y - y^3 = g(x, y)$$

where g is defined for $x^2 + y^2 \le 1$. Look for interior CPs of g :

$0 = g_1 = y^2 - 2xy = y(y - 2x)$ $\qquad\qquad$ $0 = g_2 = 2xy + 1 - x^2 - 3y^2$

Case I: $y = 0$. Then $g = 0$ and $f = 0$.

Case II: $y = 2x$. Then $4x^2 + 1 - x^2 - 12x^2 = 0$, so $9x^2 = 1$ and $x = \pm 1/3$. This case produces critical points

$\left(\frac{1}{3},\frac{2}{3},\pm\frac{2}{3}\right)$, where $f = \frac{4}{9}$, and

$\left(-\frac{1}{3},-\frac{2}{3},\pm\frac{2}{3}\right)$, where $f = -\frac{4}{9}$

Now we must consider the boundary $x^2 + y^2 = 1$ of the domain of g. Here

$$g(x, y) = xy^2 = x(1 - x^2) = x - x^3 = h(x)$$

for $-1 \le x \le 1$. At the endpoints $x = \pm 1, h = 0$, so $g = 0$ and $f = 0$. For CPs of h :

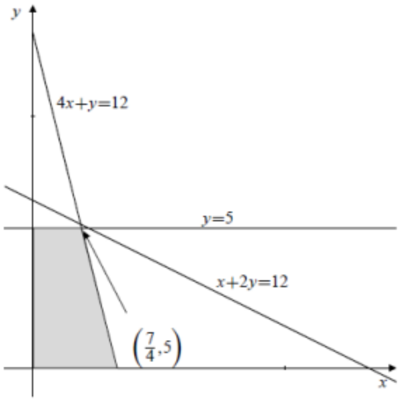
$$0 = h'(x) = 1 - 3x^2$$

so $x = \pm 1/\sqrt{3}$ and $y = \pm\sqrt{2/3}$. The value of h at such points is $\pm 2/(3\sqrt{3})$. However, $2/(3\sqrt{3}) < 4/9$, so the maximum value of f is $4/9$, and the minimum value is $-4/9$.

<u>Question 7:</u> Maximize $Q(x, y) = 2x + 3y$ subject to the constraints $x \ge 0, y \ge 0, y \le 5, x + 2y \le 12$, and $4x + y \le 12$.

To maximize $Q(x, y) = 2x + 3y$ subject to $x \ge 0,\ y \ge 0,\ y \le 5,\ x + 2y \le 12,\ 4x + y \le 12$.

The constraint region is shown in the figure.



Observe that any point satisfying $y \le 5$ and $4x + y \le 12$ automatically satisfies $x + 2y \le 12$. Since $y = 5$ and $4x + y = 12$ intersect at $\left(\frac{7}{4}, 5\right)$, the maximum value of $Q(x, y)$ subject to the given constraints is

$$Q\left(\frac{7}{4}, 5\right) = \frac{7}{2} + 15 = \frac{37}{2}.$$

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: Use the Method of Lagrange Multipliers to maximize x^3y^5 subject to the constraint $x+y=8$.

First, we observe that $f(x,y)=x^3y^5$ must have a maximum value on the line $x+y=8$ because if $x\to-\infty$ then $y\to\infty$ and if $x\to\infty$ then $y\to-\infty$. In either case $f(x,y)\to-\infty$.

Let $L=x^3y^5+\lambda(x+y-8)$. For CPS of L :

$$0=\frac{\partial L}{\partial x}=3x^2y^5+\lambda$$
$$0=\frac{\partial L}{\partial y}=5x^3y^4+\lambda$$
$$0=\frac{\partial L}{\partial \lambda}=x+y-8$$

The first two equations give $3x^2y^5=5x^3y^4$, so that either $x=0$ or $y=0$ or $3y=5x$. If $x=0$ or $y=0$ then $f(x,y)=0$. If $3y=5x$, then $x+\frac{5}{3}x=8$, so $8x=24$ and $x=3$. Then $y=5$, and $f(x,y)=3^35^5=84{,}375$. This is the maximum value of f on the line.

Question 2: Find the distance from the origin to the plane $x+2y+2z=3$,

(a) using a geometric argument (no calculus),

(b) by reducing the problem to an unconstrained problem in two variables, and

(c) using the method of Lagrange multipliers.

Let (X,Y,Z) be the point on the plane $x+2y+2z=3$ closest to $(0,0,0)$.

a) The vector $\nabla(x+2y+2z)=\mathbf{i}+2\mathbf{j}+2\mathbf{k}$ is perpendicular to the plane, so must be parallel to the vector $X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k}$ from the origin to (X,Y,Z). Thus

$$X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k}=t(\mathbf{i}+2\mathbf{j}+2\mathbf{k})$$

for some scalar t. Thus $X=t, Y=2t, Z=2t$, and, since (X,Y,Z) lies on the plane,

$$3=X+2Y+2Z=t+4t+4t=9t$$

Thus $t=\frac{1}{3}$, and we have $X=\frac{1}{3}$ and $Y=Z=\frac{2}{3}$. The minimum distance from the origin to the plane is therefore $\frac{1}{3}\sqrt{1+4+4}=1$ unit.

b) (X,Y,Z) must minimize the square of the distance from the origin to (x,y,z) on the plane. Thus, it is a critical point of $S = x^2 + y^2 + z^2$. Since

$x + 2y + 2z = 3$, we have $x = 3 - 2(y + z)$, and

$$S = S(y, z) = (3 - 2(y + z))^2 + y^2 + z^2$$

The critical points of this function are given by

$$0 = \frac{\partial S}{\partial y} = -4(3 - 2(y + z)) + 2y = -12 + 10y + 8z$$
$$0 = \frac{\partial S}{\partial z} = -4(3 - 2(y + z)) + 2z = -12 + 8y + 10z$$

Therefore $Y = Z = \frac{2}{3}$ and $X = \frac{1}{3}$, and the distance is 1 unit as in part (a).

c) The point (X, Y, Z) must be a critical point of the Lagrange function

$$L = x^2 + y^2 + z^2 + \lambda(x + 2y + 2z - 3)$$

To find these critical points we have

$$\begin{aligned} 0 &= \frac{\partial L}{\partial x} = 2x + \lambda \\ 0 &= \frac{\partial L}{\partial y} = 2y + 2\lambda \\ 0 &= \frac{\partial L}{\partial z} = 2z + 2\lambda \\ 0 &= \frac{\partial L}{\partial \lambda} = x + 2y + 2z - 3 \end{aligned}$$

The first three equations yield $y = z = -\lambda$,

$x = -\lambda/2$. Substituting these into the fourth equation we get $\lambda = -\frac{2}{3}$, so that the critical point is once again $\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$, whose distance from the origin is 1 unit.

Question 3: Use the Lagrange multiplier method to find the greatest and least distances from the point $(2,1,-2)$ to the sphere with equation $x^2 + y^2 + z^2 = 1$. (Of course, the answer could be obtained more easily using a simple geometric argument.)

The distance D from $(2,1,-2)$ to (x, y, z) is given by

$$D^2 = (x - 2)^2 + (y - 1)^2 + (z + 2)^2.$$

We can extremize D by extremizing D^2. If (x, y, z) lies on the sphere $x^2 + y^2 + z^2 = 1$, we should look for critical points of the Lagrange function

$$L = (x-2)^2 + (y-1)^2 + (z+2)^2 + \lambda(x^2+y^2+z^2-1)$$

Thus

$$0 = \frac{\partial L}{\partial x} = 2(x-2) + 2\lambda x \iff x = \frac{2}{1+\lambda}$$
$$0 = \frac{\partial L}{\partial y} = 2(y-1) + 2\lambda y \iff y = \frac{1}{1+\lambda}$$
$$0 = \frac{\partial L}{\partial z} = 2(z+2) + 2\lambda z \iff z = \frac{-2}{1+\lambda}$$
$$0 = \frac{\partial L}{\partial \lambda} = x^2+y^2+z^2-1$$

Substituting the solutions of the first three equations into the fourth, we obtain

$$\frac{1}{(1+\lambda)^2}(4+1+4) = 1$$
$$(1+\lambda)^2 = 9$$
$$1+\lambda = \pm 3$$

Thus, we must consider the two points $P = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$, and $Q = \left(-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)$ for giving extreme values for D. At $P, D = 2$. At $Q, D = 4$. Thus, the greatest and least distances from $(2,1,-2)$ to the sphere $x^2+y^2+z^2=1$ are 4 units and 2 units respectively.

Question 4: Find a, b, and c so that the volume $V = 4\pi abc/3$ of an ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$ passing through the point $(1,2,1)$ is as small as possible.

We want to minimize $V = \frac{4\pi abc}{3}$ subject to the constraint $\frac{1}{a^2}+\frac{4}{b^2}+\frac{1}{c^2} = 1$. Note that abc cannot be zero. Let $L = \frac{4\pi abc}{3} + \lambda\left(\frac{1}{a^2}+\frac{4}{b^2}+\frac{1}{c^2}-1\right)$.

For critical points of L :

$$0 = \frac{\partial L}{\partial a} = \frac{4\pi bc}{3} - \frac{2\lambda}{a^3} \iff \frac{2\pi abc}{3} = \frac{\lambda}{a^2} \qquad 0 = \frac{\partial L}{\partial b} = \frac{4\pi ac}{3} - \frac{8\lambda}{b^3} \iff \frac{2\pi abc}{3} = \frac{4\lambda}{b^2}$$

$$0 = \frac{\partial L}{\partial c} = \frac{4\pi ab}{3} - \frac{2\lambda}{c^3} \iff \frac{2\pi abc}{3} = \frac{\lambda}{c^2} \qquad 0 = \frac{\partial L}{\partial \lambda} = \frac{1}{a^2}+\frac{4}{b^2}+\frac{1}{c^2} - 1.$$

$abc \neq 0$ implies $\lambda \neq 0$, and so we must have $\frac{1}{a^2} = \frac{4}{b^2} = \frac{1}{c^2} = \frac{1}{3}$,

so $a = \pm\sqrt{3}, b = \pm 2\sqrt{3}$, and $c = \pm\sqrt{3}$.

Question 5: Find the maximum and minimum values of $f(x, y, z) = xyz$ on the sphere $x^2 + y^2 + z^2 = 12$.

Let $L = xyz + \lambda(x^2 + y^2 + z^2 - 12)$. For CPs of L:

$$0 = \frac{\partial L}{\partial x} = yz + 2\lambda x \qquad \text{(A)}$$
$$0 = \frac{\partial L}{\partial y} = xz + 2\lambda y \qquad \text{(B)}$$
$$0 = \frac{\partial L}{\partial z} = xy + 2\lambda z \qquad \text{(C)}$$
$$0 = \frac{\partial L}{\partial \lambda} = x^2 + y^2 + z^2 - 12 \qquad \text{(D)}$$

Multiplying equations (A), (B), and (C) by x, y, and z, respectively, and subtracting in pairs, we conclude that $\lambda x^2 = \lambda y^2 = \lambda z^2$, so that either $\lambda = 0$ or $x^2 = y^2 = z^2$.

If $\lambda = 0$, then (A) implies that $yz = 0$, so $xyz = 0$. If $x^2 = y^2 = z^2$, then (D) gives $3x^2 = 12$, so $x^2 = 4$.

We obtain eight points (x, y, z) where each coordinate is either 2 or -2. At four of these points $xyz = 8$, which is the maximum value of xyz on the sphere. At the other four $xyz = -8$, which is the minimum value.

Question 6: Find the maximum and minimum values of the function $f(x, y, z) = x$ over the curve of intersection of the plane $z = x + y$ and the ellipsoid $x^2 + 2y^2 + 2z^2 = 8$.

The surface has no points where any coordinate is zero, and is not bounded, so it has no farthest point from the origin. To find the closest point we look at critical points of

$$L = x^2 + y^2 + z^2 + \lambda xy^2z^4 - 32,\ x \neq 0, y \neq 0, z \neq 0$$

Thus, we calculate

$$0 = \frac{\partial L}{\partial x} = 2x + \lambda y^2 z^4 \Leftrightarrow \frac{2x}{y^2 z^4} = -\lambda$$
$$0 = \frac{\partial L}{\partial y} = 2y + 2\lambda xyz^4 \Leftrightarrow \frac{1}{xz^4} = -\lambda$$
$$0 = \frac{\partial L}{\partial z} = 2z + 4\lambda xy^2 z^3 \Leftrightarrow \frac{1}{2xy^2 z^2} = -\lambda$$
$$0 = \frac{\partial L}{\partial \lambda} = xy^2 z^4 - 1.$$

The first three equations imply that

$$\frac{2x}{y^2 z^4} = \frac{1}{xz^4} = \frac{1}{2xy^2 z^2}$$

from which it follows that $y^2 = 2x^2$ and $z^2 = 2y^2 = 4x^2$. The constraint equation then gives $x(2x^2)(16x^4) = 32$, or, simply, $x^7 = 1$. This implies $x = 1$, so $y = \pm\sqrt{2}$ and $z = \pm 2$. The distance from the surface to the origin is $\sqrt{1+2+4} = \sqrt{7}$ units.

Question 7: Find the maximum volume of a rectangular box with faces parallel to the coordinate planes if one corner is at the origin and the diagonally opposite corner lies on the plane $4x + 2y + z = 2$

We want to maximize $V = xyz$ subject to $4x + 2y + z = 2$.

Let

$$L = xyz + \lambda(4x + 2y + z - 2)$$

For critical points of L,

$$\begin{aligned}
0 &= \frac{\partial L}{\partial x} = yz + 4\lambda \iff xyz + 4\lambda x = 0 \\
0 &= \frac{\partial L}{\partial y} = xz + 2\lambda \iff xyz + 2\lambda y = 0 \\
0 &= \frac{\partial L}{\partial z} = xy + \lambda \iff xyz + \lambda z = 0 \\
0 &= \frac{\partial L}{\partial \lambda} = 4x + 2y + z - 2 = 0.
\end{aligned}$$

The first three equations imply that $z = 2y = 4x$ (since we cannot have $\lambda = 0$ if V is positive). The fourth equation then implies that $12x = 2$. Hence $x = 1/6, y = 1/3$, and $z = 2/3$.

The largest box has volume

$V = \frac{1}{6} \times \frac{1}{3} \times \frac{2}{3} = \frac{1}{27}$ cubic units.

Question 8: Find the maximum and minimum values of $xy + z^2$ on the ball $x^2 + y^2 + z^2 \leq 1$. Use Lagrange multipliers to treat the boundary case.

$f(x, y, z) = xy + z^2$ on $B = \{(x, y, z): x^2 + y^2 + z^2 \leq 1\}$ For critical points of f,

$0 = f_1(x, y, z) = y, \quad 0 = f_2(x, y, z) = x \qquad 0 = f_3(x, y, z) = 2z$

Thus, the only critical point is the interior point $(0,0,0)$, where f has the value 0, evidently neither a maximum nor a minimum. The maximum and minimum must therefore occur on the boundary of B, that is, on the sphere

$x^2 + y^2 + z^2 = 1$. Let

$$L = xy + z^2 + \lambda(x^2 + y^2 + z^2 - 1)$$

For critical points of L,

$$\begin{aligned} 0 &= \frac{\partial L}{\partial x} = y + 2\lambda x \qquad &(A)\\ 0 &= \frac{\partial L}{\partial y} = x + 2\lambda y \qquad &(B)\\ 0 &= \frac{\partial L}{\partial z} = 2z(1+\lambda) \qquad &(C)\\ 0 &= \frac{\partial L}{\partial \lambda} = x^2 + y^2 + z^2 - 1. \qquad &(D) \end{aligned}$$

From (C) either $z = 0$ or $\lambda = -1$.

CASE I. $z = 0$. (A) and (B) imply that $y^2 = x^2$ and (D) then implies that $x^2 = y^2 = 1/2$. At the four points $\left(\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, 0\right)$ and $\left(-\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, 0\right)$

f takes the values $\frac{1}{2}$ and $-\frac{1}{2}$.

CASE II. $\lambda = -1$. (A) and (B) imply that $x = y = 0$, and so by (D), $z = \pm 1$. f has the value 1 at the points $(0,0,\pm 1)$.

Thus the maximum and minimum values of f on B are 1 and $-1/2$ respectively.

The Following Questions Are Obtained From PAST EXAMS

Question 1: Use the Method of Lagrange Multipliers to find the least and greatest distance between the origin and the ellipse $17x^2 + 12xy + 8y^2 - 100 = 0$. PE. SPRING2005

Solution: Let the point be (x, y) then the distance is $d = \sqrt{x^2 + y^2}$. Now, we define

$$\begin{aligned} L(x,y,\lambda) &= x^2 + y^2 + \lambda(17x^2 + 12xy + 8y^2 - 100) \\ \frac{\partial L}{\partial x} &= 2x + \lambda(34x + 12y) = 0 \qquad &(1)\\ \frac{\partial L}{\partial y} &= 2y + \lambda(12x + 16y) = 0 \qquad &(2)\\ \frac{\partial L}{\partial \lambda} &= 17x^2 + 12xy + 8y^2 - 100 = 0 \qquad &(3) \end{aligned}$$

From the equations (1) and (2) we write $\lambda = \frac{-x}{17x+6y}$ and $\lambda = \frac{-y}{6x+8y}$, respectively.

Equating them and performing cross multiplication we get

$$2x^2 - 3xy - 2y^2 = 0 \qquad (4)$$

Multiplying, (4) by 4 , and adding with (3) we obtain, $25x^2 - 100 = 0$ or $x = \pm 2$.

For $x = 2$, (4) becomes $8 - 6y - 2y^2 = 0$ which has the roots $y = 1$ and $y = -4$.

For $x = -2$, (4) becomes $8 + 6y - 2y^2 = 0$ which has the roots $y = -1$ and $y = 4$. Therefore, we have $P_1(2,1), P_2(2,-4), P_3(-2,-1), P_4(-2,4)$.

Therefore, the least distance is $\sqrt{5}$ and the greatest distance is $2\sqrt{5}$.

Question 2: Find the set of all points on the ellipsoid $x^2 + 2y^2 + 4z^2 - 12 = 0$ at which the tangent plane to the ellipsoid is parallel to the plane $y + z = 120$. PE. SPRING2006

Let $F(x,y,z) = x^2 + 2y^2 + 4z^2 - 12$. Then $\nabla F(x,y,z)$ is normal to the level surface $F = 0$ at (x,y,z). We want ∇F to be parallel to $\vec{n} = (0,1,1)$.

$$\nabla F = (2x, 4y, 8z) = \lambda(0,1,1)$$

$$\Rightarrow \left.\begin{matrix} 2x = 0 \\ 4y = \lambda \\ 8z = \lambda \end{matrix}\right\} x = 0 \text{ and } y = 2z$$

$$0^2 + 2(2z)^2 + 4z^2 = 12 \qquad 8z^2 + 4z^2 = 12 \qquad z^2 = 1 \qquad z = \mp 1 \Rightarrow y = \mp 2$$

Points are $(0,2,1)$ and $(0,-2,-1)$

Question 3: Using Lagrange Multipliers Method, find a point of the surface $xyz^2 = 2$ that is closest to the origin.

PE. SPRING2006

Distance of a point (x,y,z) to the origin is $d(x,y,z) = \sqrt{x^2 + y^2 + z^2}$ and this is minimum if and only if $x^2 + y^2 + z^2$ is minimum. Thus, we need to find

minimum value of $f(x,y,z) = x^2 + y^2 + z^2$ subject to $g(x,y,z) = xyz^2 - 2 = 0$

set $\nabla f = \lambda \nabla g$

$$f_x = 2x = \lambda yz^2 = \lambda g_x \qquad (1)$$

$$f_y = 2y = \lambda xz^2 = \lambda g_y \qquad (2)$$

$$f_z = 2z = \lambda 2xyz = \lambda g_z \qquad (3)$$

$$xyz^2 = 2 \qquad (4)$$

Then dividing the (1) by (2), we have $\frac{x}{y} = \frac{y}{x}$ that is $x^2 = y^2 \;\rightarrow\; x = y \quad \left(\text{since } x = -y \text{ contradicts } 4^{\text{th}} \text{ equation}\right)$

Then dividing the (2) by (3), we have $\frac{y}{z} = \frac{z}{2y}$ that is $z^2 = 2y^2$.

Thus, $z^2 = 2y^2 = 2x^2$

Plug $z^2 = 2y^2 = 2x^2$ in the equation (4). Then we have $xyz^2 = 2 \quad \rightarrow (x)(x)(2x^2) = 2 \rightarrow x^4 = 1 \quad \rightarrow x = \mp 1$

since $x = \mp 1 \;\rightarrow\; y = \mp 1 \;\rightarrow\; z = \mp\sqrt{2}$

Hence points on $xyz^2 = 2$ that we need to check are

$(1,1,\sqrt{2}) \;\rightarrow\; d = 2$ $\qquad$ $(-1,-1,\sqrt{2}) \;\rightarrow\; d = 2$

$(1,1,-\sqrt{2}) \;\rightarrow\; d = 2$ $\qquad$ $(-1,-1,-\sqrt{2}) \;\rightarrow\; d = 2$

These points of the surface $xyz^2 = 2$ are the closest ones to the origin.

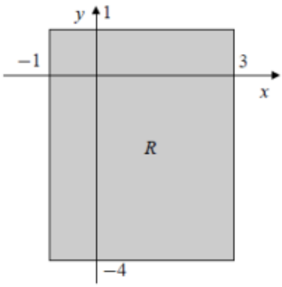
Section 5.1 Double Integrals

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: evaluate the following given double integral by inspection.

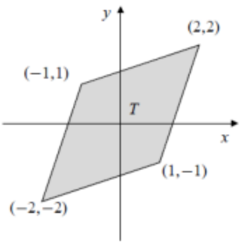
1. $\iint_R dA$, where R is the rectangle $-1 \le x \le 3, -4 \le y \le 1$

$\iint_R dA =$ area of $R = 4 \times 5 = 20.$



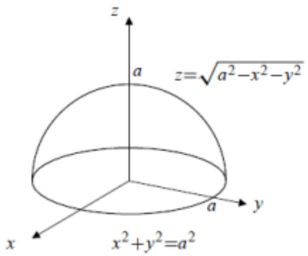
2. $\iint_T (x+y)dA$, where T is the parallelogram having the points $(2,2), (1,-1), (-2,-2)$, and $(-1,1)$ as vertices

T is symmetric about the line $x + y = 0$. Therefore, $\iint_T (x+y)dA = 0.$



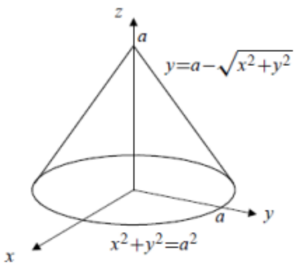
3. $\iint_{x^2+y^2\le a^2}\sqrt{a^2-x^2-y^2}\,dA$

$=$ volume of hemisphere shown in the figure $=\frac{1}{2}\left(\frac{4}{3}\pi a^3\right)=\frac{2}{3}\pi a^3$.



4. $\iint_{x^2+y^2\le a^2}\left(a-\sqrt{x^2+y^2}\right)dA$

volume of cone shown in the figure $=\frac{1}{3}\pi a^3$.



Section 5.2 Iteration of Double Integrals in Cartesian Coordinates

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: calculate the given following iterated integrals.

a) $\int_0^1 dx\int_0^x (xy+y^2)dy=\int_0^1 dx\left(\frac{xy^2}{2}+\frac{y^3}{3}\right)\Big|_{y=0}^{y=x}\int_0^1 dx\int_0^x (xy+y^2)dy=\frac{5}{6}\int_0^1 x^3dx=\frac{5}{24}$

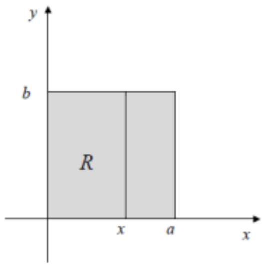
b) $\int_0^\pi\int_{-x}^x \cos y\,dydx=\int_0^\pi\int_{-x}^x \cos\ ydydx=\int_0^\pi \sin\ y\Big|_{y=-x}^{y=x}dx\ =2\int_0^\pi \sin\ xdx=-2\cos\ x|_0^\pi=4$

Question 2: evaluate the following double integrals by iteration.

$a)$ $\iint_R (x^2+y^2)dA$, where R is the rectangle $0 \le x \le a$, $0 \le y \le b$

$$\iint_R (x^2+y^2)dA = \int_0^a dx \int_0^b (x^2+y^2)dy = \int_0^a dx\left(x^2y+\frac{y^3}{3}\right)\Bigg|_{y=0}^{y=b} = \int_0^a \left(bx^2+\frac{1}{3}b^3\right)dx$$

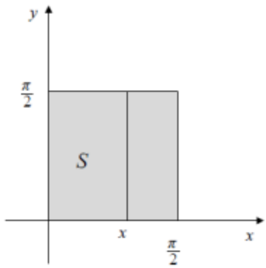
$$=\frac{1}{3}(bx^3+b^3x)\Big|_0^a = \frac{1}{3}(a^3b+ab^3)$$



$b)$ $\iint_S (\sin\, x + \cos\, y)dA$, where S is the square $0 \le x \le \pi/2, 0 \le y \le \pi/2$

$$\iint_S (\sin\, x+\cos\, y)dA = \int_0^{\pi/2} dx \int_0^{\pi/2} (\sin\, x+\cos\, y)dy = \int_0^{\pi/2} dx(y\sin\, x+\sin\, y)\Big|_{y=0}^{y=\pi/2}$$

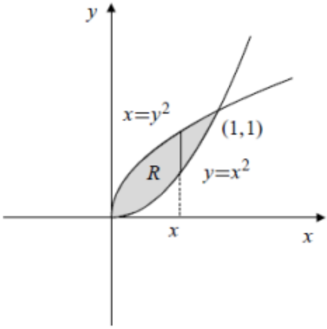
$$=\int_0^{\pi/2}\left(\frac{\pi}{2}\sin\, x+1\right)dx = \left(-\frac{\pi}{2}\cos\, x+x\right)\Big|_0^{\pi/2} = \frac{\pi}{2}+\frac{\pi}{2}=\pi.$$



$c)$ $\iint_R xy^2dA$, where R is the finite region in the first quadrant bounded by the curves $y=x^2$ and $x=y^2$

$$\iint_R xy^2dA = \int_0^1 xdx\int_{x^2}^{\sqrt{x}} y^2dy = \int_0^1 xdx\left(\frac{1}{3}y^3\right)\Bigg|_{y=x^2}^{y=\sqrt{x}} = \frac{1}{3}\int_0^1\left(x^{\frac{5}{2}}-x^7\right)dx = \frac{1}{3}\left(\frac{2}{7}x^{\frac{7}{2}}-\frac{x^8}{8}\right)\Bigg|_0^1$$

$= \frac{1}{3}\left(\frac{2}{7} - \frac{1}{8}\right) = \frac{3}{56}.$



d) $\iint_D \ln x \, dA$, where D is the finite region in the first quadrant bounded by the line $2x + 2y = 5$ and the hyperbola $xy = 1$

For intersection: $xy = 1{,}2x + 2y = 5$.

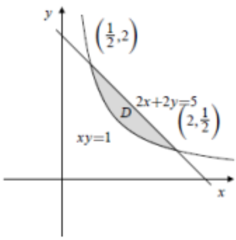
Thus $2x^2 - 5x + 2 = 0$, or $(2x - 1)(x - 2) = 0$. The intersections are at $x = 1/2$ and $x = 2$. We have

$$\iint_D \ln x \, dA = \int_{1/2}^{2} \ln x \, dx \int_{1/x}^{(5/2)-x} dy = \int_{1/2}^{2} \ln x \left(\frac{5}{2} - x - \frac{1}{x}\right) dx = \int_{1/2}^{2} \ln x \left(\frac{5}{2} - x\right) dx - \frac{1}{2}(\ln x)^2\Big|_{1/2}^{2}$$

$$U = \ln x \; dV = \left(\frac{5}{2} - x\right) dx \qquad dU = \frac{dx}{x} \; V = \frac{5}{2}x - \frac{x^2}{2}$$

$$= -\frac{1}{2}\left((\ln\ 2)^2 - \left(\ln\frac{1}{2}\right)^2\right) + \left(\frac{5}{2}x - \frac{x^2}{2}\right)\ln x\Big|_{\frac{1}{2}}^{2} - \int_{\frac{1}{2}}^{2}\left(\frac{5}{2} - \frac{x}{2}\right) dx = (5-2)\ln\ 2 - \left(\frac{5}{4} - \frac{1}{8}\right)\ln\ \frac{1}{2} - \frac{15}{4} + \frac{15}{16}$$

$$= \frac{33}{8}\ln 2 - \frac{45}{16}.$$

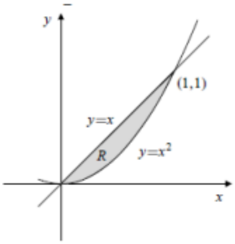


e) $\iint_R \frac{x}{y} e^y dA$, where R is the region $0 \le x \le 1, x^2 \le y \le x$

$$\iint_R \frac{x}{y} e^y dA = \int_0^1 \frac{e^y}{y} dy \int_y^{\sqrt{y}} x dx = \frac{1}{2}\int_0^1 (1-y)e^y dy$$

$U = 1 - y \; dV = e^y dy \qquad dU = -dy \; V = e^y$

$$= \frac{1}{2}\left[(1-y)e^y|_0^1 + \int_0^1 e^y dy\right] = -\frac{1}{2} + \frac{1}{2}(e-1) = \frac{e}{2} - 1.$$

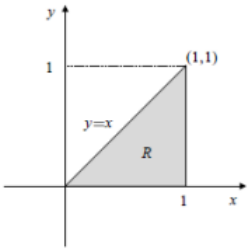


<u>Question 3:</u> sketch the domain of the following integrations and evaluate the given following iterated integrals.

a. $\int_0^1 dy \int_y^1 e^{-x^2} dx$

$$\int_0^1 dy \int_y^1 e^{-x^2} dx = \int_R e^{-x^2} dx \text{ (}R\text{ as shown) } = \int_0^1 e^{-x^2} dx \int_0^x dy = \int_0^1 x e^{-x^2} dx \text{ Let } u = x^2$$

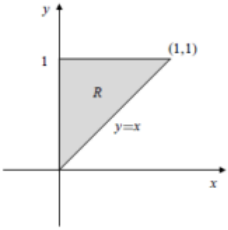
$$= \frac{1}{2}\int_0^1 e^{-u} du = -\frac{1}{2} e^{-u}\Big|_0^1 = \frac{1}{2}\left(1 - \frac{1}{e}\right)$$



b. $\int_0^1 dx \int_x^1 \frac{y^\lambda}{x^2+y^2} dy\ (\lambda > 0)$

$$\int_0^1 dx \int_x^1 \frac{y^\lambda}{x^2+y^2} dy\ (\lambda > 0) = \iint_R \frac{y^\lambda}{x^2+y^2} dA\ (R \text{ as shown }) = \int_0^1 y^\lambda dy \int_0^y \frac{dx}{x^2+y^2}$$

$$= \int_0^1 y^\lambda dy \frac{1}{y}\left(\tan^{-1}\frac{x}{y}\right)\Big|_{x=0}^{x=y} = \frac{\pi}{4}\int_0^1 y^{\lambda-1} dy = \frac{\pi y^\lambda}{4\lambda}\Big|_0^1 = \frac{\pi}{4\lambda}$$



Question 4: Find the volumes of the following indicated solids.

a. Under $z = 1 - x^2$ and above the region $0 \le x \le 1, 0 \le y \le x$

$$V = \int_0^1 dx \int_0^x (1 - x^2) dy = \int_0^1 (1 - x^2) x dx = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \text{ cu. units.}$$

b. Under $z = 1 - x^2 - y^2$ and above the region $x \ge 0, y \ge 0, x + y \le 1$

$$V = \int_0^1 dx \int_0^{1-x} (1 - x^2 - y^2) dy = \int_0^1 \left((1 - x^2)y - \frac{y^3}{3}\right)\Big|_{y=0}^{y=1-x} dx$$

$$= \int_0^1 \left((1 - x^2)(1 - x) - \frac{(1-x)^3}{3}\right) dx = \int_0^1 \left(\frac{2}{3} - 2x^2 + \frac{4x^3}{3}\right) dx = \frac{2}{3} - \frac{2}{3} + \frac{1}{3} = \frac{1}{3} \text{ cu. units.}$$

c. Under the surface $z = 1/(x + y)$ and above the region in the xy-plane bounded by $x = 1, x = 2, y = 0$, and $y = x$

$$V = \int_1^2 dx \int_0^x \frac{1}{x+y} dy = \int_1^2 dx\left(\ln(x+y)|_{y=0}^{y=x}\right) = \int_1^2 (\ln 2x - \ln x) dx = \ln 2 \int_1^2 dx = \ln 2 \text{ cu. units.}$$

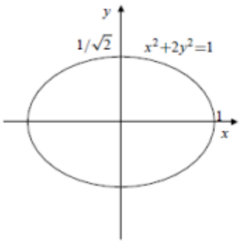
d. Above the xy-plane and under the surface $z = 1 - x^2 - 2y^2$

$$\text{Vol} = \iint_E (1 - x^2 - 2y^2) dA = 4\int_0^1 dx \int_0^{\sqrt{\frac{(1-x^2)}{2}}} (1 - x^2 - 2y^2) dy$$

$$= 4\int_0^1 \left(\frac{1}{\sqrt{2}}(1-x^2)^{3/2} - \frac{2}{3}\frac{(1-x^2)^{3/2}}{2\sqrt{2}}\right)dx = \frac{4\sqrt{2}}{3}\int_0^1 (1-x^2)^{3/2}dx \text{ Let } x = \sin\ \theta$$

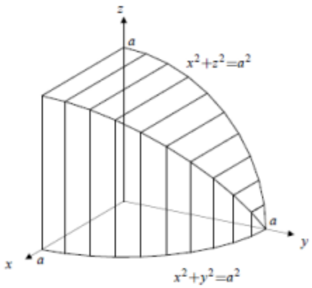
$$= \frac{4\sqrt{2}}{3}\int_0^{\pi/2} \cos^4\ \theta d\theta = \frac{4\sqrt{2}}{3}\int_0^{\pi/2}\left(\frac{1+\cos\ 2\theta}{2}\right)^2 d\theta = \frac{\sqrt{2}}{3}\int_0^{\pi/2}\left(1 + 2\cos\ 2\theta + \frac{1+\cos\ 4\theta}{2}\right)d\theta$$

$$= \frac{\sqrt{2}}{3}\left[\frac{3\theta}{2} + \sin\ 2\theta + \frac{1}{8}\sin\ 4\theta\right]\Bigg|_0^{\pi/2} = \frac{\pi}{2\sqrt{2}}\text{cu. units.}$$



e. Inside the two cylinders $x^2 + y^2 = a^2$ and $y^2 + z^2 = a^2$ Vol $= 8 \times$ part in the first octant

$$= 8\int_0^a dx \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2}dy = 8\int_0^a (a^2-x^2)dx = 8\left(a^2x - \frac{x^3}{3}\right)\Bigg|_0^a = \frac{16}{3}a^3\text{cu. units.}$$



Section 5.3 Double Integrals in Polar Coordinates

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: evaluate the given following double integrals over the disk D given by $x^2 + y^2 \leq a^2$, where $a > 0$.

1. $\iint_D (x^2+y^2)dA = \int_0^{2\pi} d\theta \int_0^a r^2 r dr = 2\pi\frac{a^4}{4} = \frac{\pi a^4}{2}$

2. $\iint_D \dfrac{dA}{\sqrt{x^2+y^2}} = \int_0^{2\pi} d\theta \int_0^a \dfrac{rdr}{r} = 2\pi a$

3. $\iint_D x^2 dA = \frac{\pi a^4}{4}$; by symmetry the value of this integral is half of that in question 1.

<u>Question 2:</u> evaluate the given following double integrals over the quarter-disk Q given by $x \geq 0, y \geq 0$, and $x^2+y^2 \leq a^2$, where $a > 0$.

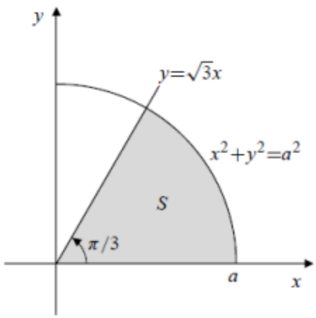
1. $\displaystyle\iint_Q ydA = \int_0^{\pi/2} d\theta \int_0^a r\sin\theta\, rdr \; = (-\cos\;\theta)|_0^{\pi/2} \frac{a^3}{3} = \frac{a^3}{3}$

2. $\displaystyle\iint_Q e^{x^2+y^2} dA \; = \int_0^{\pi/2} d\theta \int_0^a e^{r^2} rdr = \frac{\pi}{2}\left(\frac{1}{2}e^{r^2}\right)\Big|_0^a = \frac{\pi\left(e^{a^2}-1\right)}{4}$

<u>Question 3:</u> Evaluate $\iint_S (x+y)dA$, where S is the region in the first quadrant lying inside the disk $x^2+y^2 \leq a^2$ and under the line $y = \sqrt{3}x$.

$$\iint_S (x+y)dA = \int_0^{\pi/3} d\theta \int_0^a (r\cos\;\theta + r\sin\;\theta)rdr = \int_0^{\pi/3} (\cos\;\theta + \sin\;\theta)d\theta \int_0^a r^2 dr$$

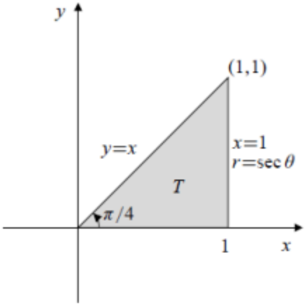
$$= \frac{a^3}{3}(\sin\;\theta - \cos\;\theta)\Bigg|_0^{\pi/3} = \left[\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) - (-1)\right]\frac{a^3}{3} = \frac{(\sqrt{3}+1)a^3}{6}$$



<u>Question 4:</u> Evaluate $\iint_T (x^2+y^2)dA$, where T is the triangle with vertices (0,0), (1,0), and (1,1).

$$\iint_T (x^2+y^2)dA = \int_0^{\frac{\pi}{4}} d\theta \int_0^{\sec\theta} r^3 dr = \frac{1}{4}\int_0^{\frac{\pi}{4}} \sec^4\theta d\theta = \frac{1}{4}\int_0^{\frac{\pi}{4}} (1+\tan^2\theta)\sec^2\theta d\theta$$

Let $u = \tan\theta$ $\qquad = \dfrac{1}{4}\displaystyle\int_0^1 (1+u^2)du = \frac{1}{4}\left(u + \frac{u^3}{3}\right)\Bigg|_0^1 = \frac{1}{3}$



Question 5: For what values of k, and to what value, does the integral $\iint_{x^2+y^2\leq 1} \frac{dA}{(x^2+y^2)^k}$ converge?

If D is the disk $x^2 + y^2 \leq 1$, then

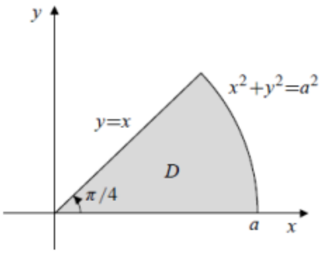
$$\iint_D \frac{dA}{(x^2+y^2)^k} = \int_0^{2\pi} d\theta \int_0^1 r^{-2k} r dr = 2\pi \int_0^1 r^{1-2k} dr$$

which converges if $1 - 2k > -1$, that is, if $k < 1$. In this case the value of the integral is

$$2\pi \frac{r^{2-2k}}{2-2k}\bigg|_0^1 = \frac{\pi}{1-k}$$

Question 6: Evaluate $\iint_D xydA$, where D is the plane region satisfying $x \geq 0, 0 \leq y \leq x$, and $x^2 + y^2 \leq a^2$.

$$\iint_D xydA = \int_0^{\pi/4} d\theta \int_0^a r\cos\theta r\sin\theta r dr = \frac{1}{2}\int_0^{\pi/4} \sin 2\theta d\theta \int_0^a r^3 dr = \frac{a^4}{8}\left(-\frac{\cos 2\theta}{2}\right)\bigg|_0^{\pi/4} = \frac{a^4}{16}$$



Question 7: Find the volume lying between the paraboloids $z = x^2 + y^2$ and $3z = 4 - x^2 - y^2$.

The paraboloids $z = x^2 + y^2$ and $3z = 4 - x^2 - y^2$ intersect where $3(x^2 + y^2) = 4 - (x^2 + y^2)$, i.e., on the cylinder $x^2 + y^2 = 1$. The volume they bound is given by

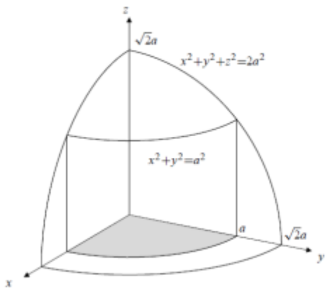
$$V = \iint_{x^2+y^2\le 1} \left[\frac{4 - x^2 - y^2}{3} - (x^2 + y^2)\right] dA = \int_0^{2\pi} d\theta \int_0^1 \left[\frac{4 - r^2}{3} - r^2\right] r dr = \frac{8\pi}{3}\int_0^1 (r - r^3) dr$$

$$= \frac{8\pi}{3}\left(\frac{r^2}{2} - \frac{r^4}{4}\right)\Big|_0^1 = \frac{2\pi}{3} \text{ cu. units.}$$

Question 8: Find the volume lying inside both the sphere $x^2 + y^2 + z^2 = 2a^2$ and the cylinder $x^2 + y^2 = a^2$.

The volume inside the sphere $x^2 + y^2 + z^2 = 2a^2$ and the cylinder $x^2 + y^2 = a^2$ is

$$V = 8\int_0^{\pi/2} d\theta \int_0^a \sqrt{2a^2 - r^2} r dr \text{ Let } u = 2a^2 - r^2 = 2\pi\int_{a^2}^{2a^2} u^{1/2} du = \frac{4\pi a^3}{3}(2\sqrt{2} - 1) \text{ cu. units.}$$



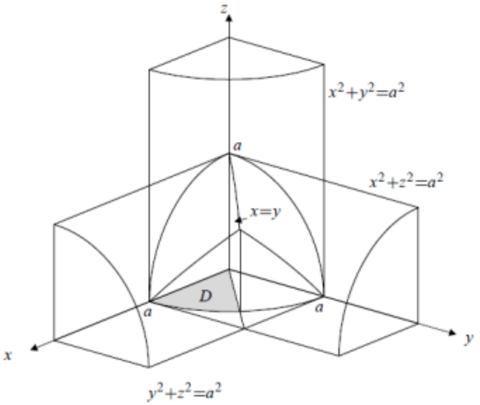
Question 9: Find the volume of the region lying inside all three of the circular cylinders $x^2 + y^2 = a^2, x^2 + z^2 = a^2$, and $y^2 + z^2 = a^2$. Hint: Make a good sketch of the first octant part of the region and use symmetry whenever possible.

One eighth of the required volume lies in the first octant. This eighth is divided into two equal parts by the plane $x = y$. One of these parts lies above the circular sector D in the xy-plane specified in polar coordinate by $0 \le r \le a, 0 \le \theta \le \pi/4$, and beneath the cylinder $z = \sqrt{a^2 - x^2}$. Thus, the total volume lying inside all three cylinders is

$$V = 16\iint_D \sqrt{a^2 - x^2} dA = 16\int_0^{\pi/4} d\theta \int_0^a \sqrt{a^2 - r^2\cos^2\theta} r dr \qquad \text{Let } u = a^2 - r^2\cos^2\theta$$

$$= 8\int_0^{\pi/4} \frac{d\theta}{\cos^2\theta}\int_{a^2\sin^2\theta}^{a^2} u^{1/2} du = \frac{16a^3}{3}\int_0^{\pi/4} \frac{1 - \sin^3\theta}{\cos^2\theta} d\theta = \frac{16a^3}{3}\int_0^{\pi/4}\left(\sec^2\theta - \frac{1 - \cos^2\theta}{\cos^2\theta}\sin\theta\right) d\theta$$

$$= \frac{16a^3}{3}\left(\tan\,\theta - \frac{1}{\cos\,\theta} - \cos\,\theta\right)\Bigg|_0^{\pi/4} = \frac{16a^3}{3}\left(1 - 0 - \sqrt{2} + 1 - \frac{1}{\sqrt{2}} + 1\right) = 16\left(1 - \frac{1}{\sqrt{2}}\right)a^3 \text{cu. units.}$$

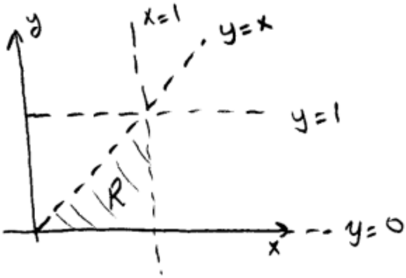


The Following Questions Are Obtained From PAST EXAMS

Question 1: a) Evaluate $\int_0^1 \int_y^1 sin(x^2)\,dx\,dy$ PE. SPRING2006

$$\iint_R sin(x^2)\,dA = \int_0^1 \int_0^x sin(x^2)\,dy\,dx = \int_0^1 y\sin(x^2)|_0^x\,dx = \int_0^1 x\sin(x^2)\,dx \Rightarrow \begin{cases} u = x^2 & x = 0 \rightarrow u = 0 \\ du = 2x\,dx & x = 1 \rightarrow u = 1 \end{cases}$$

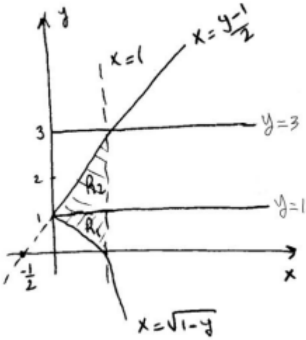
$$= \frac{1}{2}\int_0^1 \sin u\,du = -\frac{1}{2}\cos u|_0^1 = -\frac{1}{2}(\cos 1 - \cos 0) = \frac{1}{2}(1 - \cos 1)$$



b) After drawing the region for the integral, express $\int_0^1 \int_{\sqrt{1-y}}^1 f(x,y)\,dx\,dy + \int_1^3 \int_{\frac{y-1}{2}}^1 f(x,y)\,dx\,dy$ as one double integral.

$$x = \sqrt{1-y} \Rightarrow x^2 = 1-y \quad \Rightarrow y = 1-x^2 \qquad\qquad x = \frac{y-1}{2} \Rightarrow y = 2x+1$$

$$\iint_{R_1} f(x,y)\,dxdy + \iint_{R_2} f(x,y)\,dxdy = \int_0^1 \int_{1-x^2}^{2x+1} f(x,y)\,dy\,dx$$

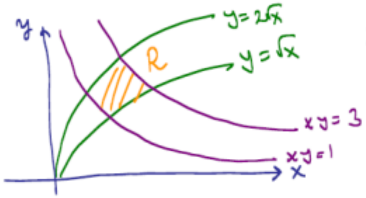


Question 2: Let R be the finite region, bounded by the line $y = x$ and the curve $y = x^2$. Find the volume of the solid over R bounded from below by the xy –plane and the graph of $f(x,y) = xy$. PE. SPRING2006

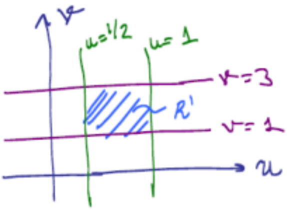
$$V = \iint_R f(x,y)dA = \int_0^1 \int_{x^2}^x xydydx = \frac{1}{2}\int_0^1 xy^2\Big|_{x^2}^x dx = \frac{1}{2}\int_0^1 x(x^2 - x^4)dx$$

$$= \frac{1}{2}\int_0^1 x^3 - x^5 dx = \frac{1}{2}\left[\frac{x^4}{4} - \frac{x^6}{6}\Big|_0^1\right] = \frac{1}{2}\left(\frac{1}{4} - \frac{1}{6}\right) = \frac{1}{24}$$

Question 3: Evaluate the double integral $\iint_R \frac{\sqrt{x}}{y} e^{-\frac{\sqrt{x}}{y}} dA$ where R is the region on the $xy-$plane bounded by the curves $y = 2\sqrt{x}, y = \sqrt{x}, xy = 1, xy = 3$. PE. SPRING2016



Let $u = \frac{\sqrt{x}}{y}$, $v = xy$



$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{\left|\frac{\partial(u,v)}{\partial(x,y)}\right|} = \frac{1}{\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}} = \frac{1}{\begin{vmatrix} \frac{1}{2\sqrt{xy}} & -\frac{\sqrt{x}}{y^2} \\ y & x \end{vmatrix}} = \frac{1}{\frac{3}{2}\frac{\sqrt{x}}{y}} = \frac{2}{3u}$$

$$\iint_R f(x\cdot y)dydx = \iint_{R'} f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dvdu = \int_{1/2}^{1}\int_{1}^{3} ue^{-u}\frac{2}{3u}dvdu = \int_{1/2}^{1}\frac{2}{3}e^{-u}v\Big|_{v=1}^{v=3} du$$

$$= \int_{1/2}^{1}\frac{4}{3}e^{-u}du = \frac{4}{3}(-e^{-u})\Big|_{1/2}^{1} = \frac{4}{3}\left(e^{-1/2} - e^{-1}\right)$$

Section 5.4 Triple Integrals

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: In following questions, evaluate the triple integrals over the indicated region. Be alert for simplifications and auspicious orders of iteration.

1. $\iiint_B xyzdV$, over the box B given by $0 \le x \le 1, -2 \le y \le 0, 1 \le z \le 4$

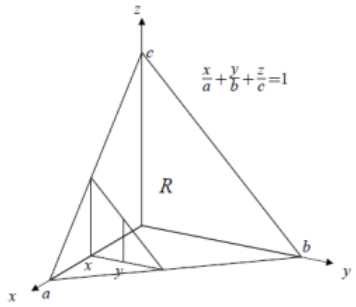
$$\iiint_B xyzdV = \int_0^1 x\,dx \int_{-2}^{0} y\,dy \int_1^4 z\,dz = \frac{1}{2}\left(-\frac{4}{2}\right)\left(\frac{16-1}{2}\right) = -\frac{15}{2}$$

2. $\iiint_R xdV$, over the tetrahedron bounded by the coordinate planes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

$$\iiint_R xdV = \int_0^a xdx \int_0^{b\left(1-\frac{x}{a}\right)} dy \int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dz = c\int_0^a xdx\int_0^{b\left(1-\frac{x}{a}\right)}\left(1-\frac{x}{a}-\frac{y}{b}\right)dy$$

$$= c\int_0^a x\left[b\left(1-\frac{x}{a}\right)^2 - \frac{b^2}{2b}\left(1-\frac{x}{a}\right)^2\right]dx = \frac{bc}{2}\int_0^a \left(1-\frac{x}{a}\right)^2 x dx$$ Let $u = 1-(x/a)$

$$= \frac{a^2bc}{2}\int_0^1 u^2(1-u)du = \frac{a^2bc}{24}$$

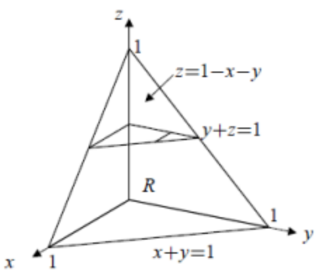


3. $\iiint_R (xy+z^2)dV$, over the set $0 \le z \le 1-|x|-|y|$

The set $R: 0 \le z \le 1-|x|-|y|$ is a pyramid, one quarter of which lies in the first octant and is bounded by the coordinate planes and the plane $x+y+z=1$. (See the figure.) By symmetry, the integral of xy over R is 0. Therefore,

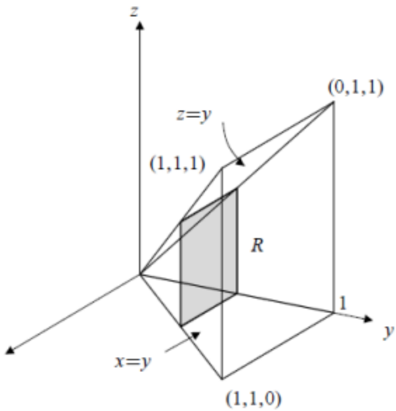
$$\iiint_R (xy+z^2)dV = \iiint_R z^2 dV = 4\int_0^1 z^2dz\int_0^{1-z}dy\int_0^{1-z-y}dx = 4\int_0^1 z^2dz\int_0^{1-z}(1-z-y)dy$$

$$= 4\int_0^1 z^2\left[(1-z)^2 - \frac{1}{2}(1-z)^2\right]dz = 2\int_0^1 (z^2-2z^3+z^4)dz = \frac{1}{15}$$



4. $\iiint_R \sin(\pi y^3)dV$, over the pyramid with vertices $(0,0,0)$, $(0,1,0)$, $(1,1,0)$, $(1,1,1)$, and $(0,1,1)$

$$\iiint_R \sin(\pi y^3)dV = \int_0^1 \sin(\pi y^3)dy\int_0^y dz\int_0^y dx = \int_0^1 y^2\sin(\pi y^3)dy = -\frac{\cos(\pi y^3)}{3\pi}\bigg|_0^1 = \frac{2}{3\pi}$$



5. $\iiint_R ydV$, over that part of the cube $0 \le x, y, z \le 1$ lying above the plane $y + z = 1$ and below the plane $x + y + z = 2$.

$$\iiint_R ydV = \int_0^1 ydy \int_{1-y}^1 dz \int_0^{2-y-z} dx = \int_0^1 ydy \int_{1-y}^1 (2-y-z)dz = \int_0^1 ydy\left((2-y)z - \frac{z^2}{2}\right)\Big|_{z=1-y}^{z=1}$$

$$= \int_0^1 y\left((2-y)y - \frac{1}{2}(1-(1-y)^2)\right)dy = \int_0^1 \frac{1}{2}(2y^2 - y^3)dy = \frac{5}{24}.$$

Question 2: Find the volume of the region lying inside the cylinder $x^2 + 4y^2 = 4$, above the xy-plane, and below the plane $z = 2 + x$.

Let E be the elliptic disk bounded by $x^2 + 4y^2 = 4$. Then E has area $\pi(2)(1) = 2\pi$ square units.

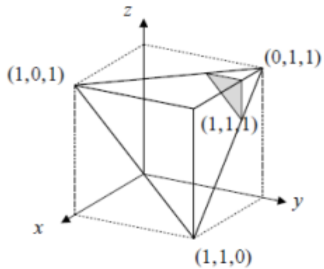
The volume of the region of 3 -space lying above E and beneath the plane $z = 2 + x$ is

$V = \iint_E (2 + x)dA = 2\iint_E dA = 4\pi$ cu. units, since $\iint_E xdA = 0$ by symmetry.

Question 3: Find $\iiint_T xdV$, where T is the tetrahedron bounded by the planes $x = 1, y = 1, z = 1$, and $x + y + z = 2$.

$$\iiint_T xdV = \int_0^1 xdx \int_{1-x}^1 dy \int_{2-x-y}^1 dz = \int_0^1 xdx \int_{1-x}^1 (x+y-1)dy = \int_0^1 x\left[\frac{(x-1)^2}{2} + x - \frac{1}{2}\right]dx$$

$$= \int_0^1 \frac{x^3}{2}dx = \frac{1}{8}$$



Section 5.5 Change of Variables in Triple Integrals

The Following Questions Are Obtained From **Calculus - A Complete Course 7th ed - R. Adams, C. Essex**

Question 1: In following questions, find the volumes of the indicated regions.

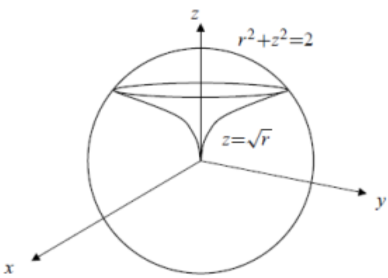
1. Above the surface $z = (x^2 + y^2)^{1/4}$ and inside the sphere $x^2 + y^2 + z^2 = 2$

The surface $z = \sqrt{r}$ intersects the sphere $r^2 + z^2 = 2$ where $r^2 + r - 2 = 0$. This equation has positive root

$r = 1$. The required volume is

$$V = \int_0^{2\pi} d\theta \int_0^1 rdr \int_{\sqrt{r}}^{\sqrt{2-r^2}} dz = \int_0^{2\pi} d\theta \int_0^1 \left(\sqrt{2-r^2} - \sqrt{r}\right) rdr \; = 2\pi\left(\left(\int_0^1 r\sqrt{2-r^2}dr - \frac{2}{5}\right)\right.$$

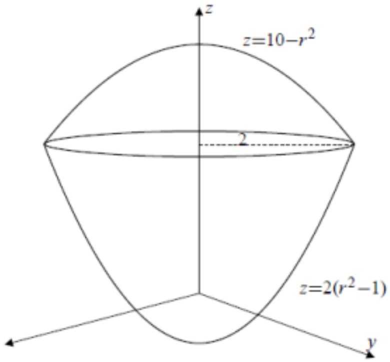
Let $u = 2 - r^2$ $\quad = \pi \int_1^2 u^{1/2} du - \frac{4\pi}{5} \; du = -2rdr = \frac{2\pi}{3}(2\sqrt{2} - 1) - \frac{4\pi}{5} = \frac{4\sqrt{2}\pi}{3} - \frac{22\pi}{15}$ cu. units.



2. Between the paraboloids $z = 10 - x^2 - y^2$ and $z = 2(x^2 + y^2 - 1)$

The paraboloids $z = 10 - r^2$ and $z = 2(r^2 - 1)$ intersect where $r^2 = 4$, that is, where $r = 2$. The volume lying between these surfaces is

$$V = \int_0^{2\pi} d\theta \int_0^2 [10 - r^2 - 2(r^2 - 1)]rdr = 2\pi \int_0^2 (12r - 3r^3)dr = 24\pi \text{ cu. units.}$$

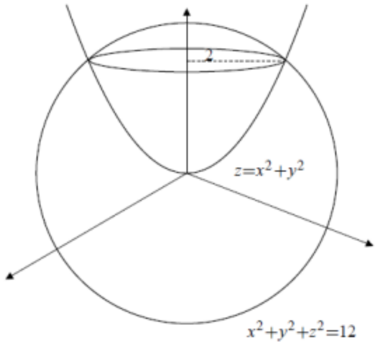


3. Inside the paraboloid $z = x^2 + y^2$ and inside the sphere $x^2 + y^2 + z^2 = 12$

The paraboloid $z = r^2$ intersects the sphere $r^2 + z^2 = 12$ where $r^4 + r^2 - 12 = 0$, that is, where $r = \sqrt{3}$. The required volume is

$$V = \int_0^{2\pi} d\theta \int_0^{\sqrt{3}} \left(\sqrt{12 - r^2} - r^2\right) rdr \ = 2\pi \int_0^{\sqrt{3}} r\sqrt{12 - r^2}dr - \frac{9\pi}{2} \qquad \text{Let } u = 12 - r^2$$

$$= \pi \int_9^{12} u^{1/2}du - \frac{9\pi}{2} \ du = -2rdr \ = \frac{2\pi}{3}\left(12^{3/2} - 27\right) - \frac{9\pi}{2} = 16\sqrt{3}\pi - \frac{45\pi}{2} \text{ cu. units.}$$



4. Above the xy-plane, under the paraboloid $z = 1 - x^2 - y^2$, and in the wedge $-x \le y \le \sqrt{3}x$

The required volume V lies above $z = 0$, below $z = 1 - r^2$, and between $\theta = -\pi/4$ and $\theta = \pi/3$. Thus

$$V = \int_{-\pi/4}^{\pi/3} d\theta \int_0^1 (1 - r^2) r dr = \frac{7\pi}{12}\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{7\pi}{48} \text{cu. units.}$$

Question 2: Evaluate $\iiint_R (x^2 + y^2 + z^2) dV$, where R is the cylinder $0 \le x^2 + y^2 \le a^2, 0 \le z \le h$.

$$\iiint_R (x^2 + y^2 + z^2) dV = \int_0^{2\pi} d\theta \int_0^a r dr \int_0^h (r^2 + z^2) dz = 2\pi \int_0^a \left(r^3 h + \frac{1}{3} r h^3\right) dr$$

$$= 2\pi \left(\frac{a^4 h}{4} + \frac{a^2 h^3}{6}\right) = \frac{\pi a^4 h}{2} + \frac{\pi a^2 h^3}{3}$$

Question 3: Find $\iiint_B (x^2 + y^2 + z^2) dV$, where B is the ball of $x^2 + y^2 + z^2 \le a^2$.

$$\iiint_B (x^2 + y^2 + z^2) dV = \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi \, d\phi \int_0^a R^4 dR = \frac{4\pi a^5}{5}$$

Question 4: Find $\iiint_R x dV$ and $\iiint_R z dV$, over that part of the hemisphere $0 \le z \le \sqrt{a^2 - x^2 - y^2}$ that lies in the first octant.

By symmetry, both integrals have the same value:

$$\iiint_R x dV = \iiint_R z dV = \int_0^{\pi/2} d\theta \int_0^{\pi/2} \cos\phi \sin\phi d\phi \int_0^a R^3 dR = \frac{\pi}{2}\left(\frac{1}{2}\right)\frac{a^4}{4} = \frac{\pi a^4}{16}$$

Section 6.1 Vector Functions of One Variable

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: In following questions, find the velocity, speed, and acceleration at time t of the particle whose position is $\mathbf{r}(t)$. Describe the path of the particle.

1. $\mathbf{r} = a\cos\ \omega t\mathbf{i} + b\mathbf{j} + a\sin\ \omega t\mathbf{k}$

Position: $\mathbf{r} = a\cos\ \omega t\mathbf{i} + b\mathbf{j} + a\sin\ \omega t\mathbf{k}$ Velocity: $\mathbf{v} = -a\omega\sin\ \omega t\mathbf{i} + a\omega\cos\ \omega t\mathbf{k}$

Speed: $v = |a\omega|$ Acceleration: $\mathbf{a} = -a\omega^2\cos\ \omega t\mathbf{i} - a\omega^2\sin\ \omega t\mathbf{k}$

Path: the circle $x^2 + z^2 = a^2, y = b$.

2. $\mathbf{r} = 3\cos\ t\mathbf{i} + 4\sin\ t\mathbf{j} + t\mathbf{k}$

Position: $\mathbf{r} = 3\cos\ t\mathbf{i} + 4\sin\ t\mathbf{j} + t\mathbf{k}$ Velocity: $\mathbf{v} = -3\sin\ t\mathbf{i} + 4\cos\ t\mathbf{j} + \mathbf{k}$

Speed: $v = \sqrt{9\sin^2\ t + 16\cos^2\ t + 1} = \sqrt{10 + 7\cos^2\ t}$

Acceleration : $\mathbf{a} = -3\cos\ t\mathbf{i} - 4\sin\ t\mathbf{j} = t\mathbf{k} - \mathbf{r}$

Path: a helix (spiral) wound around the elliptic cylinder $(x^2/9) + (y^2/16) = 1$

Question 2: A particle moves to the right along the curve $y = 3/x$. If its speed is 10 when it passes through the point $\left(2, \frac{3}{2}\right)$, what is its velocity at that time?

When its x-coordinate is x, the particle is at position $\mathbf{r} = x\mathbf{i} + (3/x)\mathbf{j}$, and its velocity and speed are

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} - \frac{3}{x^2}\frac{dx}{dt}\mathbf{j} \qquad v = \left|\frac{dx}{dt}\right|\sqrt{1 + \frac{9}{x^4}}$$

We know that $dx/dt > 0$ since the particle is moving to the right. When $x = 2$, we have

$10 = v = (dx/dt)\sqrt{1 + (9/16)} = (5/4)(dx/dt)$. Thus $dx/dt = 8$. The velocity at that time is $\mathbf{v} = 8\mathbf{i} - 6\mathbf{j}$.

Question 3: An object moves along the curve $y = x^2, z = x^3$, with constant vertical speed $dz/dt = 3$. Find the velocity and acceleration of the object when it is at the point (2,4,8).

The position of the object when its x-coordinate is x is $\mathbf{r} = x\mathbf{i} + x^2\mathbf{j} + x^3\mathbf{k}$

so, its velocity is $\mathbf{v} = \frac{dx}{dt}[\mathbf{i} + 2x\mathbf{j} + 3x^2\mathbf{k}]$. Since $dz/dt = 3x^2 dx/dt = 3$, when $x = 2$ we have $12dx/dt = 3$, so $dx/dt = 1/4$. Thus, $\mathbf{v} = \frac{1}{4}\mathbf{i} + \mathbf{j} + 3\mathbf{k}$

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: In following questions, find the required parametrization of the first quadrant part of the circular arc $x^2+y^2=a^2$.

1. In terms of the y-coordinate, oriented counterclockwise

On the first quadrant part of the circle $x^2+y^2=a^2$, we have $x=\sqrt{a^2-y^2}, 0\le y\le a$. The required parametrization is

$$\mathbf{r}=\mathbf{r}(y)=\sqrt{a^2-y^2}\mathbf{i}+y\mathbf{j},\ (0\le y\le a).$$

2. In terms of the x-coordinate, oriented clockwise

On the first quadrant part of the circle $x^2+y^2=a^2$, we have $y=\sqrt{a^2-x^2}, 0\le x\le a$. The required parametrization is

$$\mathbf{r}=\mathbf{r}(x)=x\mathbf{i}+\sqrt{a^2-x^2}\mathbf{j},\ (0\le x\le a)$$

3. In terms of the angle between the tangent line and the positive x-axis, oriented counterclockwise

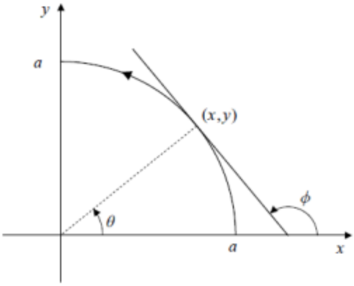
From the figure we see that

$$\phi=\theta+\frac{\pi}{2}, \qquad 0\le\theta\le\frac{\pi}{2}$$

$$x=a\cos\theta=a\cos\left(\phi-\frac{\pi}{2}\right)=a\sin\phi \qquad\qquad y=a\sin\,\theta=a\sin\,\left(\phi-\frac{\pi}{2}\right)=-a\cos\,\phi$$

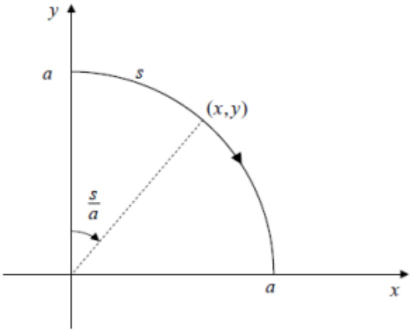
The required parametrization is

$$\mathbf{r}=a\sin\,\phi\mathbf{i}-a\cos\,\phi\mathbf{j},\ \left(\frac{\pi}{2}\le\phi\le\pi\right).$$



4. In terms of arc length measured from $(0, a)$, oriented clockwise

$x = a\sin\frac{s}{a},\ y = a\cos\frac{s}{a},\ 0 \le \frac{s}{a} \le \frac{\pi}{2}$ $\qquad$ $\mathbf{r} = a\sin\frac{s}{a}\mathbf{i} + a\cos\frac{s}{a}\mathbf{j},\ \left(0 \le s \le \frac{a\pi}{2}\right)$



5. The plane $x + y + z = 1$ intersects the cylinder $z = x^2$ in a parabola. Parametrize the parabola using $t = x$ as parameter.

$z = x^2, x + y + z = 1$. If $t = x$, then

$z = t^2$ and $y = 1 - t - t^2$. The parametrization is $\mathbf{r} = t\mathbf{i} + (\mathbf{l} - t - t^2)\mathbf{j} + t^2\mathbf{k}$

Question 2: parametrize the curve of intersection of the given surfaces. Note: the answers are not unique.

$z = \sqrt{1 - x^2 - y^2}$ and $x + y = 1$

$x + y = 1, z = \sqrt{1 - x^2 - y^2}$. If $x = t$, then $y = 1 - t$ and $z = \sqrt{1 - t^2 - (1 - t)^2} = \sqrt{2(t - t^2)}$. One possible parametrization is $\mathbf{r} = t\mathbf{i} + (1 - t)\mathbf{j} + \sqrt{2(t - t^2)}\mathbf{k}$

Question 3: Find the length of the conical helix $\mathbf{r} = t\cos\ t\mathbf{i} + t\sin\ t\mathbf{j} + t\mathbf{k}, (0 \le t \le 2\pi)$. Why is the curve called a conical helix?

$\mathbf{r} = t\cos t\mathbf{i} + t\sin t\mathbf{j} + t\mathbf{k},\ 0 \le t \le 2\pi$

$\mathbf{v} = (\cos\ t - t\sin\ t)\mathbf{i} + (\sin\ t + t\cos\ t)\mathbf{j} + \mathbf{k}$

$v = |\mathbf{v}| = \sqrt{(1 + t^2) + 1} = \sqrt{2 + t^2}$

The length of the curve is $L = \int_0^{2\pi} \sqrt{2 + t^2}dt$ Let $t = \sqrt{2}\tan\ \theta$

$$= 2\int_{t=0}^{t=2\pi} \sec^3 \theta d\theta = (\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|)|_{t=0}^{t=2\pi} \sec^2\ \theta d\theta = \frac{t\sqrt{2 + t^2}}{2} + \ln\left(\frac{\sqrt{2 + t^2}}{\sqrt{2}} + \frac{t}{\sqrt{2}}\right)\Bigg|_0^{2\pi}$$

$= \pi\sqrt{2 + 4\pi^2} + \ln\left(\sqrt{1 + 2\pi^2} + \sqrt{2}\pi\right)$ units.

The curve is called a conical helix because it is a spiral lying on the cone $x^2 + y^2 = z^2$.

Question 4: Describe the intersection of the sphere $x^2 + y^2 + z^2 = 1$ and the elliptic cylinder $x^2 + 2z^2 = 1$. Find the total length of this intersection curve.

One-eighth of the curve C lies in the first octant. That part can be parametrized

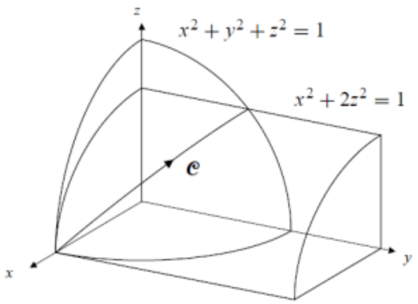
$x = \cos t,\ z = \frac{1}{\sqrt{2}}\sin t,\ \left(0 \le t \le \frac{\pi}{2}\right)$

$y = \sqrt{1 - \cos^2\ t - \frac{1}{2}\sin^2\ t} = \frac{1}{\sqrt{2}}\sin\ t$

Since the first octant part of $\mathcal{C}$ lies in the plane $y = z$, it must be a quarter of a circle of radius 1. Thus, the length of all of $\mathcal{C}$ is $8 \times (\pi/2) = 4\pi$ units.

If you wish to use an integral, the length is

$$8\int_0^{\pi/2} \sqrt{\sin^2\ t + \frac{1}{2}\cos^2\ t + \frac{1}{2}\cos^2\ t}dt = 8\int_0^{\pi/2} dt = 4\pi \text{ units.}$$



Question 5: reparametrize the given curve in the same orientation in terms of arc length measured from the point where $t=0$.

$\mathbf{r}=e^t\mathbf{i}+\sqrt{2}t\mathbf{j}-e^{-t}\mathbf{k}$

$\mathbf{v}=e^t\mathbf{i}+\sqrt{2}\mathbf{j}+e^{-t}\mathbf{k}$ $\qquad v=|\mathbf{v}|=\sqrt{e^{2t}+2+e^{-2t}}=e^t+e^{-t}.$

The arc length from the point where $t=0$ to the point corresponding to arbitrary t is

$$s=s(t)=\int_0^t(e^u+e^{-u})du=e^t-e^{-t}=2\sinh t.$$

Thus $t=\sinh^{-1}(s/2)=\ln\left(\frac{s+\sqrt{s^2+4}}{2}\right)$, and $e^t=\frac{s+\sqrt{s^2+4}}{2}$. The required parametrization is

$$\mathbf{r}=\frac{s+\sqrt{s^2+4}}{2}\mathbf{i}+\sqrt{2}\ln\left(\frac{s+\sqrt{s^2+4}}{2}\right)\mathbf{j}-\frac{2\mathbf{k}}{s+\sqrt{s^2+4}}$$

Section 7.1 Line Integrals

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: Let $\mathcal{C}$ be the conical helix with parametric equations $x=t\cos t, y=t\sin t, z=t, (0\le t\le 2\pi)$. Find $\int_{\mathcal{C}} zds$.

$\mathcal{C}$ is given by $\mathbf{r}=t^2\mathbf{i}+t\mathbf{j}+t^2\mathbf{k}$, for $0\le t\le 1$. Thus,

$\frac{d\mathbf{r}}{dt}=2t\mathbf{i}+\mathbf{j}+2t\mathbf{k}$ $\qquad \left|\frac{d\mathbf{r}}{dt}\right|=\sqrt{4t^2+1+4t^2}=\sqrt{1+8t^2}$ $\qquad \int_{\mathcal{C}} yds=\int_0^1 t\sqrt{1+8t^2}dt$

Let $1+8t^2=u$ $\qquad = \dfrac{1}{16}\displaystyle\int_1^9 \sqrt{u}du = \dfrac{1}{16}\cdot\dfrac{2}{3}\cdot u^{3/2}\Big|_1^9 = \dfrac{27-1}{24} = \dfrac{13}{12}$

Question 2: Find $\int_C \sqrt{1+4x^2z^2}ds$, where C is the curve of intersection of the surfaces $x^2+z^2=1$ and $y=x^2$.

C: $\mathbf{r} = e^t\,cos\,t\,\mathbf{i} + e^t\,sin\,t\,\mathbf{j} + t\,\mathbf{k}$, for $0 \le t \le 2\pi$

We have $\displaystyle\int_C e^z ds = \int_0^{2\pi} e^t\sqrt{1+2e^{2t}}dt$ $\qquad$ Let $\sqrt{2}e^t = \tan\,\theta$

$$= \frac{1}{\sqrt{2}}\int_{t=0}^{t=2\pi} \sec^3\theta\, d\theta = \frac{1}{2\sqrt{2}}[\sec\theta\tan\theta + \sec^2\theta d\theta = \frac{\sqrt{2}e^t\sqrt{1+2e^{2t}} + \ln\left(\sqrt{2}e^t+\sqrt{1+2e^{2t}}\right)}{2\sqrt{2}}\Bigg|_0^{2\pi}$$

$$= \frac{e^{2\pi}\sqrt{1+2e^{4\pi}}-\sqrt{3}}{2} + \frac{1}{2\sqrt{2}}\ln\frac{\sqrt{2}e^{2\pi}+\sqrt{1+2e^{4\pi}}}{\sqrt{2}+\sqrt{3}}$$

Question 3: Find $\int_C xds$ along the first octant part of the curve of intersection of the cylinder $x^2+y^2=a^2$ and the plane $z=x$.

$\mathbf{r} = e^t\mathbf{i} + \sqrt{2}t\mathbf{j} + e^{-t}\mathbf{k}$, $(0 \le t \le 1)$ $\qquad \mathbf{v} = e^t\mathbf{i} + \sqrt{2}\mathbf{j} - e^{-t}\mathbf{k}$ $\qquad v = \sqrt{e^{2t}+2+e^{-2t}} = e^t + e^{-t}$

$$\int_C (x^2+z^2)ds = \int_0^1 (e^{2t}+e^{-2t})(e^t+e^{-t})dt = \int_0^1 (e^{3t}+e^t+e^{-t}+e^{-3t})dt = \frac{e^3}{3} + e - \frac{1}{e} - \frac{1}{3e^3}.$$

Section 7.2 Vector and Scalar Fields

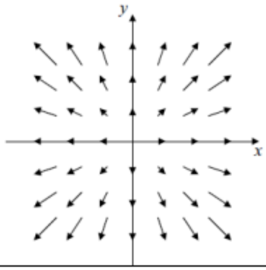
The Following Questions Are Obtained From **Calculus - A Complete Course 7th ed - R. Adams, C. Essex**

Question 1: In following questions, sketch the given plane vector field and determine its field lines.

1. $\mathbf{F}(x,y) = x\mathbf{i} + y\mathbf{j}$

The field lines satisfy $\frac{dx}{x} = \frac{dy}{y}$.

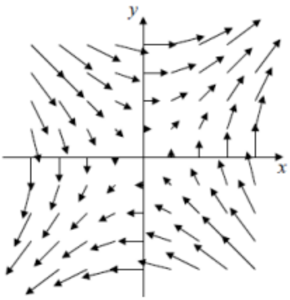
Thus, $\ln\,y = \ln\,x + \ln\,C$, or $y = Cx$. The field lines are straight half-lines emanating from the origin.



2. $\mathbf{F}(x, y) = y\mathbf{i} + x\mathbf{j}$

The field lines satisfy $\frac{dx}{y} = \frac{dy}{x}$.

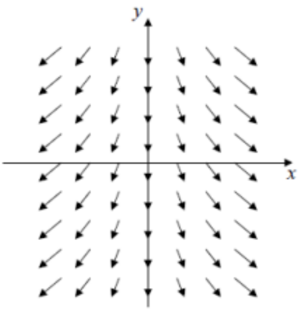
Thus, $x\,dx = y\,dy$. The field lines are the rectangular hyperbolas (and their asymptotes) given by $x^2 - y^2 = C$.



3. $\mathbf{F}(x, y) = \nabla(x^2 - y)$

$$\mathbf{F} = \nabla(x^2 - y) = 2x\mathbf{i} - \mathbf{j}.$$

They are the curves $y = -\frac{1}{2}\ln x + C$.



Section 7.3 Gradient, Divergence, and Curl

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: In following questions, calculate div **F** and curl **F** for the given vector fields.

1. $\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$

$$\text{div F} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(x) = 0$$

$$\text{curl F} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -\mathrm{i} - \mathrm{j} - \mathrm{k}$$

2. $\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

$$\text{div } \mathbf{F} = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy) = 0$$

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = (x - x)\mathbf{i} + (y - y)\mathbf{j} + (z - z)\mathbf{k} = \mathbf{0}$$

Section 7.4 Conservative Fields

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: In following questions, determine whether the given vector field is conservative, and find a potential if it is.

1. $\mathbf{F}(x, y, z) = y\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$

$F_1 = y, F_2 = x, F_3 = z^2$. We have

$$\frac{\partial F_1}{\partial y} = 1 = \frac{\partial F_2}{\partial x} \qquad \frac{\partial F_1}{\partial z} = 0 = \frac{\partial F_3}{\partial x} \qquad \frac{\partial F_2}{\partial z} = 0 = \frac{\partial F_3}{\partial y}$$

Therefore, $\mathbf{F}$ may be conservative. If $\mathbf{F} = \nabla\phi$, then

$$\frac{\partial \phi}{\partial x} = y, \ \frac{\partial \phi}{\partial y} = x, \ \frac{\partial \phi}{\partial z} = z^2.$$

Therefore,

$$\phi(x, y, z) = \int y\,dx = xy + C_1(y, z) \qquad x = \frac{\partial \phi}{\partial y} = x + \frac{\partial C_1}{\partial y} \Rightarrow \frac{\partial C_1}{\partial y} = 0$$

$C_1(y,z) = C_2(z)$, $\phi(x,y,z) = xy + C_2(z)$ $\quad\quad$ $z^2 = \dfrac{\partial\phi}{\partial z} = C_2'(z) \Rightarrow C_2(z) = \dfrac{z^3}{3}$

Thus, $\phi(x,y,z) = xy + \frac{z^3}{3}$ is a potential for $\mathbf{F}$, and $\mathbf{F}$ is conservative on $\mathbb{R}^3$.

2. $\mathbf{F}(x,y,z) = e^{x^2+y^2+z^2}(xz\mathbf{i} + yz\mathbf{j} + xy\mathbf{k})$

$F_1 = xze^{x^2+y^2+z^2}$ $\quad$ $F_2 = yze^{x^2+y^2+z^2}$ $\quad$ $F_3 = xye^{x^2+y^2+z^2}$. We have

$\dfrac{\partial F_1}{\partial y} = 2xyze^{x^2+y^2+z^2} = \dfrac{\partial F_2}{\partial x}$ $\quad$ $\dfrac{\partial F_1}{\partial z} = (x + 2xz^2)e^{x^2+y^2+z^2}$ $\quad$ $\dfrac{\partial F_3}{\partial x} = (y + 2x^2y)e^{x^2+y^2+z^2} \neq \dfrac{\partial F_1}{\partial z}$

Thus $\mathbf{F}$ cannot be conservative.

Question 2: Show that the vector field

$$\mathbf{F}(x,y,z) = \frac{2x}{z}\mathbf{i} + \frac{2y}{z}\mathbf{j} - \frac{x^2+y^2}{z^2}\mathbf{k}$$

is conservative and find its potential. Describe the equipotential surfaces. Find the field lines of $\mathbf{F}$.

$F_1 = \dfrac{2x}{z}$ $\quad$ $F_2 = \dfrac{2y}{z}$ $\quad$ $F_3 = -\dfrac{x^2+y^2}{z^2}$. We have

$\dfrac{\partial F_1}{\partial y} = 0 = \dfrac{\partial F_2}{\partial x}$, $\quad$ $\dfrac{\partial F_1}{\partial z} = -\dfrac{2x}{z^2} = \dfrac{\partial F_3}{\partial x}$, $\quad$ $\dfrac{\partial F_2}{\partial z} = -\dfrac{2y}{z^2} = \dfrac{\partial F_3}{\partial y}$.

Therefore, $\mathbf{F}$ may be conservative in $\mathbb{R}^3$ except on the plane $z = 0$ where it is not defined. If $\mathbf{F} = \nabla\phi$, then

$$\frac{\partial\phi}{\partial x} = \frac{2x}{z},\ \frac{\partial\phi}{\partial y} = \frac{2y}{z},\ \frac{\partial\phi}{\partial z} = -\frac{x^2+y^2}{z^2}$$

Therefore,

$\phi(x,y,z) = \displaystyle\int \frac{2x}{z}dx = \frac{x^2}{z} + C_1(y,z)$ $\quad\quad$ $\dfrac{2y}{z} = \dfrac{\partial\phi}{\partial y} = \dfrac{\partial C_1}{\partial y} \Rightarrow C_1(y,z) = \dfrac{y^2}{z} + C_2(z)$

$\phi(x,y,z) = \dfrac{x^2+y^2}{z} + C_2(z)$ $-\dfrac{x^2+y^2}{z^2} = \dfrac{\partial\phi}{\partial z} = -\dfrac{x^2+y^2}{z^2} + C_2'(z) \Rightarrow C_2(z) = 0.$

Thus $\phi(x,y,z) = \frac{x^2+y^2}{z}$ is a potential for $\mathbf{F}$, and $\mathbf{F}$ is conservative on $\mathbb{R}^3$ except on the plane $z = 0$.

The equipotential surfaces have equations

$$\frac{x^2+y^2}{z} = C, \text{ or } Cz = x^2+y^2$$

Thus, the equipotential surfaces are circular paraboloids.

The field lines of **F** satisfy

$$\frac{dx}{\frac{2x}{z}} = \frac{dy}{\frac{2y}{z}} = \frac{dz}{-\frac{x^2+y^2}{z^2}}$$

From the first equation, $\frac{dx}{x} = \frac{dy}{y}$, so $y = Ax$ for an arbitrary constant A. Therefore

$$\frac{dx}{2x} = \frac{zdz}{-(x^2+y^2)} = \frac{zdz}{-x^2(1+A^2)}$$

so $-(1+A^2)xdx = 2zdz$. Hence

$$\frac{1+A^2}{2}x^2 + z^2 = \frac{B}{2}$$

or $x^2+y^2+2z^2 = B$, where B is a second arbitrary constant. The field lines of F are the ellipses in which the vertical planes containing the z-axis intersect the ellipsoids $x^2+y^2+2z^2 = B$. These ellipses are orthogonal to all the equipotential surfaces of F.

Section 7.5 Line Integrals of Vector Fields

The Following Questions Are Obtained From Calculus - A Complete Course 7th ed - R. Adams, C. Essex

Question 1: In following questions, evaluate the line integral of the tangential component of the given vector field along the given curve.

1. $\mathbf{F}(x,y,z) = z\mathbf{i} - y\mathbf{j} + 2x\mathbf{k}$ along the curve $x = t, y = t^2, z = t^3$ from $(0,0,0)$ to $(1,1,1)$

$\mathcal{C}: \mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}, (0 \le t \le 1)$.

$$\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} = \int_0^1 [t^3 - t^2(2t) + 2t(3t^2)]dt = \int_0^1 5t^3 dt = \frac{5t^4}{4}\Bigg|_0^1 = \frac{5}{4}$$

2. $\mathbf{F}(x,y,z) = (x-z)\mathbf{i} + (y-z)\mathbf{j} - (x+y)\mathbf{k}$ along the polygonal path from $(0,0,0)$ to $(1,0,0)$ to $(1,1,0)$ to $(1,1,1)$

$$\mathbf{F} = (x-z)\mathbf{i} + (y-z)\mathbf{j} - (x+y)\mathbf{k} = \nabla\left(\frac{x^2+y^2}{2} - (x+y)z\right)$$

$\mathcal{C}$ is a given polygonal path from $(0,0,0)$ to $(1,1,1)$ (but any other piecewise smooth path from the first point to the second would do as well).

$$\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} = \left(\frac{x^2+y^2}{2} - (x+y)z\right)\Bigg|_{(0,0,0)}^{(1,1,1)} = 1-2=-1$$

Question 2: Evaluate $\oint_{\mathcal{C}} x^2y^2dx + x^3ydy$ counterclockwise around the square with vertices $(0,0), (1,0), (1,1)$, and $(0,1)$.

C is made up of four segments as shown in the figure.

On $\mathcal{C}_1, y=0, dy=0$, and x goes from 0 to 1.

On $\mathcal{C}_2, x=1, dx=0$, and y goes from 0 to 1.

On $\mathcal{C}_3, y=1, dy=0$, and x goes from 1 to 0.

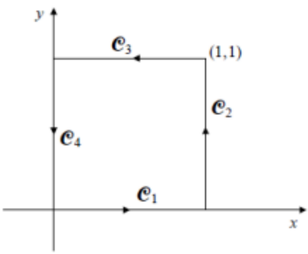
On $\mathcal{C}_4, x=0, dx=0$, and y goes from 1 to 0.

Thus, $\int_{\mathcal{C}_1} x^2y^2dx + x^3ydy = 0$ $\quad$ $\int_{\mathcal{C}_2} x^2y^2dx + x^3ydy = \int_0^1 ydy = \frac{1}{2}$

$$\int_{\mathcal{C}_3} x^2y^2dx + x^3ydy = \int_1^0 x^2dx = -\frac{1}{3} \qquad \int_{\mathcal{C}_4} x^2y^2dx + x^3ydy = 0$$

Finally, therefore,

$$\int_{\mathcal{C}} x^2y^2dx + x^3ydy = 0 + \frac{1}{2} - \frac{1}{3} + 0 = \frac{1}{6}$$



Question 3: Evaluate $\int_{\mathcal{C}} e^{x+y}\sin(y+z)dx + e^{x+y}(\sin(y+z)+\cos(y+z))dy + e^{x+y}\cos(y+z)dz$

along the straight-line segment from (0,0,0) to $\left(1,\frac{\pi}{4},\frac{\pi}{4}\right)$.

Observe that if $\phi = e^{x+y}\sin(y+z)$, then

$$\nabla\phi = e^{x+y}\sin(y+z)\mathbf{i} + e^{x+y}(\sin(y+z)+\cos(y+z))\mathbf{j} + e^{x+y}\cos(y+z)\mathbf{k}$$

Thus, for any piecewise smooth path from (0,0,0) to $\left(1,\frac{\pi}{4},\frac{\pi}{4}\right)$, we have

$$\int_{\mathcal{C}} e^{x+y}\sin(y+z)\,dx + e^{x+y}(\sin(y+z)+\cos(y+z)) + e^{x+y}\cos(y+z)\,dz$$

$$= \int_{\mathcal{C}} \nabla\phi\cdot d\mathbf{r} = \phi(x,y,z)\Big|_{(0,0,0)}^{(1,\pi/4,\pi/4)} = e^{1+(z/4)}$$

Question 4: If $\mathcal{C}$ is the intersection of $z=\ln(1+x)$ and $y=x$ from (0,0,0) to $(1,1,\ln 2)$, evaluate

$$\int_{\mathcal{C}} (2x\sin(\pi y)-e^z)dx + (\pi x^2\cos(\pi y)-3e^z)dy - xe^z dz$$

For $z=\ln(1+x), y=x$, from $x=0$ to $x=1$, we have

$$\int_{\mathcal{C}}[(2x\sin(\pi y)-e^z)dx + (\pi x^2\cos(\pi y)-3e^z)dy - xe^z dz] = \int_{\mathcal{C}} \nabla(x^2\sin(\pi y)-xe^z)\cdot d\mathbf{r} - 3\int_{\mathcal{C}} e^z dy$$

$$= (x^2\sin(\pi y)-xe^z)\Big|_{(0,0,0)}^{(1,1,\ln 2)} - 3\int_0^1 (1+x)dx = -2-3\left(x+\frac{x^2}{2}\right)\Big|_0^1 = -2-\frac{9}{2} = -\frac{13}{2}$$

Question 5: Evaluate

$$\frac{1}{2\pi}\oint_{\mathcal{C}} \frac{-ydx+xdy}{x^2+y^2}$$

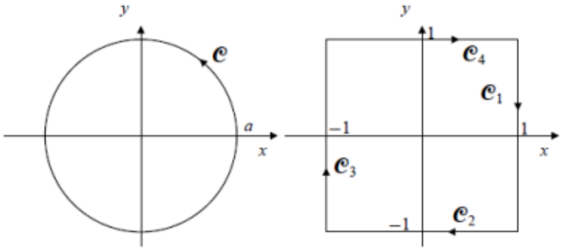
(a) counterclockwise around the circle $x^2+y^2=a^2$,

(b) clockwise around the square with vertices $(-1,-1)$, $(-1,1)$, $(1,1)$, and $(1,-1)$,

(c) counterclockwise around the boundary of the region $1\le x^2+y^2\le 4, y\ge 0$.

a) $\mathcal{C}: x = a\cos t, x = a\sin t, 0\le t\le 2\pi$.

$$\frac{1}{2\pi}\oint_{\mathcal{C}} \frac{-ydx+xdy}{x^2+y^2} = \frac{1}{2\pi}\int_0^{2\pi} \frac{a^2\cos^2 t + a^2\sin^2 t}{a^2\cos^2 t + a^2\sin^2 t}dt = 1$$



b) See the figure. $\mathcal{C}$ has four parts.

On $C_1, x = 1, dx = 0, y$ goes from 1 to -1.

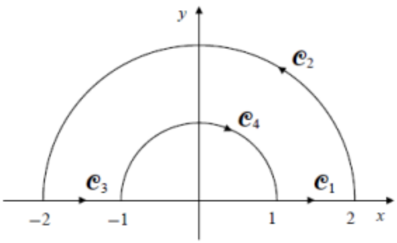
On $C_2, y = -1, dy = 0, x$ goes from 1 to -1.

On $C_3, x = -1, dx = 0, y$ goes from -1 to 1.

On $C_4, x = 1, dx = 0, y$ goes from 1 to -1.

$$\frac{1}{2\pi}\oint_{\mathcal{C}}\frac{xdy - ydx}{x^2 + y^2} = \frac{1}{2\pi}\left[\int_1^{-1}\frac{dy}{1+y^2} + \int_1^{-1}\frac{dx}{x^2+1} + \int_{-1}^{1}\frac{-dy}{1+y^2} + \int_{-1}^{1}\frac{-dx}{x^2+1}\right] = -\frac{2}{\pi}\int_{-1}^{1}\frac{dt}{1+t^2}$$

$$= -\frac{2}{\pi}\tan^{-1} t\,\Big|_{-1}^{1} = -\frac{2}{\pi}\left(\frac{\pi}{4}+\frac{\pi}{4}\right) = -1$$



See the figure. $\mathcal{C}$ has four parts.

On $\mathcal{C}_1, y = 0, dy = 0, x$ goes from 1 to 2.

On $\mathcal{C}_2, x = 2\cos\ t, y = 2\sin\ t, t$ goes from 0 to π.

On $\mathcal{C}_3, y = 0, dy = 0, x$ goes from -2 to -1.

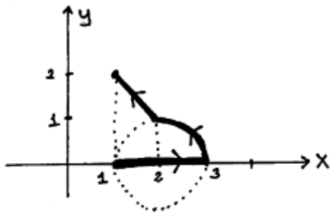
On $\mathcal{C}_4, x = \cos\ t, y = \sin\ t, t$ goes from π to 0.

$$\frac{1}{2\pi}\oint_{\mathcal{C}}\frac{xdy - ydx}{x^2+y^2} = = \frac{1}{2\pi}\left[0 + \int_0^{\pi}\frac{4\cos^2\ t + 4\sin^2\ t}{4\cos^2\ t + 4\sin^2\ t}dt\ + 0 + \int_{\pi}^{0}\frac{\cos^2\ t + \sin^2\ t}{\cos^2\ t + \sin^2\ t}dt\right] = \frac{1}{2\pi}(\pi - \pi) = 0.$$

Question 1: It is given that $F(x,y) = (P(x,y), Q(x,y))$ where PE. SPRING2006

$P(x,y) = e^x \sin(y) + 2y,$ $\qquad Q(x,y) = e^x \cos(y) + 2x - 2y$ and

C_1 is the curve from the point $(1,0)$ to $(1,2)$ as in the figure:



a) Prove that $Pdx + Qdy$ is exact (that is (P,Q) is conservative), by finding a potential function $\phi(x,y)$such that $\phi_x = P, \phi_y = Q$.

$P_y = e^x \cos y + 2 = Q_x$ and P, Q are polynomials so (P,Q) is exact on $\mathbb{R}^2$.

$$\phi_x = P = e^x \sin y + 2y \qquad \therefore \phi = e^x \sin y + 2yx + c(y)$$

$$\phi_y = e^x \cos y + 2x + c'(y) = Q = e^x \cos(y) + 2x - 2y \qquad \therefore c'(y) = -2y \ \rightarrow c(y) = -y^2 + c$$

$$\phi(x,y) = e^x \sin y + 2yx - y^2 + c$$

b) Using ϕ, evaluate $\int_{C_1} Pdx + Qdy$. Since (P,Q) is conservative, the given the integral depands only on the end points of C_1.

Hence $\int_c Pdx + Qdy = \Phi$ (terminal point of C_1) $-\Phi$ (initial point of C_1) $= \phi(1,2) - \phi(1,2) = e \ sin \ 2$

c) Evaluate $\int_{C_2} Pdx + Qdy$ where C_2 is the line segment from the point (1,0) to the point (1,2).

First way: since (P,Q) is conservative, integral depends only on the end points of C_2 which are the same as of C_1, hence

$$\int_{C_2} Pdx + Qdy = \int_{C_1} Pdx + Qdy = e \ sin \ 2$$

Second way: C_2: from $(1,0)$ to $(1,2)$ $\qquad x(t) = 1 \ and \ \ y(t) = 2t \ \ t \in [0,1] \quad \frac{dx}{dt} = 0 \quad \frac{dy}{dt} = 2$

$\int_{c_2} P\,dx + Q\,dy = \int_0^1 (e\sin 2t + 4t)\cdot 0 + (e\cos 2t + 2 - 4t)\cdot 2 = \int_0^1 2e\cos 2t + 4 - 8t\,dt$

$$= 2\,e\sin\frac{2t}{2} + 4t - \frac{8t^2}{2}\Big|_0^1 = e\,sin\;2$$

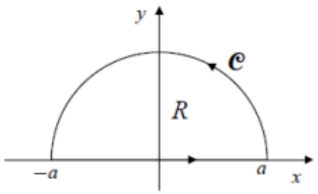
Section 8.1 Green's Theorem in the Plane

The Following Questions Are Obtained From **Calculus - A Complete Course 7th ed - R. Adams, C. Essex**

Question 1: Evaluate $\oint_{\mathcal{C}}(\sin\, x + 3y^2)dx + \left(2x - e^{-y^2}\right)dy$, where $\mathcal{C}$ is the boundary of the half-disk $x^2 + y^2 \leq a^2, y \geq 0$, oriented counterclockwise.

$$\oint_{\mathcal{C}}(\sin\, x + 3y^2)dx + \left(2x - e^{-y^2}\right)dy = \iint_R \left[\frac{\partial}{\partial x}\left(2x - e^{-y^2}\right) - \frac{\partial}{\partial y}(\sin\, x + 3y^2)\right]dA$$

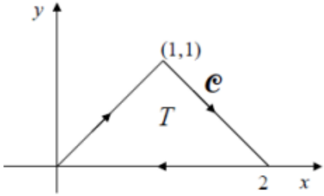
$$= \iint_R (2 - 6y)dA = \int_0^{\pi} d\theta \int_0^a (2 - 6r\sin\,\theta)rdr = \pi a^2 - 6\int_0^{\pi} \sin\,\theta d\theta \int_0^a r^2 dr = \pi a^2 - 4a^3.$$



Question 2: Evaluate $\oint_{\mathcal{C}}(x^2 - xy)dx + (xy - y^2)dy$ clockwise around the triangle with vertices (0,0), (1,1), and (2,0).

$$\oint_{\mathcal{C}}(x^2 - xy)dx + (xy - y^2)dy = -\iint_T \left[\frac{\partial}{\partial x}(xy - y^2) - \frac{\partial}{\partial y}(x^2 - xy)\right]dA = -\iint_T (y + x)dA$$

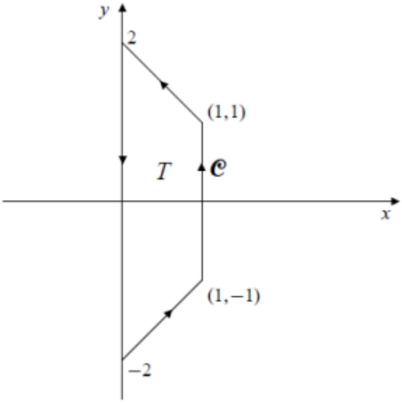
$$= -(\bar{y} + \bar{x}) \times (\text{ area of } T) = -\left(\frac{1}{3} + 1\right) \times 1 = -\frac{4}{3}$$



Question 3: Evaluate $\oint_{\mathcal{C}}(x\sin(y^2) - y^2)dx + (x^2y\cos(y^2) + 3x)dy$, where $\mathcal{C}$ is the counterclockwise boundary of the trapezoid with vertices $(0,-2), (1,-1), (1,1)$, and $(0,2)$.

$$\oint_{\mathcal{C}}(x\sin(y^2) - y^2)dx + (x^2y\cos(y^2) + 3x)dy = \iint_T [2xy\cos\, y^2 + 3 - (2xy\cos\, y^2 - 2y)]dA$$

$$= \iint_T (3 + 2y)dA = 3\iint_T dA + 0 = 3 \times 3 = 9$$



Question 4: Evaluate $\oint_{\mathcal{C}} x^2y dx - xy^2 dy$, where $\mathcal{C}$ is the clockwise boundary of the region $0 \leq y \leq \sqrt{9 - x^2}$.

Let D be the region $x^2 + y^2 \leq 9, y \geq 0$. Since $\mathcal{C}$ is the clockwise boundary of D,

$$\oint_{\mathcal{C}} x^2y dx - xy^2 dy = -\iint_D \left[\frac{\partial}{\partial x}(-xy^2) - \frac{\partial}{\partial y}(x^2y)\right]dxdy = \iint_D (y^2 + x^2)dA = \int_0^{\pi} d\theta \int_0^3 r^3 dr = \frac{81\pi}{4}$$

Question 5: Use a line integral to find the plane area enclosed by the curve $\mathbf{r} = a\cos^3 t\mathbf{i} + b\sin^3 t\mathbf{j}$, $(0 \le t \le 2\pi)$.

$$\text{Area} = \frac{1}{2}\oint_{\mathcal{C}} xdy - ydx = \frac{1}{2}\int_0^{2\pi} [a\cos^3 t\,(3b\sin^2 \mathrm{t}\cos t) - b\sin^3 t\,(-3a\cos^2 t\sin t)]dt$$

$$= \frac{3ab}{2}\int_0^{2\pi} \sin^2 t\cos^2 t dt = \frac{3ab}{2}\int_0^{2\pi} \frac{\sin^2(2t)}{4}dt = \frac{3\pi ab}{8}$$

The Following Questions Are Obtained From PAST EXAMS

Question 1: Evaluate $\oint_{\mathcal{C}}(-10y^3x^2 + \cos(x^2))dx + (3x^5 + 15y^4x + \sin(y^2))dy$ where $\mathcal{C}$ is counterclockwise oriented perimeter of the region bounded by $y = x, x = 0, x^2 + y^2 = 4$ in the first quadrant.

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Let $\left.\begin{aligned} P(x,y) &= -10y^3x^2 + \cos\ x^2 \\ Q(x,y) &= 3x^5 + 15y^4x + \sin\ y^2 \end{aligned}\right\}$ $\qquad \begin{aligned} P_y &= -30y^2x^2 \\ Q_x &= 15x^4 + 15y^4 \end{aligned}$

- since $P_y \neq Q_x$, (P,Q) is <u>not</u> exact (i.e., $\nexists f(x,y)$ with $\nabla f = (P,Q)$)
- P & Q are polynomials, hence have all partial derivalives of all ordes. bet $\mathcal{C} = \partial R$ (boundary curve of the region R) we can use Green`s Theorem.

$$\oint_{\mathcal{C}=\partial R} Pdx + Qdy = \iint_R (Q_x - P_y)dA = \iint_R 15x^4 + 15y^4 + 30x^2y^2 dA = 15\iint_R (x^2+y^2)^2 dA$$

$$= 15\int_0^{\pi/4}\int_0^2 r^4 r dr d\theta = 15\int_0^{\pi/4} \frac{r^6}{6}\bigg|_0^2 d\theta = \frac{15}{6}2^6 \cdot \frac{\pi}{4} = 40\pi$$

Question 2: Compute the line integral

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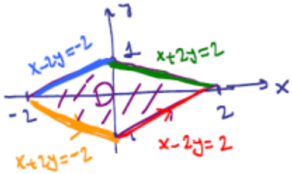
$$\oint_{\mathcal{C}} \left(x^3\sin\sqrt{x^2+4} - xe^{x+2y}\right)dx + (\cos(y^3+y) - 4ye^{x+2y})dy$$

on the curve $\mathcal{C}$ which is the boundary of the parallelogram with vertices $(2,0), (0,1), (-2,0), (0,-1)$ with counterclockwise orientation.

$\vec{F}(x,y) = (P(x,y), Q(x,y))$ where

$P(x,y) = x^3\sin\sqrt{x^2+4} - xe^{x+2y}$ $\qquad$ $Q(x,y) = \cos(y^3+y) - 4ye^{x+2y}$

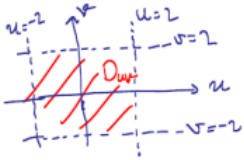
P & Q have continuous partial derivatives on D, $\mathcal{C} = \partial D$ is the counterclockwise oriented boundary curve of D. Moreover, $\mathcal{C}$ is closed and piecewise smooth curve.



Hence, by Greens Theorem.

$$\oint\limits_{\mathcal{C}=\partial D} P(x,y)dx + Q(x,y)dy = \iint_D [Q_x(x,y) - P_y(x,y)]dA = \iint_D [-4ye^{x+2y} - (-xe^{x+2y} \cdot 2)]dA =$$

$\iint_D 2(x-2y)e^{x+2y}dxdy$ Let $\begin{aligned} u &= x - 2y \\ v &= x + 2y \end{aligned}$



$$\iint\limits_{D_{xy}} 2(x-2y)e^{x+2y}dxdy = \iint_{\mathrm{D_{uv}}} 2ue^{v} \cdot \frac{1}{4}dudv$$

$$= \int_{-2}^{2}\int_{-2}^{2} 2u \cdot e^{v}\frac{1}{4}dudv = \int_{-2}^{2} \frac{u^2}{4}e^{v}\Big|_{u=-2}^{u=+2} dv = \int_{-2}^{2} 0\, dv = 0$$