Sequences → function with domain $\mathbb{N}$

Ex. $a_n = \cos(n\pi) = (-1)^n$ divergent

$f: \mathbb{N} \to \mathbb{R}$

a_n and b_n are convergent

→ $\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n$

→ $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim a_n}{\lim b_n}$, if $\lim b_n \neq 0$

Sequeeze theorem can be used for sequences.

If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$

Theorem: r^n is convergent, if $-1 < r \leq 1$

is divergent for all other values of r.

Sequences is called monotonic if it is either increasing and dec.

A sequence a_n is bounded above if there is a number s.t.

$a_n \leq M$ for all $n \geq 1$

A sequence a_n is bounded below

$a_n \geq m$ for all $n \geq 1$

If a_n is bounded from above and below, then a_n is a bounded sequence

Monotonic sequence theorem: Every bounded monotonic sequences are convergent. (increasing and bounded above)

least upper bound $\to L$

Given $\varepsilon > 0$, $L-\varepsilon$ is not an upper bound for S. Therefore $a_n > L-\varepsilon$ for some integer N

But the sequence is increasing so $a_n > a_N$ for every $n > N$

Thus, if $n > N$ we have $a_n > L-\varepsilon$ so $0 \leq L - a_n < \varepsilon$ since $a_n \leq L$. Thus $|L-a_n| < \varepsilon$ whenever $n > N$ so $\lim_{n\to\infty} a_n = L$

Similarly one can show that a decreasing sequence that is bounded below is convergent.

Ex. $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2+a_n}$

show that a_n is increasing

we are going to prove that $a_n < a_{n+1}$

for $n=1$ ✓

Assume that $a_k < a_{k+1}$, then using this assumption try to prove that $a_{k+1} < a_{k+2}$

$$a_{k+2} = \sqrt{2+a_{k+1}} > \sqrt{2+a_k} = a_{k+1}$$

Hence, a_n is increasing

Show that a_n is bounded from above by 5 which means $a_n < 5$

$a_1 = \sqrt{2} < 5$ ✓

Assume $a_k < 5$ $\quad a_{k+1} = \sqrt{2+a_k} < \sqrt{2+5} = \sqrt{7} < 5$

Hence, a_n is bounded from above by 5

Therefore, by monotonic sequence theorem, a_n must be convergent

$\lim_{n\to\infty} a_n = L$ exists $\qquad a_{n+1} = \sqrt{2 + a_n}$

$\lim_{n\to\infty} a_{n+1} = L \qquad L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2 + a_n}$

$$L = \sqrt{2+L} \rightarrow L^2 = 2 + L$$

$$L^2 - L - 2 = 0 \qquad \boxed{L = 2} \quad \cancel{L = -1}$$

Subsequences

If a sequence converges then all subsequences also converge to the same number.

If any subsequence of a sequence diverges or two subsequences have different limits, the the original sequence diverges.

e.g $(-1)^n$ diverges

Limits of some sequences

(i) $\lim_{n\to\infty} \frac{\ln n}{n} = 0$

(ii) $\lim_{n\to\infty} \sqrt[n]{n} = 1$

(iii) $\lim_{n\to\infty} x^{1/n} = 1 \quad (x>0)$

(iv) $\lim_{n\to\infty} x^n = 0 \quad (|x|<1)$

(v) $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$ (any x)

(vi) $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ (any x)

x remains fixed while $n \to \infty$

ex. $a_n = \dfrac{n}{n+1}$ $\quad a_{n+1} - a_n = \dfrac{-1}{(n+1)(n+2)} < 0$

$0 < n$ $\quad\downarrow$

$a_n \leq 1 \rightarrow$ bounded $\quad$ strictly increasing

ex. $a_n = \dfrac{1}{n}$ $\quad \dfrac{1}{n} > \dfrac{1}{n+1} \rightarrow$ strictly Increasing

also bounded by 1

ex. $a_n = \dfrac{(-1)^n}{n}$ is neither decreasing nor increasing

ex. $\lim \dfrac{5^n}{n!}$

$$\frac{5^n}{n!} = \frac{5.5.5.5.5.5.\cdots 5}{\underbrace{1.2.3.4.5.}_{\frac{5^5}{5!}}6.7\ldots n-1} \cdot 1.1.1.1.$$

$$\begin{array}{ccccc} 0 \leq & \dfrac{5^n}{n!} & \leq & \dfrac{5^5}{5!} \cdot \dfrac{5}{n} & \rightarrow \text{sequeeze} \\ \downarrow & \downarrow & & \downarrow & \\ 0 & 0 & & 0 & \end{array}$$

Basics of Series

$$\sum_{n=1}^{\infty} ar^{n-1} \qquad S_n = \frac{a(1-r^N)}{1-r}$$

ex.

$$\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n} = \sum_{n=1}^{\infty} \underbrace{\frac{1}{4}}_{a} \Big(\underbrace{-\frac{3}{4}}_{r}\Big)^{n-1} = \frac{1/4}{1-\left(-\frac{3}{4}\right)} = \frac{1}{7}$$

$|r| < 1$

Ex. $$\sum_{n=1}^{\infty} \frac{2}{(n+1)(n+3)} = \sum_{n=1}^{\infty} \frac{1}{n+1} - \frac{1}{n+3} \qquad \frac{A^{1}}{n+1} + \frac{B^{-1}}{n+3} = \frac{2}{(n+1)(n+3)}$$

$$S_N = \sum_{i=1}^{N} \frac{1}{i+1} - \frac{1}{i+3} = \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\frac{1}{3} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}\right) \cdots \cdots$$
$$\cdots \left(\cancel{\frac{1}{N}} - \frac{1}{N+2}\right) + \left(\cancel{\frac{1}{N+1}} - \frac{1}{N+3}\right)$$

$$S_N = \frac{1}{2} + \frac{1}{3} - \frac{1}{N+2} - \frac{1}{N+3} = \frac{5}{6}$$

Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n} \qquad 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots$$
$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)$$
$$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$
$$= 1 + \frac{\infty}{2} = \infty$$ so divergent

Theorem: If S_n is convergent then $\lim_{n\to\infty} a_n = 0$

Converse of the theorem is false (ex. $1/n$)

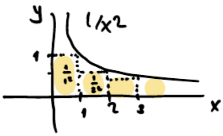
Test for divergence

- If $\lim_{n\to\infty} a_n$ dne or $\neq 0$, then $\sum_{n=1}^{\infty} a_n$ is divergent

Ex. $\sum_{n=1}^{\infty} \ln\left(\frac{n}{2n+5}\right) = \ln\left(\frac{1}{2}\right) \neq 0$ so it is divergent

The Integral Test (IT)

$\sum_{n=1}^{\infty} \frac{1}{n^2}$



$S_N < 1 + \int_1^{\infty} \frac{1}{x^2}\,dx$ for all N

$\Downarrow$

converges by P-test

no simple formula for S_N

so S_N is convergent

S_n is an increasing sequence that is bounded from above

by MST, $\{S_N\}$ is convergent

Hence $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent

α

For $p > 0$: $f(x) = \frac{1}{x^p}$ cont, pos and dec on $[1, \infty)$

$\int_1^{\infty} \frac{1}{x^p}\,dx$ converges if $p > 1$ and diverges $p \leq 1$

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p>1$ and divergent if $p \leq 1$.

Ex. $\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$ $\quad f(x) = \frac{\ln x}{x^2}$ cont, pos

$\lim_{x\to\infty} f(x) = 0$ by L'H

$\lim_{n\to\infty} \frac{\ln n}{n^2} = 0$

$f'(x) = \frac{1-2\ln x}{x^4}$ $\quad$ for $x \geq 2 \rightarrow f'(x) < 0$

So we can apply IT

$\int_2^{\infty} \frac{\ln x}{x^2}\, dx$ $\quad u = \ln x \quad dv = 1/x^2$

$du = \frac{dx}{x} \quad v = -1/x$

$\Downarrow$

$\frac{1}{2} + \frac{\ln 2}{2} \Rightarrow$ therefore $\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$ is convergent

cont, positive and decreasing means that we can apply integral test.

The Comparison test

$a_n \leq b_n$ $\quad$ if $\sum b_n$ is convergent, $\sum a_n$ is convergent.

$a_n \geq b_n$ $\quad$ if $\sum b_n$ is divergent, $\sum a_n$ is divergent

Ex. $\sum_{n=2}^{\infty} \frac{1}{n-\sqrt{n}}$

1 hep yap

$\lim_{n\to\infty} \frac{1}{n-\sqrt{n}} = \frac{1}{n\left(1-\frac{1}{\sqrt{n}}\right)} = 0$

$\sum_{n=2}^{\infty} \frac{1}{n}$ = divergent

$\frac{1}{n-\sqrt{n}} > \frac{1}{n} > 0$

$\downarrow$

divergent

The Limit Comparison Test

a_n and b_n have positive terms and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$

(i) $L=0 \Rightarrow \sum b_n$ conv $\Rightarrow \sum a_n$ conv

$\sum a_n$ dive $\Rightarrow \sum b_n$ div

(ii) $L=\infty \Rightarrow \sum a_n$ conv $\Rightarrow \sum b_n$ conv

$\sum b_n$ div $\Rightarrow \sum a_n$ div

(iii) $0 \neq L < \infty$ both series converge or diverge

Ex. $\sum_{n=1}^{\infty} \frac{1+2^n}{1+3^n}$

$\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n \rightarrow$ convergent

$\lim_{n\to\infty} \frac{\frac{1+2^n}{1+3^n}}{\frac{2^n}{3^n}} = \frac{3^n+6^n}{2^n+6^n} = \frac{\cancel{6^n}\left(\overset{0}{\frac{1}{2^n}}+1\right)}{\cancel{6^n}\left(\underset{0}{\frac{1}{3^n}}+1\right)} = 1$ therefore answer is convergent

Ex. $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$ $\qquad$ $\sum_{n=1}^{\infty} \frac{1}{n} \to$ divergent

$$\lim_{n\to\infty} \frac{\frac{1}{n^{1+\frac{1}{n}}}}{\frac{1}{n}} = \frac{1}{n^{1/n}} = 1$$

$y = x^{1/x}$ indeterminacy

$\ln y = \frac{\ln x}{x}$ $\frac{\infty}{\infty}$ $\xrightarrow{L'H}$

$\ln \lim_{x\to\infty} y = \lim_{x\to\infty} \frac{1}{x}$

$\lim_{x\to\infty} y = 1$

answer is divergent

The Ratio Test

$\sum a_n$ is series with positive terms ($a_n \geq 0$ for some $n > N$)

(i) If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L < 1$ then $\sum a_n$ is convergent

(ii) If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L > 1$ or $= \infty$, then $\sum a_n$ is divergent

(iii) If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$, the ratio test inconclusive

Ex. $\sum_{n=1}^{\infty} \frac{n^2 2^n}{n!}$ $\qquad$ $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{(n+1)^{\cancel{2}} \cdot 2^{n+1}}{\cancel{(n+1)!}} \cdot \frac{\cancel{n!}}{n^2 \cdot \cancel{2^n}} = \frac{2 \cdot n+1}{n^2}$

$= 0 < 1$

therefore it is convergent

$\left(1 - \frac{1}{n}\right)^n = e^{-1}$

Ex. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

Recall: $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{\cancel{(n+1)}!}{\cancel{n+1}\,{}^{n+1}} \cdot \frac{n^n}{\cancel{n!}} = \frac{n^n}{n+1^n} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = e^{-1} < 1$$

Root Test

$\sum a_n$ is series with positive terms

(i) $\lim_{n\to\infty} \sqrt[n]{a_n} = L < 1$, then $\sum a_n$ is convergent

(ii) $\lim_{n\to\infty} \sqrt[n]{a_n} = L > 1$ or $= \infty$, then $\sum a_n$ is divergent

(iii) if $\lim = 1$, then the root test is inconclusive

Ex. $\sum_{n=12}^{\infty} \frac{1}{(\ln n)^n}$ $\quad \lim_{n\to\infty} \sqrt[n]{\frac{1}{(\ln n)^n}} = \frac{1}{\ln n} = 0 < 1$

Absolute Convergence

If $\sum |a_n|$ is convergent, $\sum a_n$ is absolutely convergent

If $\sum |a_n|$ is absolutely convergent, then it is convergent

Proof $0 \le a_n + |a_n| \le 2|a_n|$

a_n is called conditionally convergent if it is convergent but it is not absolutely convergent.

$\sum a_n = S$ if it is absolutely convergent, any rearrangement of $\sum a_n = S$.
it is conditionally converget, any rearrangement of $\sum a_n = r$

Ex. $\sum_{n=17}^{\infty} \frac{\cos(n\frac{\pi}{3})}{n!}$ $\sum_{n=17}^{\infty} \frac{1}{n!}$ $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0 < 1$ so it is conv.

$\sum_{n=17}^{\infty} \frac{|\cos(n\frac{\pi}{3})|}{n!}$ $0 \leq \frac{|\cos(n\frac{\pi}{3})|}{n!} \leq \frac{1}{n!}$

therefore it is absolutely convergence

Ex. $\frac{2}{5} + \frac{2}{5}\cdot\frac{6}{8} + \frac{2}{5}\cdot\frac{6}{8}\cdot\frac{10}{11} + \frac{2}{5}\cdot\frac{6}{8}\cdot\frac{10}{11}\cdot\frac{14}{14} + \dots$

$= \sum_{n=1}^{\infty} \frac{2\cdot 6 \dots (4n-2)}{5\cdot 8 \dots (3n+2)}$ apply ratio test $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

$\lim_{n\to\infty} \frac{2\cdot 6 \dots (4n-2)(4n+2)}{5\cdot 8 \dots (3n+2)(3n+5)} \cdot \frac{5\cdot 8 \dots (3n+2)}{2\cdot 6 \dots (4n-2)} = \frac{4n+2}{3n+5} = \frac{4}{3} \underline{> 1}$

diverges to ∞

Ex. $\sum_{n=1}^{\infty} n^2 \sin\left(\frac{1}{4n}\right)$
$n^2 \to +$, $\sin\left(\frac{1}{4n}\right) \to +$
positive always

$\lim_{n\to\infty} \underbrace{\frac{\sin\left(\frac{1}{4n}\right)}{\frac{1}{n^2}}}_{a_n} \cdot \underbrace{\frac{1}{n}}_{\frac{1}{b_n}} = \frac{\sin\left(\frac{1}{4n}\right)}{\frac{1}{n}\cdot\frac{4}{4}} = \frac{1}{4}$

by LCT $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1}{4}$ $\sum_{n=1}^{\infty} n \to$ divergent

answer is divergent

or: $\lim_{n\to\infty} n^2 \sin\left(\frac{1}{4}\right) = \infty$

Alternating Series

mutlak değerine bak

1 → positive
2 → decreasing
3 → limit = 0

} conv

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 + b_4 - \dots \quad (b_n > 0)$$

$|b_{n+1}| \leq |b_n|$ or $a_n \cdot a_{n+1} < 0$

(i) $\lim_{n\to\infty} b_n = 0$

then the series is convergent

(ii) $b_{n+1} \leq b_n$ for all n

Alternating Series Estimation Theorem:

$S = \sum (-1)^{n-1} b_n$ is the sum of an alternating series

that satisfies $\lim_{n\to\infty} b_n = 0$ and $b_{n+1} \leq b_n$

then $R_N = |S - S_N| \leq b_{N+1}$ $\quad S_N = b_1 - b_2 + b_3 \dots + (-1)^{N-1} b_N$

error

S lies between S_N and S_{N+1} $\Rightarrow$ $|S - S_N| \leq |S_{N+1} - S_N|$

$= b_{N+1}$

Ex. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\ln n}{n}$

(i) $\lim_{n\to\infty} \frac{\ln n}{n} = 0$

$\rightarrow \lim_{x\to\infty} \frac{\ln x}{x} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \frac{1}{x} = 0$

(ii) $b_n = \frac{\ln n}{n}$

$f(x) = \frac{\ln x}{x}$ $\quad f'(x) = \frac{1-\ln x}{x^2} < 0$

for $2 \leq x < \infty$

our series satisfies the theorem so it is convergent by AST

$\sum_{n=2}^{\infty} \frac{\ln n}{n}$ $\qquad \sum_{n=2}^{\infty} \frac{1}{n}$ divergent

LCT

$$\lim_{n\to\infty} \frac{\frac{\ln n}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \ln n = \infty$$

$f(x) = \frac{\ln x}{x}$ positive, continuos, decreasing so we can apply IT

$$\int_2^{\infty} \frac{\ln x}{x}\,dx = \lim_{t\to\infty} \int_2^{t} \frac{\ln x}{x}\,dx = \infty$$

$u = \ln x$
$du = 1/x\,dx$

Ex. How many terms do we need so that (at least)

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^4} \qquad \underset{|\text{error}|}{R_N} < 0.001 = \frac{1}{1000} \qquad \lim_{n\to\infty} b_N = 0$$

$\frac{1}{n^4} \downarrow b_N$

$R_7 = |S - S_7| < \frac{1}{8^4} = \frac{1}{4036}$ $\qquad S_5 = 1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \frac{1}{5^4}$

$R_5 = |S - S_5| < \frac{1}{6^4} = \frac{1}{1296}$

$R_4 = \frac{1}{625}$

Power Series

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 \dots$$

power series in $(x-a)$
" " centered at a
" " about a

$$\boxed{\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1 \quad \Downarrow \quad \text{interval of convergence}}$$

Ex. For which values of x does $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}$ converge?

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = |x| \frac{n+1}{n+2} = |x|$$

by Ratio test when $|x| < 1 \rightarrow$ convergent

$x=-1$:

$= \sum_{n=0}^{\infty} \frac{1}{n+1} \rightarrow \text{div}$

$x=1$:

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \rightarrow$ convergent by AST

$\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n+1}$ converges when $x \in (-1, 1]$

Theorem:

For a given series $\sum_{n=0}^{\infty} c_n (x-a)^n$ there are only three possibilities.

(i) The series converges only when $x=a$

(ii) The series converges for all x.

(iii) There is a positive number R s.t.

the series converges if $|x-a| < R$

" " diverges if $|x-a| > R$

R is called the radius of convergence

Ex. Find the radius of convergence and interval of convergence

$\sum_{n=2}^{\infty} (-1)^n \frac{(2x+3)^n}{n \ln n}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = |2x+3| \lim_{n\to\infty} \frac{n \cdot \ln n}{\underbrace{(n+1)}_{1} \ln(n+1)}$$

$$= |2x+3|$$

convergent if $|2x+3| < 1$ | $-1 < 2x+3 < 1$

divergent if $|2x+3| > 1$ | $-2 < x < -1$

$\widetilde{\text{aralık}}$ 1 uzunluğunda

$\underline{R = \frac{1}{2}}$

$x = -2$

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$ → divergent by IT or LCT

$x = -1$

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$$ → convergent by AST

↓

alternating series

Interval of convergence → $(-2, -1]$

Ex. $$\sum_{n=2}^{\infty} \underbrace{\frac{x^n}{(\ln n)^n}}_{a_n}$$

so $\lim_{n\to\infty} \frac{x^n}{(\ln n)^n} = 0$

Root test

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{|x|}{\ln n} = 0 < 1$$ for all x → convergent

so $I = (-\infty, \infty)$ $R = \infty$

Ex.

$$\sum_{n=1}^{\infty} \underbrace{\frac{2.4.6.\ldots(2n)}{1.3.5.\ldots(2n-1)}}_{c_n} x^n$$

our center is 0

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} |x| \frac{2n+2}{2n+1} = |x|$$

if $|x| < 1$, convergent $(-1, 1) \rightarrow R=1$

$x=-1$

$$\underbrace{\frac{2.4.6..2n}{1.3.5...2n-1}}_{b_n}(-1)^n \qquad \lim_{n\to\infty} b_n \neq 0$$

so the series is divergent

$b_n > 1$ for all n $\frac{2}{1}.\frac{4}{3}.\frac{6}{5}\ldots$ hepsi 1'den büyük

$x=1$ için de aynı divergent

Representations of Functions as Power Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1+x+x^2+\ldots+x^n+\ldots \quad \text{for } |x|<1$$

Ex. Find a power series representation for $f(x) = \frac{1}{1+9x^2}$ and determine the interval of convergence.

$$\frac{1}{1+9x^2} = \frac{1}{1-(-9x^2)} = \sum_{n=0}^{\infty}(-9x^2)^n \qquad \text{for } |-9x^2|<1$$

$$-\frac{1}{3} < x < \frac{1}{3}$$

$$= \sum_{n=0}^{\infty} (-1)^n . 3^{2n} . x^{2n} \qquad -\frac{1}{3} < x < \frac{1}{3}$$

Ex. $f(x) = \dfrac{x}{4x+1}$

$$\frac{x}{4x+1} = x . \frac{1}{1-(-4x)} = x . \sum_{n=0}^{\infty} (-4x)^n \qquad |-4x| < 1$$

$$= \sum_{n=0}^{\infty} (-1)^n . 2^{2n} . x^{n+1} \qquad \text{for } -\frac{1}{4} < x < \frac{1}{4}$$

Differentation and Integration of Power Series

If the power series $\sum c_n (x-a)^n$ has radius > 0 then the function f defined by

$$f = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \dots$$

differentable (cont) on the interval $(a-R, a+R)$ and

(i) $f'(x) = c_1 + 2c_2 (x-a) + 3c_3 (x-a)^2 + \; = \displaystyle\sum_{n=1}^{\infty} n \, c_n (x-a)^{n-1}$

(ii) $\displaystyle\int f(x) = C + c_0 (x-a) + c_1 \frac{(x-a)^2}{2} + c_2 \frac{(x-a)^3}{3} +$

$\hookrightarrow = \displaystyle\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$ $\qquad$ C_n integral alınmaz sadece power kısmı

$\propto$ Radius does not change. Interval might change.

Ex. Find a power series representation for $f(x) = \frac{1}{(1+x)^2}$

① $$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$-1 < x < 1 \quad R=1$

② $$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n . x^n$$

$-1 < x < 1$

③ differentiate both sides

$$\frac{-1}{(1+x)^2} = \sum_{n=1}^{\infty} (-1)^n . n . x^{n-1}$$

④ $$\frac{1}{(1+x)^2} = \sum_{n=1}^{\infty} (-1)^{n+1} . n . x^{n-1}$$

for $x=-1$ it is divergent

for $x=1$ it is divergent

⑤ $$\alpha \quad \frac{x}{(1+x)^2} = \sum_{n=1}^{\infty} (-1)^{n+1} . n . x^n$$

$-1 < x < 1$

$\downarrow x = \frac{1}{3}$

$$\rightarrow \sum_{n=1}^{\infty} (-1)^{n+1} . \frac{n}{3^n} = \frac{1/3}{\left(1+\frac{1}{3}\right)^2}$$

$$\rightarrow \sum_{n=2}^{\infty} (-1)^{n+1} . \frac{n}{3^n} = \frac{1/3}{\left(1+\frac{1}{3}\right)^2} - \frac{1}{3}$$

Ex. $f(x) = \ln(1+x)$

$$\ln(1+x) \Rightarrow \int \frac{1}{1+x}\,dx = \int \left[\sum_{n=0}^{\infty} (-1)^n x^n\right] dx$$

$$= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \qquad \underset{?}{-1 <} x \underset{?}{< 1}$$

$x = 0$

$\ln 1 = 0 = C + 0$
$C = 0$

$$\boxed{\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \qquad -1 < x \leq 1}$$

for $x = 1$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \qquad b_n = \frac{1}{n+1}$$

it is conv by AST

for $x = -1$

$$\sum_{n=0}^{\infty} \frac{-1}{n+1} = -\sum_{n=0}^{\infty} \frac{1}{n+1} \quad \text{div to } -\infty$$

$$\ln(2) \approx 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}$$

$|\text{error}| < \frac{1}{6}$

Taylor Series

$$f(x) = f(x_0) + f'(x_0)\cdot(x - x_0) + \frac{f''(x_0)(x-x_0)^2}{2!} + \frac{f'''(x_0)(x-x_0)^3}{3!} + \dots$$

- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \dots$ Yakınsak aralığı $-\infty < x < \infty$

- $\sin x = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \quad -\infty < x < \infty$
- $\cos x = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \quad -\infty < x < \infty$
- $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \quad -1 < x < 1$
- $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad -1 < x \leq 1$
- $\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \quad -1 \leq x \leq 1$

- $\int \left(\frac{1}{1+x^2}\right) dx$

$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n \cdot x^{2n} \quad -1 < x < 1$

Yerine Koyma Metodu

Ex. e^{x^2} maclurin serisi =?

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

Ex. $x^3 \cdot e^{4x}$ $\quad x^3 \sum_{n=0}^{\infty} \frac{(4x)^n}{n!} = \sum_{n=0}^{\infty} x^3 \cdot \frac{(4x)^n}{n!}$

Ex. $\int e^{x^3} dx$ integralini maclurin serisinin ilk 2 terimin kullanarak hesaplayınız.

$$e^{x^3} = \sum_{n=0}^{\infty} \frac{x^{3n}}{n!} = 1 + x^3 + \dots \qquad \int 1 + x^3 \, dx = x + \frac{x^4}{4} + c$$

Ex. $x^2 \cdot \sin 5x$ maclurin serisi = ?

$$x^2 \sum_{n=0}^{\infty} (-1)^n \cdot \frac{(5x)^{2n+1}}{(2n+1)!}$$

- $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ convergent for all x by ratio test

$$= e^x$$

Ex. Represent $f(x) = \sin x$ as Taylor Series centered at $a = \frac{\pi}{2}$

$$t = x - \frac{\pi}{2} \Rightarrow x = t + \frac{\pi}{2}$$

$$\sin x = \sin\left(t + \frac{\pi}{2}\right) = \cos(-t) = \cos t = (\sin t)'$$

Write maclaurin series for $\cos t$

$$\frac{d}{dt} \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!} \qquad t \in (-\infty, \infty)$$

$$\cos t = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{2n!} \qquad t \in (-\infty, \infty)$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{(x - \pi/2)^{2n}}{2n!} \quad \text{for all } x.$$

Analytic Geometry

(i) $|\vec{v}\cdot\vec{w}| \leq \|\vec{v}\|\,\|\vec{w}\|$ Cauchy-Schwarz inequality

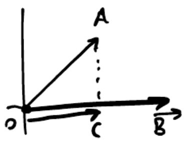
(ii) $\|\vec{v}+\vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$ Triangle inequality

(iii) $\vec{v}\cdot\vec{w} = |\vec{v}|\cdot|\vec{w}|\cdot\cos\alpha$

If $\cos\alpha = 0$ then $\vec{v}\cdot\vec{w} = 0$
$\alpha = \frac{\pi}{2}$

or if $\vec{v},\vec{w} \neq 0$ and $\vec{v}\cdot\vec{w} = 0$
then $\alpha = \frac{\pi}{2}$

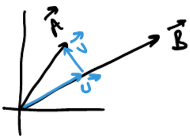
Projection



$\vec{OC} = \lambda . \vec{B}$

$\vec{OC} = \vec{A} . \frac{\vec{B}}{|B|} . \frac{\vec{B}}{|B|}$.

Ex. Write $\vec{A} = (1,2,1)$ as a sum of two vectors $\vec{v},\vec{w}$
s.t. $u \parallel (1,-1,0) = B$
$v \perp (1,-1,0)$



$\vec{v} = \vec{A} - \vec{u}$

$\vec{u} = \vec{A} . \frac{\vec{B} . \vec{B}}{(\|\vec{B}\|)^2} = \frac{(1,2,1)\cdot(1,-1,0)(1,-1,0)}{1^2+(-1)^2+0^2}$

$\vec{u} = \frac{-1}{2}(1,-1,0)$

$\vec{v} = (1,2,1) - \left(-\frac{1}{2}(1,-1,0)\right)$

OR: $u = \lambda.(1,-1,0)$

$v.(1,-1,0) = 0$ (because they are perpendicular)

Say $\vec{v} = (a,b,c)$

$a - b + 0 = 0$

$a = b$

$\vec{u} + \vec{v} = \vec{A} = (1,2,1)$

$\to (a,a,c)$

$\to \lambda(1,-1,0)$

$a + \lambda = 1$ $\qquad a = 3/2$

$a - \lambda = 2$ $\qquad \lambda = -1/2$

$c + \lambda.0 = 1$ $\qquad c = 1$

CROSS (VECTOR) PRODUCT

$$\begin{bmatrix} a & 1-a \\ 1+a & -a \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$\begin{bmatrix} b & a \\ -b^2/a & -b \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \neq 0$$

$a \neq 0$

DETERMINANTS

for 2×2, 3×3

$\det a_{11} = a_{11}$

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\det \begin{array}{ccc} \left[\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right] \\ \hdashline a_{11} \quad a_{12} \quad a_{13} \\ a_{21} \quad a_{22} \quad a_{23} \end{array} = (a_{11}a_{22}a_{33})+(a_{21}a_{32}a_{13})+(a_{31}a_{12}a_{23}) - (a_{31}a_{22}a_{13})+(a_{11}a_{32}a_{23})+(a_{21}a_{12}a_{33})$$

If two rows(columns) in a square matrix are interchanged then the value of the determinant is multiplied by -1.

If multiplied by a scalar λ then the value of the determinant is multiplied by λ.

If a row is multiplied by a constant and added to another row, then the value of the determinant does not change.

$\propto$ If $\vec{w} = \lambda \vec{v}$ (i.e. $\vec{w} \parallel \vec{v}$) then $\vec{w} \times \vec{v} = \vec{v} \times \vec{w} = \vec{0}$ $(\vec{v} \times \vec{v} = \vec{0})$

$\propto$ $\vec{v} \times \vec{w} \perp \vec{v}$ and $\vec{v} \times \vec{w} \perp \vec{w}$

verified by $(\vec{v} \times \vec{w}) \cdot \vec{v} = 0 = (\vec{v} \times \vec{w}) \cdot \vec{w}$

$\propto$ $\vec{i} \times \vec{j} = \vec{k}$, $\vec{j} \times \vec{k} = \vec{i}$, $\vec{k} \times \vec{i} = \vec{j}$

$$\underbrace{\underbrace{(\vec{i} \times \vec{j})}_{\vec{k}} \times \vec{j}}_{-\vec{i}} \neq \vec{i} \times \underbrace{(\vec{j} \times \vec{j})}_{\vec{0}} = 0$$

So places of parentheses are important.

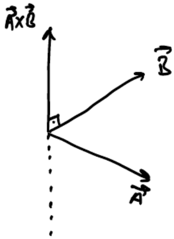
Theorem:

$$\|\vec{A}\times\vec{B}\| = \|\vec{A}\|\,\|\vec{B}\|\sin\alpha$$

Proof: $\|\vec{A}\times\vec{B}\|^2 = \|A\|^2\|B\|^2 - \underbrace{(A.B)^2}_{(\|A\|.\|B\|\cos\alpha)^2}$

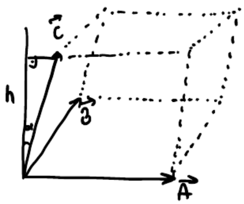
$$\vec{A}\times\vec{B} = \vec{0} \iff \vec{A}\,//\,\vec{B}$$

Remark : The direction of $\vec{A}\times\vec{B}$ obeys the right hand rule



$$\vec{i}\times\vec{j} = \vec{k}$$

Geometric Applications



$$\begin{array}{rl} & \text{Area} = \|\vec{A}\times\vec{B}\| \\ \times & \text{height} = \|\vec{C}\|.\cos\alpha \\ \hline & = |(\vec{A}\times\vec{B}).\vec{C}| \end{array}$$

Ex. Find the area of the triangle $P_1(1,0)$ $P_2(2,1)$ $P_3(3,3)$

$\frac{1}{2}.\|P_2P_1 \times P_2P_3\|$

$P_2P_1 = P_1 - P_2 = (-1,-1)$

$P_2P_3 = P_3 - P_2 = (1,2)$

$$\begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ -1 & -1 & 0 \\ 1 & 2 & 0 \\ \hdashline \vec{i} & \vec{j} & \vec{k} \\ -1 & -1 & 0 \end{array}$$

$(0 - 2\vec{k} + 0) - (-\vec{k} + 0 + 0) = -\vec{k}$

Thus the area $= \frac{1}{2}\|-\vec{k}\| = \frac{1}{2}$

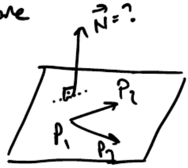
PLANES

Ex. Find the plane

$P_1\ (0,1,0)$

$P_2\ (2,1,1)$

$P_3\ (1,0,1)$



$\vec{N} = \overrightarrow{P_1P_3} \times \overrightarrow{P_1P_2}$

$(P_3 - P_1) \times (P_2 - P_1)$

$$\begin{array}{ccc} i & j & k \\ 1 & -1 & 1 \\ 2 & 0 & 1 \\ \hdashline i & j & k \\ 1 & -1 & 1 \end{array} = (-1,1,2)$$

$\downarrow$

$\vec{N}$

The eqn $= \overrightarrow{P_0P} \cdot \underset{\vec{N}}{(-1,1,2)} = 0$

$(x,y,z) - \underbrace{(0,1,0)}_{P_1 \text{ or } P_2 \text{ or } P_3}$

$\downarrow$

$= -x + y - 1 + 2z = 0$

(P_1, P_2, P_3 are not collinear
if they are, $\overrightarrow{P_1P_3} \times \overrightarrow{P_1P_2} = \vec{0}$)

Ex. Find the eqn. of the plane which is // to the plane $z = 2 - x - y$ passing through $P_0 = (0,1,2)$

we should find $\vec{N}$ as any perpendicular vector to given plane.

$x+y+z=2$

so a perpendicular vector is $(1,1,1)$

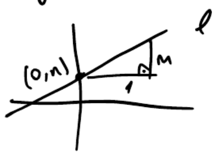
The wanted eqn $\vec{P_0P}(1,1,1)=0$

$$(x-0,y-1,z-2)(1,1,1)=0$$

$$x+y+z=3$$

Explanation: The plane eqn in $\mathbb{R}^3$ is general form of line eqn. in $\mathbb{R}^2$.

Say a line has slope m and passes through $P_0=(0,n)$



So $(1,m) \parallel \ell$

then $(-m,1) \perp \ell$

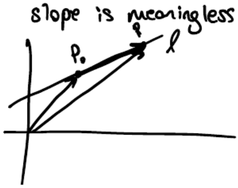
(by dot product check)

So eqn of ℓ = $\vec{P_0P}(-m,1)=0$

$$[(x,y)-(0,n)](-m,1) \quad \text{so} \quad y=mx+n$$

Lines in space

slope is meaningless in $\mathbb{R}^3$ then we use vectors.



$P = P_0 + \vec{PP_0} \qquad v \parallel \ell$

$P = P_0 + t.v$ is the eqn of ℓ

(vector form)

$(x, y, z) = (x_0, y_0, z_0) + (ta, tb, tc)$ where $\vec{v} = (a, b, c)$

Doğru denklemi yazma

$x = x_0 + ta$
$y = y_0 + tb$
$z = z_0 + tc$

in parametric form, eliminate t

$$t = \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$ symmetric form

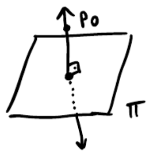
(If $a = 0$, $x = x_0$)

Observe that a line eqn is actually 2 scalar eqns each of which is a plane eqn. because a line is intersection of two planes.

Plane denklemi $ax + by + cz + d = 0$

Ex. Find the eqn of

1) the line passing through $P_0 = (1,1,1)$ and $\perp$ to the plane $\pi : x + y - 3z = 3$



$\ell = ?$ Since $\vec{N} \parallel \ell$ we can take $\vec{v}_\ell$ to be $\vec{N}$

So $\ell : P = (1,1,1) + t \underbrace{(1,1,-3)}_{\text{perpendicular to plane}}$

Ex. Find the intersection point of the line $\ell : x - 1 = \frac{y-2}{2} = \frac{z}{2}$ and the plane $\pi : x + y + z = 1$

$t = x - 1 = \frac{y-2}{2} = \frac{z}{2}$ so $x = t + 1$, $y = 2t + 2$, $z = 2t$) put in eqn of π

$(t+1) + (2t+2) + 2t = 1$

$5t = -2 \quad t = -2/5$

$x_0 = -2/5 + 1$
$y_0 = 2(-2/5) + 2$
$z_0 = 2(-2/5)$

Ex. Find the line of intersection of the planes

$\Pi_1 : x+y+2z=2$

$\Pi_2 : x-y+3z=1$

they are not parallel

parametric form:

let take $x=t$

then

$t=2-y-2z$

$t=1+y-3z$

$2t=3-5z \rightarrow z=\frac{3-2t}{5}$

so $t=\frac{3-5z}{2}$

$t=1+y-3\left(\frac{3-2t}{5}\right)$

$y=t+3\left(\frac{3-2t}{5}\right)-1$

symmetric form

$t=x$ by subtraction

$2y-z=1$

$\Pi_1 \rightarrow x+y+2z=2$

$2(2y-z)=2$

$+\ t+y+2z=2$

$t+5y=4$

from Π_2

$x=4-5y$

$t-y+3z=1$

$t=1+y-3z$

$y=\frac{4-t}{5}$

t in terms of z

vector form

there are normal vectors N_1, N_2

$N_1 \perp \ell$
$N_2 \perp \ell$ So take $V_\ell = N_1 \times N_2$

$$V_\ell = \begin{vmatrix} i & j & k \\ 1 & 1 & 2 \\ 1 & -1 & 3 \end{vmatrix} = (5, -1, -2)$$

we need a point P_0 on ℓ

say $z=0 \rightarrow \begin{matrix} x+y=2 \\ x-y=1 \end{matrix} \quad \begin{matrix} x = 3/2 \\ y = 1/2 \end{matrix} \rightarrow P_0 : (3/2, 1/2, 0)$

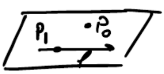
So the equation of ℓ is:

$P = (3/2, 1/2, 0) + t\,(5, -1, -2)$

Ex. Find the plane containing
$\ell: x-1 = 1-2y = \frac{z}{2}$ and $P_0(1,0,1)$

- we already know a point, we need a perpendicular vector to the plane.

P_1 on ℓ and $\vec{P_0P_1}$, V_ℓ are // to the plane



So $\vec{N} = \vec{P_0P_1} \times V_\ell$ (can be taken) taking $t=0 \rightarrow x=1 \quad y=1/2 \quad z=0$

so $P_1(1, 1/2, 0)$ is on ℓ

to find a V_ℓ we bring the eqn in standart form

$$\left(\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c} \right) \rightarrow V=(a,b,c)$$

$$\frac{x-1}{1} = \frac{-1/2+y}{-1/2} = \frac{z}{2} \quad \text{so } V_\ell = (1, -1/2, 2)$$

$$\vec{N} = P_0P_1 \times V_\ell = \begin{vmatrix} i & j & k \\ 0 & 1/2 & -1 \\ 1 & -1/2 & 2 \end{vmatrix} = \left(\frac{1}{2}, -1, -1/2\right)$$

$$\begin{matrix} i & j & k \\ 0 & 1/2 & -1 \end{matrix}$$

- So $P_0P \cdot \vec{N} = 0$

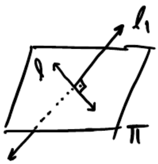
$$\boxed{\frac{1}{2}(x-1) - (y-0) - \frac{1}{2}(z-1) = 0}$$

Definition: A line ℓ is said to be normal to a plane, if it is $\perp$ to the plane that is $V_\ell \parallel \vec{N}$

Ex. normal line to $\pi : x+y+z=2$
passing through $P_0=(1,1,2)$

take $V_\ell=\vec{N}=(1,1,1)$ so $P=(1,1,2)+t(1,1,1)$

Ex. line ℓ on the plane $\pi : x+y-z=2$
$\perp$ to ℓ_1 where $\ell_1 : 2x-1=\frac{y+1}{3}=\frac{z}{2}$



$\ell \perp \ell_1$ they intersect and perpendicular

$V_\ell \perp N$
$V_\ell \perp V_{\ell_1}$ thus we can take $V_\ell = N \times V_{\ell_1}$

$$\begin{array}{r} \\ N \leftarrow \\ \ell_1 \leftarrow \\ \\ \\ \end{array}\begin{vmatrix} i & j & k \\ 1 & 1 & -1 \\ 1/2 & 3 & 2 \end{vmatrix} = (5, -3/2, 5/2) = V_\ell$$
$$\begin{array}{ccc} i & j & k \\ 1 & 1 & -1 \end{array}$$

to find P_0 on ℓ (since $N \cdot V_{\ell_1} \neq 0$ N is not $\perp$ to V_{ℓ_1} so ℓ_1 does not lie on or // to π. So ℓ_1 intersects π exactly at one point which by defn ($\ell \perp \ell_1$) must be on ℓ as well.)

it can be found by simultaneous solution of 3 eqn and is
$P_0=(4/3, 4, 10/3)$

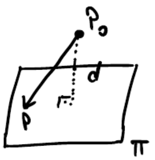
So $\ell: P = (4/3, 4, 10/3) + t(5, -3/2, 5/2)$

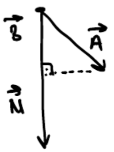
Doğru ve düzlem kesişimi bulma

doğruyu parametrik hale getir düzlemde yerine yaz

(Shortest) Distances

between a point P_0 and a Plane



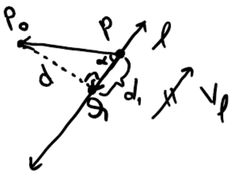


Vectorially, $\vec{B}$ is the projection of $\vec{A}$ on $\vec{N}$.
$\overrightarrow{P_0P}$

$$\vec{B} = \left(\overrightarrow{P_0P} \cdot \frac{\vec{N}}{\|N\|}\right) \cdot \frac{\vec{N}}{\|N\|} \qquad d = \|\vec{B}\| = \left| P_0P \cdot \frac{N}{\|N\|} \right|$$

herhangi bir P noktası bul $\overrightarrow{P_0P}$'nin $\vec{N}$ üzerindeki iz düşümü.

between a point P_0 and a line



$\vec{PQ}$ is the projection of PP_0 on ℓ (V_ℓ)

$$PQ = \left(PP_0 \cdot \frac{V_\ell}{\|V_\ell\|}\right) \cdot \frac{V_\ell}{\|V_\ell\|}$$

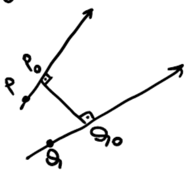
$$d_1 = \|PQ\| = \left| PP_0 \cdot \frac{V_\ell}{\|V_\ell\|} \right|$$

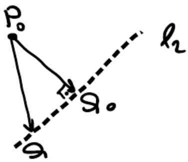
$$d = \sqrt{\|PP_0\|^2 - d_1^2} = \|PP_0\| \sqrt{1-\cos^2\alpha}$$

$$= \|P_0P\| . \sin\alpha = \|P_0P\| \underbrace{\left\| \frac{V_\ell}{\|V_\ell\|} \right\|}_{\to 1} . \sin\alpha = \| P_0P \times u_\ell \| \ , \ u_\ell = \frac{V_\ell}{\|V_\ell\|}$$

distance between two skew lines

they do not intersect and are not parallel





P_0Q_0 is the projection of P_0Q on P_0Q_0

$V_{\ell_2} \perp P_0Q_0 \perp V_{\ell_1}$

so $P_0Q_0 \parallel (V_{\ell_1} \times V_{\ell_2})$

$$P_0Q_0 = \left(P_0Q \cdot \frac{\vec{A}}{\|A\|} \right) \frac{\vec{A}}{\|A\|} \qquad \vec{A} = \text{u2un } P_0Q_0$$

take $\vec{A} = V_{\ell_1} \times V_{\ell_2}$ and $\vec{u} = \dfrac{V_{\ell_1} \times V_{\ell_2}}{\|V_{\ell_1} \times V_{\ell_2}\|}$

so $P_0Q_0 = (P_0Q . \vec{u}) \underbrace{\vec{u}}_{\text{unit vector}}$

$$d = \|P_0Q_0\| = |P_0Q . \vec{u}|$$

$$\Downarrow$$

$$P_0Q - PQ + PQ \qquad = |(P - P_0 + PQ) \cdot \vec{u}|$$

$$\underbrace{Q-P_0+P-Q}_{P-P_0} \qquad \underbrace{}_{//\, l_1} \xrightarrow{\qquad} \underbrace{}_{\perp V_{l_1}}$$

$= |PQ \cdot \vec{u}|$ çarpımları 0

Quadric Surfaces

A quadric surface is a generalization of conic sections (parabolas, hyperbolas, ellipses) in plane.

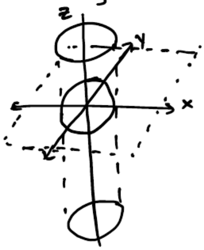
Thus a quadric surface is the set of all points $(x,y,z) \in \mathbb{R}^3$ satisfying a polynomial of degree 2.

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

to sketch a surface in space, we consider its intersections with planes parallel to the coordinate planes. The resulting curves are called "level curves".

Ex. $x^2+y^2=4$ (in $\mathbb{R}^3$) → it is not a curve ; it is a surface

intersecting this surface with horizontal planes, we get:



$z>0$ we get $x^2+y^2=4$

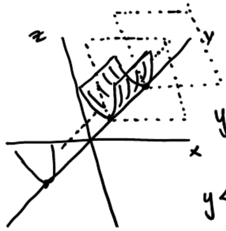
$z=0$ we get $x^2+y^2=4$

$z<0$ " " "

we obtain an unbounded cylinder. (∞)

Ex. $z = x^2$

considering intersections with $y =$ constant planes



$y = y_2 > y_1$, $z = x^2$

$y = y_1$, $z = x^2$

$y < 0$, $z = x^2$

it is also called cylinder (meaning that a variable does not appear in the formula)

Ex. $z = x^2 + y^2$

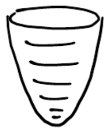
intersect with horizontal planes

$z_2 > 0$, $x^2 + y^2 = (\sqrt{z})^2$

$z_1 = 0$, $x^2 + y^2 = 0 \rightarrow$ the point $(0,0)$

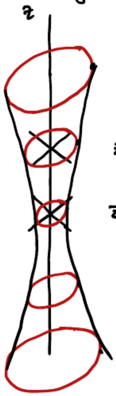
$z < 0$, cannot be true

paraboloid



$x = 0$, $z = y^2$

Ex. $x^2+y^2-z^2=1$

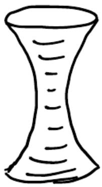


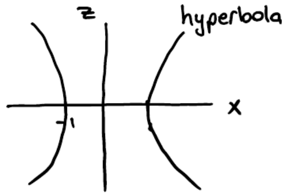
$z>0$, $x^2+y^2=1+z^2$ $(\ell>1)$

$z=0$, $x^2+y^2=1$

Γ is the intersection of xz-plane $(y=0)$ with the surface so $x^2+0-z^2=1$

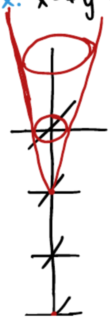
hyperboloid of one sheet





Ex. $x^2+y^2-z^2=-1$ (consider with $z=$ constant)

$z\geq 1$



$z>1$ $\quad x^2+y^2=z^2-1>0$

$z=1$ $\quad x^2+y^2=0$

$z=0$ $\quad x^2+y^2=-1$ not possible

$z=-1$ $\quad x^2+y^2=0$

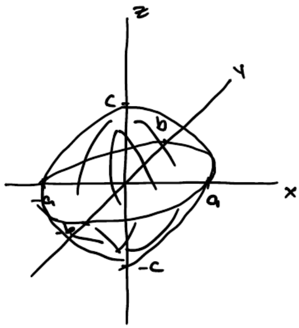
Γ is the intersection of the surface with zx-plane $(y=0)$ $z^2 - x^2 = 1$ hyperbola

we get

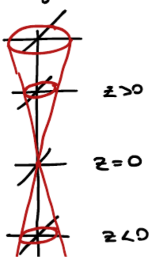
hyperboloid of 2 sheets

Ex. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

ellipsoid



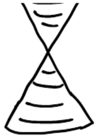
Ex. $x^2 + y^2 = z^2$



Γ is the intersection of the surface with the plane zx $(y=0)$

so $z^2 = x^2$ $z = x$ or $z = -x$

we get cone

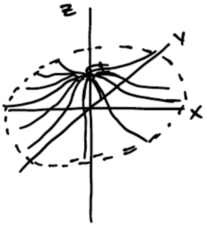


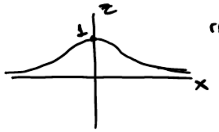
Remark: If in the expression of a quadric surface one only has "x^2+y^2" term,
then the surface is obtained by rotating a curve about z.
Thus,

$z = e^{-(x^2+y^2)}$

if $y=c$, so on zx-plane

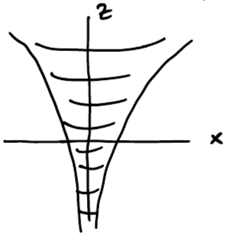
$z = e^{-x^2}$





rotate about z-axis

$z = \ln(x^2+y^2)$ rotate $z = \ln x^2$



Taylor Example.

Estimate $\cos 69°$ $\quad f(x) = \cos x$

$69° = \frac{\pi}{3} + \frac{\pi}{20}$ $\quad a = \frac{\pi}{3}$

$n = 3$

$f\left(\frac{\pi}{3}\right) = \frac{1}{2}$

$f'(x) = -\sin x$ $\quad f''(x) = -\cos x$ $\quad f^3(x) = \sin x$

$f'\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$ $\quad f''\left(\frac{\pi}{3}\right) = -\frac{1}{2}$ $\quad f^3\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$$T_3(x) = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{4}\left(x - \frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{3}\right)^3$$

$$f\left(\frac{\pi}{3} + \frac{\pi}{20}\right) \approx T_3\left(\frac{\pi}{3} + \frac{\pi}{20}\right) =$$

Functions of Several Variables

The properties

LIMIT of $f: \mathbb{R}^2 \to \mathbb{R}$:

If $f: \mathbb{R}^2 \to \mathbb{R}$

$\lim_{P \to P_0} f(P) = L$ means $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t.

if $0 < \underbrace{|P - P_0|}_{\text{length of the vector}} < \delta$

then $|f(P) - L| < \varepsilon$.

α the limit properties are exactly the same with one variable.

PROPERTIES

toplama, çıkarma, çarpma, bölme, squeezing theorem is valid

5) $\lim_{P \to P_0} |f(P)| = 0 \iff \lim_{P \to P_0} f(P) = 0$

Ex. Show that (using defn)

$$\lim_{\underset{\downarrow}{(x,y)} \to \underset{\downarrow}{(1,1)}} (x-y+2) = 2$$

$P \quad P_0$

given $\varepsilon > 0$ find δ s.t.

if $0 < |(x,y)-(1,1)| < \delta$

then $|\underbrace{x-y+2}_{f(P)} - 2| < \underset{?}{\varepsilon}$

$\delta > |(x,y)-(1,1)| =$

$\sqrt{(x-1)^2 + (y-1)^2}$

$\geq |x-1|, |y-1|$

$|x-y| = |x-1-(y-1)|$

$\leq |x-1| + |y-1|$

so;

$|f(x,y) - 2| \leq |x-1| + |y-1| < \delta + \delta = 2\delta \leq \varepsilon$

since $\delta = \frac{\varepsilon}{2}$

Ex. Show $\lim_{(x,y) \to (0,0)} \frac{x^2 y}{x^2+y^2} = 0$

let us use squeezing thm:

$$0 \le \left| \frac{x^2 y}{x^2+y^2} \right| = \underbrace{\frac{x^2}{x^2+y^2}}_{\le 1} |y| \le |y| \to 0$$

by squeezing thm this also goes 0

so by property 5, without abs value, we also have

$$\lim_{p \to (0,0)} f(x,y) = 0$$

Remark: $\mathbb{R} \to \mathbb{R}$ için sağdan ve soldan yaklaşılır.

for $f: \mathbb{R}^2 \to \mathbb{R}$

there are ∞-many ways $P \to P_0$

Thus, if $P \to P_0$ on any curve (through P_0) and

$\lim_{P \to P_0} f(P) = L$ on some fixed curve.

So, $\forall$ such curves we get L

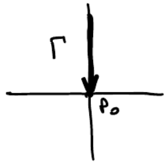
then, $\lim_{P \to P_0} f(P) = L$

However to show that $\lim_{P \to P_0} f(P)$ d.n.e.

it suffies to find just 2 curves giving diff. limit values (or, some curve giving no limit)

Ex. show limit dne

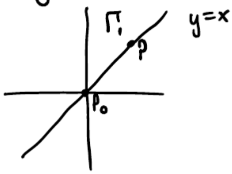
$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$



$\lim_{(x,y)\to(0,0)}$ on y axis (so, $x=0$)

$$\lim_{\substack{(0,y)\to(0,0)\\ y\to 0}} \frac{0}{0+y^2} = 0 \qquad y\neq 0$$

say we check



$$\lim_{(x,y)\to(0,0)} = \lim_{(x,x)\to(0,0)} \frac{x^2}{x^2+x^2} = \frac{1}{2}$$

(x,y) is on Γ_1 so $x=y$ $\qquad x\neq 0$

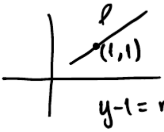
So limit d.n.e.

\# we could "economize", by checking all lines at the same time.
An arbitrary line through $P_0=(0,0)$ (except y-axis)
is : $y=mx$

$$\text{So } \lim_{\substack{(x,y)\to(0,0)\\ y=mx}} f(x,y) = \lim_{(x,mx)\to(0,0)} \frac{x\cdot mx}{x^2+m^2x^2} = \lim_{\substack{x\to 0\\ x\neq 0}} \frac{mx^2}{(1+m^2)x^2} = \frac{m}{1+m^2}$$

→ this means that if you change the m limit value is diff.
so limit d.n.e.

Ex. $\displaystyle\lim_{(x,y)\to(1,1)} \frac{(x-1)(y-1)^2}{(x-1)^2+(y-1)^4}$ do not try L'H çünkü olmaz



$y-1=m(x-1)$

$$\Rightarrow \lim_{(x,\underbrace{m(x-1)+1}_{y})\to(1,1)} \frac{(x-1)\,m^2(x-1)^2}{(x-1)^2+m^4(x-1)^4} =$$

$$\Rightarrow \lim_{x\to1} \frac{m^2(x-1)^{\cancel{3}}}{\cancel{(x-1)^2}(1+m^4(x-2)^2)} = 0$$

this does not say $\lim = 0$
because we did not check all curves
but just lines.

Let us try $(x-1)=(y-1)^2$ (parabola passes thr. (1,1))

$$\lim_{(x,\underbrace{\pm\sqrt{x-1}+1}_{y})\to(1,1)} \frac{(y-1)^2(y-1)^2}{(y-1)^4+(y-1)^4} =$$

$$\downarrow$$

$\displaystyle\lim_{y\to1} 1/2 = 1/2 \neq 0$ so $\displaystyle\lim_{p\to(1,1)} f(p)$ d.n.e.

Continuity

For $f:\mathbb{R}^2\to\mathbb{R}$ (or $\mathbb{R}^n\to\mathbb{R}$) we say that f is continuous at P_0 if $f(P_0)$, $\displaystyle\lim_{P\to P_0} f(P)$ exist and equal.

α If $f,g:\mathbb{R}^2\to\mathbb{R}$ then $f\circ g$ or $g\circ f$ are undefined.
However $\mathbb{R}^2\to\mathbb{R}\to\mathbb{R}$

$g \circ f : \mathbb{R}^2 \xrightarrow{f} \mathbb{R} \xrightarrow{g}$ is cont. if f, g are cont.

Ex. $e^{x \sin(xy)}$ is cont on $\mathbb{R}^2$

xy - polyn $(\mathbb{R}^2 \to \mathbb{R})$ is cont

$\sin u$ - $\mathbb{R} \to \mathbb{R}$ is cont

$\sin xy$ $\mathbb{R}^2 \to \mathbb{R}$ is cont

$x.\sin xy$ (product of cont funcs) or ratio, sum, diff

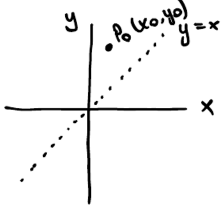
$e^u : \mathbb{R} \to \mathbb{R}$ is cont

So by remark $h(x,y) = e^{x\sin(xy)} : \mathbb{R}^2 \to \mathbb{R}$ is cont every

Ex. Let $f(x,y) = \begin{cases} x^2+y^2 & \text{if } x \neq y \\ 2x & \text{if } x = y \end{cases}$

when is f cont.?

A) $P_0 = (x_0, y_0)$ is not on $x=y$:



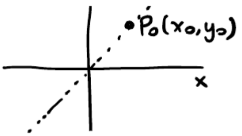
$\lim_{P \to P_0} f(P) = \lim_{P \to P_0} (x^2+y^2) = x_0^2 + y_0^2$

f is cont. at. P_0

B) $P_0 = (x_0, y_0)$ is on $y=x$ line:

Firstly $y_0 = x_0$

① P_0'a $y=x$ üzerinden yaklaşabiliriz



ya da başka bir curve üzerinden ②

1 → $\lim\limits_{\substack{P\to P_0 \\ \text{on } \Gamma}} f(P) = \lim\limits_{\substack{(x,y)\to(x_0,y_0) \\ x=y}} 2x = 2x_0$

2 → $\lim\limits_{\substack{P\to P_0 \\ \text{not on } \Gamma}} f(P) = \lim\limits_{P\to P_0} x^2+y^2 = x_0^2+y_0^2 = 2x_0^2$ ($x_0=y_0$)

to have a limit at $P_0\,(x_0,x_0)$ we need $2x_0 = 2x_0^2$

that is $x_0 = 0, 1$

$P_0 = (0,0)$, $P_1(1,1)$

- Contuniouty -

Since $x_0 = y_0$, by defn,

$f(x_0,x_0) = 2x_0 = \lim\limits_{P\to(x_0,x_0)} f(P)$

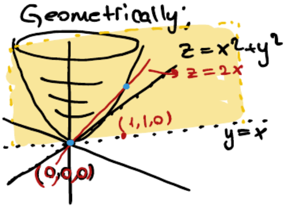
if $x_0 = 0,1$

so f is cont at (0,0) and (1,1)

is not cont any other point on $y=x$.

(since limit d.n.e. these points)

Geometrically:



DERIVATIVE of $f: \mathbb{R}^2 \to \mathbb{R}$

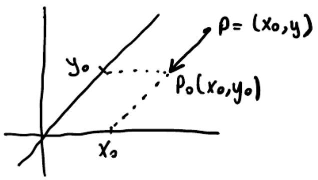
Recall

$$\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$$

for $f: \mathbb{R}^2 \to \mathbb{R}$ this is meaningless since we cannot divide by the vector $P-P_0$

- $f(x,y) = x+y$ does not have a derivative

A) Then $P \to P_0$ means $x \to x_0$ (resp $y \to y_0$)



$$\lim_{y \to y_0} \frac{f(P)-f(P_0)}{y-y_0} = \lim_{y \to y_0} \frac{f(x_0,y)-f(x_0,y_0)}{y-y_0}$$

this is called the partial derivative of f wrt y shown by $\left.\dfrac{\partial f}{\partial y}\right|_{P_0}$

∂ "del"

B) Similarly,

$$\lim_{\substack{P \to P_0 \\ \left(\text{on the line // to the x passing } P_0\right)}} \frac{f(P)-f(P_0)}{x-x_0} = \lim_{x \to x_0} \frac{f(x,y_0)-f(x_0,y_0)}{x-x_0} =$$ partial derivative of f wrt x at P_0 shown by $\left.\frac{\partial f}{\partial x}\right|_{P_0}$

Remarks and Notations:

1) Let $g(x) = f(x,y_0)$

Then $g'(x_0) = \lim_{x \to x_0} \frac{g(x)-g(x_0)}{x-x_0} = \lim_{x \to x_0} \frac{f(x,y_0)-f(x_0,y_0)}{x-x_0} = \left.\frac{\partial f}{\partial x}\right|_{P_0}$

Thus $\left.\frac{\partial}{\partial x}(x\sin(xy))\right|_{P_0} = \left.\frac{d\,g(x)}{dx}\right|_{x_0} = \left.\frac{d}{dx}\, x\sin(xy_0)\right|_{x=x_0}$

$= \left.\sin(xy_0) + x\cos(xy_0)\,y_0\right|_{x=x_0}$

2) Notations:

$$f_x(x_0,y_0) = \overbrace{D_x}^{D_1 \cdots} f(x_0,y_0) = f_1(x_0,y_0) = \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}$$

$$f_y(x_0,y_0) = \underbrace{D_y}_{D_2 \cdots} f(x_0,y_0) = f_2(x_0,y_0) = \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}$$

α $\frac{\partial}{\partial x} x^2 \sin y = 2x\sin y$

$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} x^2\sin y\right) = 2x\cos y = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} x^2\sin y\right)$ → hey böyle olmaz

$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} x^2\sin y\right) = 2\sin y$

NOTATIONS for higher-order partial derivatives

$$\frac{\partial}{\partial x}\Big(\underbrace{\frac{\partial}{\partial y} f(x,y)}_{f_y}\Big) = f_{yx} = f_{21} = D_{21}f = \frac{\partial^2}{\partial x \partial y} f(x,y)$$

CLAIRAUT'S THM: If f has continuous 2^{nd} order mixed partials then $f_{xy} = f_{yx}$

$$f_{xxy} = f_{xyx} = f_{yxx}$$

REMARK: Even if both partials exist at P_0 this does not say f is cont. at P_0.

Consider $f(x,y) = \begin{cases} 1 & \text{if } x \neq 0 \text{ and } y \neq 0 \\ 2 & \text{if } x = 0 \text{ or } y = 0 \end{cases}$

$\lim\limits_{P \to (0,0)} f = \text{d.n.e.}$ so f is not cont. at $P = (0,0)$

But in finding $\frac{\partial}{\partial x} f(0,0) = \frac{\partial}{\partial} 2 = 0$

we consider what happens above the x-axis so

$$\frac{\partial}{\partial y} f(0,0) = 0$$

Also at $(0,0,2)$ a plane which we could call "tangent plane" d.n.e

İlginç şeyler

(1) $$\lim_{x \to x_0} \frac{f(x) - f(x_0) - L(x - x_0)}{x - x_0} = 0$$

which can be generalized to $\mathbb{R}^2 \to \mathbb{R}$

$$\lim_{P \to P_0} \frac{f(x,y) - f(x_0,y_0) - L_1(x - x_0) - L_2(y - y_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0$$

(2) Epsilon form

$$\underbrace{\frac{f(x) - f(x_0) - L(x - x_0)}{x - x_0}}_{= \varepsilon(x - x_0)} \underset{(x \to x_0)}{\to} 0$$

so, $f(x) = f(x_0) + L(x - x_0) + \varepsilon(x - x_0)$ and $\varepsilon \to 0$ as $x \to x_0$

generalizing to $f: \mathbb{R}^2 \to \mathbb{R}$:

$\exists\, L_1, L_2$ s.t.

$$f(x,y) = f(x_0,y_0) + L_1(x - x_0) + L_2(y - y_0) + \varepsilon_1(x - x_0) + \varepsilon_2(y - y_0)$$

where, $\varepsilon_1(\Delta x, \Delta y)$, $\varepsilon_2(\Delta x, \Delta y) \to 0$

as $\begin{matrix} \Delta x \to 0 \\ \Delta y \to 0 \end{matrix} \equiv (P \to P_0)$

$f: \mathbb{R}^2 \to \mathbb{R}$ is called differentiable at $P_0(x_0, y_0)$

if 1) or 2) holds. (Actually 1) and 2) are equivalent)

Warning: There is no derivative concept for $f: \mathbb{R}^2 \to \mathbb{R}$

Theorem: If f is differentiable at $P_0 = (x_0, y_0)$ then

a) f is cont. at P_0

b) the constants L_1, L_2 are actually $f_x(P_0)$, $f_y(P_0)$

$$f_x(P_0) = \lim_{x \to x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0}$$

Remarks:

1) $f(x) = f(x_0) + f'(x_0)(x - x_0) + \varepsilon (x - x_0)$

and $\varepsilon \to 0$ as $x \to x_0$

it is differentiability of $f: \mathbb{R} \to \mathbb{R}$

$f(x) \approx f(x_0) + f'(x_0)(x - x_0) \to$ tangent line approximation

If $f: \mathbb{R}^2 \to \mathbb{R}$ is diff at P_0 then

$f(x,y) = f(P_0) + f_x(P_0)\Delta x + f_y(P_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$

$\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$

(differentiability expression)

we get the approximation

$f(P) \approx f(P_0) + f_x(P_0)\Delta x + f_y(P_0)\Delta y$

$\to$ tangent plane approximation

Note that: the differential functions

$\Delta f = f(x) - f(x_0) \approx f'(x_0)(x - x_0)$ 1-dim

$$\Delta f = f(P) - f(P_0) \approx f_x(P_0)(x-x_0) + f_y(P_0)(y-y_0) \quad \text{2-dim}$$

$df(\Delta x)$ resp $df(\Delta x, \Delta y)$

Ex. show that $f(x,y) = xy$ is diff. everywhere

say we use limit form:

$$\frac{f(x,y) - f(x_0,y_0) - f_x(P_0)\Delta x - f_y(P_0)\Delta y}{\|P-P_0\|} \to 0$$

we have

$$\frac{xy - x_0y_0 - \overset{f_x(P_0)}{y_0}(x-x_0) - x_0(y-y_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}}$$

$$= \frac{(x-x_0)(y-y_0)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = \frac{\Delta x \cdot \Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}} \to 0$$

$$\underset{\downarrow \atop 0}{0} \leq \left| \underbrace{\frac{\Delta x}{\sqrt{(\Delta x)^2+(\Delta y)^2}}}_{\leq 1} \Delta y \right| \leq \underset{\downarrow \atop 0}{|\Delta y|}$$

so by lim form, $f(x,y) = xy$ is diff at any point (x_0, y_0)

THEOREM (total derivative formula, chain rule):

Let $f: \mathbb{R}^2 \to \mathbb{R}$ be differentiable at P_0 and $x(t), y(t): \mathbb{R} \to \mathbb{R}$ have derivatives. Then

$$\left.\frac{d\, f(x(t), y(t))}{dt}\right|_{t=t_0} = f_x(\underset{\substack{\| \\ (x(t_0),\, y(t_0))}}{P_0}) \cdot x'(t_0) + f_y(P_0) \cdot y'(t_0)$$

Summary:

f is diff at P_0
- → $f_x(P_0)$, $f_y(P_0)$ exist
- → f is cont at P_0
- → linear approx makes sense ($\text{error} = \underbrace{\varepsilon_1}_{\to 0} \Delta x + \varepsilon_2 \Delta y$)
- → total derivative formula (hence chain rule) is valid.

There is a PRACTICAL way of showing differentiability:

THEOREM (fundamental lemma):

If $f_x(x,y)$, and $f_y(x,y)$ exist in a disc about P_0 and they are cont. at P_0 then f is differentiable at P_0.

Ex. Let $f(x,y) = xy\sin x$

then

$$\left.\begin{aligned} \frac{\partial f}{\partial x} \underset{y=\text{const.}}{=} y\sin x + xy\cos x \\ \frac{\partial f}{\partial y} \underset{x=\text{const.}}{=} x\sin x \end{aligned}\right\} \text{cont. everywhere}$$

By fundamental lemma, f is differentiable everywhere.

Note: showing that f is diff. at $P_0 \not\Rightarrow f_x, f_y$ are cont.
converse of fundamental lemma is false

Ex.

Let $f(x,y) = \begin{cases} (x^2+y^2)\sin\frac{1}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

$P_0 = (0,0)$

1) Find $f_x(x,y)$

If $(x_0,y_0) \neq (0,0)$: $\forall (x,y)$ near (x_0,y_0),

So $f_x(x_0,y_0) = 2x_0 \cdot \sin\frac{1}{\sqrt{x_0^2+y_0^2}} + (x_0^2+y_0^2) \cdot \cos\frac{1}{\sqrt{x_0^2+y_0^2}} \cdot \left(-\frac{1}{2}\right) \ldots$

$$f_x(0,0) = \lim_{x\to 0} \frac{f(x,0) - \overset{0}{f(0,0)}}{x-0} = \lim_{x\to 0} \frac{x^{\cancel{2}} \sin\frac{1}{\sqrt{x^2}}}{\cancel{x}} = 0$$

(by sequeeze theorem)

$$\text{So } f_x(x,y) = \begin{cases} 2x\sin\frac{1}{\sqrt{x^2+y^2}} - \frac{1}{2} \cdot 2x \cdot \frac{1}{\sqrt{x^2+y^2}} \cdot \cos\frac{1}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

α $\lim_{(x,y)\to(0,0)} f_x(x,y) =$ d.n.e. because as $(x,y) \to (0,0)$ on $y=0$
we get

$$\lim_{x\to 0} f_x(x,0) = \lim_{x\to 0}\Big(\underbrace{2x.\sin\frac{1}{\sqrt{x^2}}}_{\downarrow\, 0} - \underbrace{\frac{1}{2}.\frac{2x}{\sqrt{x^2}}.\cos\frac{1}{\sqrt{x^2}}}_{\text{d.n.e}}\Big)$$

so f_x at $(0,0)$ is not cont. (similerly f_y at $(0,0)$ is not)

but they exist, $f_x(0,0) = f_y(0,0) = 0$

f is diff. at $(0,0)$ because, by limit form

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - \overset{\nearrow 0}{f_x(0,0)}(x-0) - \overset{\nearrow 0}{f_y(0,0)}(y-0)}{\sqrt{(x-0)^2+(y-0)^2}}$$

$$= \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)\sin\frac{1}{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2}.\sin\frac{1}{\sqrt{x^2+y^2}} = 0$$

(by sequeezing theorem)

so f is diff at $(0,0)$

$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

Examples for linear approximations (tangent plane):

1) Find $\sqrt{1,02.(1,99)^2}$ by using linear approximation

define $f(x,y) = \sqrt{xy^2}$, $P_0 = (1,2)$

$f_x(x,y) = \frac{1}{2\sqrt{x}}.\sqrt{y^2}$

$f_y(x,y) = \sqrt{x} \cdot 1$ (near $y=2$)

$\sqrt{1{,}02 \cdot (1{,}99)^2} \approx f(1,2) + f_x(1,2)(1{,}02-1) + f_y(1,2)(1{,}99-2)$

$= 2 + 1.(0{,}02) + 1.(-0{,}01)$

$= 2.01$

2) $e^{\frac{1}{100} \cdot \sqrt[3]{7{,}99}}$

consider $f(x,y) = e^{x \cdot \sqrt[3]{y}}$ $\quad P_0(0,8)$

$f_x = \sqrt[3]{y} \cdot e^{x \cdot y^{1/3}}$

$f_y = x \cdot e^{x \cdot 1/3} \cdot \frac{1}{3} \cdot y^{-2/3}$

So the value $\approx f(0,8) + f_x(0,8)\underbrace{(\tfrac{1}{100}-0)}_{(x-x_0)} + f_y(0,8)\underbrace{(7{,}99-8)}_{(y-y_0)}$

$= 1 + \frac{2}{100} = 1.02$

THE CHAIN RULE

Recall the total derivative formula:

$$\frac{d}{dt} f(x(t), y(t)) = f_x \cdot x'(t) + f_y \cdot y'(t)$$

(if f, x, y are diff)

Remarks:

1) Let $f: \mathbb{R}^2 \to \mathbb{R}$ then

$\underbrace{\frac{d}{dx} f}_{\text{means } y=g(x)}$, $\frac{\partial}{\partial x} f$ do not mean the same thing.

so, $\frac{d}{dx} f(x, g(x)) = f_1 \cdot \frac{d}{dx} x + f_2 \frac{d}{dx} g(x) = f_1 + f_2 \cdot g'(x)$

$\frac{\partial f}{\partial x}$

(by important derivative formula)

$\frac{\partial f}{\partial x}$ means y is constant.

2) Let $f: \mathbb{R}^2 \to \mathbb{R}$

$x = x(s,t)$, $y = y(s,t)$

$z(s,t) = f(x(\underbrace{s,t}_{u}), y(\underbrace{s,t}_{v}))$

$$\frac{\partial z}{\partial s} \underset{t=\text{const.}}{=} \underset{f_u}{f_1} \cdot \frac{\partial x}{\partial s} + \underset{f_v}{f_2} \cdot \frac{\partial y}{\partial s}$$

Ex. $z = f(x,y) = xy^2 + y\sin x$; $x = t^2$, $y = \cos t$ (diff by F. lemma)

$z = f(t^2, \cos t)$

$$\frac{dz}{dt} = \underset{f_x}{f_1} \cdot \frac{d}{dt} t^2 + \underset{f_y}{f_2} \frac{d}{dt} \cos t$$

$= (y^2 + y\cos x) \cdot 2t + (2xy + \sin x)(-\sin t)$

t'yi yerine yaz.

So for example, $\left.\frac{dz}{dt}\right|_{t=0} = 0$ (thus $x(0)=0$ $y(0)=1$)

Ex. $f(x,y) = x^y$; $x = x(u,v) = u\tan v$; $u(t) = t^2$
$y = y(u,v) = v\sin u$ $\quad v(t) = \ln t$

$$\frac{d}{dt} f(x(u(t),v(t)),\, y(u(t),v(t)))$$

$$= f_1 \cdot \frac{d}{dt} x(u(t),v(t)) + f_2 \cdot \frac{dy(u(t),v(t))}{dt}$$

$f_1 \downarrow f_x = y \cdot x^{y-1}$ $\qquad$ $f_2 \downarrow f_y = x^y \cdot \ln x$

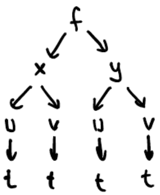
$$\frac{d}{dt} x(u(t),v(t)) = x_1 \cdot u'(t) + x_2 \cdot v'(t)$$

$x_1 \downarrow x_u = \tan v$ $\qquad$ $x_2 \searrow x_v = u \cdot \sec^2 v$

$$\frac{d}{dt} y(u(t),v(t)) = y_u \cdot u'(t) + y_v \cdot v'(t)$$

$y_u \downarrow v \cdot \cos u$ $\qquad$ $y_v \downarrow \sin u$

NOTE: The "tree" formulation



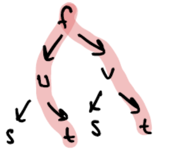
go thru all possible brunches that lead to t

$$\frac{d}{dt} f = f_x \cdot x_u \cdot u'(t) + f_x \cdot x_v \cdot v'(t) + f_y \cdot y_u \cdot u'(t) + f_y \cdot y_v \cdot v'(t)$$

(this method works only for first order derivatives, not higher ones)

Ex. Let $f:\mathbb{R}\to\mathbb{R}$ be diff twice (second order cont. partials)

$z(s,t) = f(st, s^2t^2)$ Find $\dfrac{\partial^2 z}{\partial s\,\partial t}$



$u = st \quad v = s^2t^2$

$$\frac{\partial z}{\partial t} = \overset{f_u}{\overset{\uparrow}{f_1}} \cdot u_t + \overset{f_v}{\overset{\uparrow}{f_2}} \cdot v_t$$

$$\frac{\partial}{\partial s}\left(\frac{\partial z}{\partial t}\right) = \frac{\partial}{\partial s}\left(\underset{f_u(u,v)}{\underset{\downarrow}{f_1}} \cdot \overset{s}{\overset{\uparrow}{u_t}} + f_2 \cdot \overset{2ts^2}{\overset{\uparrow}{v_t}}\right)$$

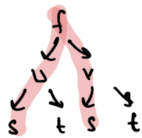
NOTE: to find f_u, v = const. and v does not disappear: say $f(u,v) = u^2v^3$ then $f_u = 2uv^3$ so $f_u = f_u(u,v)$

so by product derivative

$$= \left[\left(\frac{\partial}{\partial s} f_1\right)\cdot u_t + f_1 \cdot \underbrace{\frac{\partial}{\partial s} u_t}_{u_{ts}=1}\right] + \left[\left(\frac{\partial}{\partial s} f_2\right) v_t + f_2 \underbrace{\frac{\partial}{\partial s} v_t}_{4st}\right]$$

$$\alpha \quad \frac{\partial}{\partial s} f_1 = \frac{\partial}{\partial u} f_1 \cdot \frac{\partial u}{\partial s} + \frac{\partial f_1}{\partial v} \cdot \frac{\partial v}{\partial s} = f_{11} \cdot \frac{\partial u}{\partial s} + f_{12} \cdot \frac{\partial v}{\partial s}$$

(or by using total derivative formula)



similarly; $$\frac{\partial}{\partial s} f_2 = \underset{f_{vu}}{\underset{\downarrow}{f_{21}}} \cdot \frac{\partial u}{\partial s} + \underset{f_{uv}}{\underset{\downarrow}{f_{22}}} \cdot \frac{\partial v}{\partial s}$$

Implicit Differentiation

Ex. Suppose $z = f(x,y)$ is impl. defined by

$zx + \sin z - xyz + xy = 1 \quad P_0 = (1,1,0)$ Find $\left.\frac{\partial z}{\partial x}\right|_{P_0}$ (y = const)

$$\frac{\partial z}{\partial x} = \left[\frac{\partial z}{\partial x} \cdot x + z\right] + \cos z \cdot \frac{\partial z}{\partial x} - y \cdot \left[\frac{\partial z}{\partial x} \cdot x + z\right] + y = 0$$

$$\frac{\partial z}{\partial x} = \left.\frac{-z + zy - y}{x + \cos z - xy}\right|_{P_0} = -1$$

Now suppose $F(x,y) = 0$ defines a diff funct. $y = f(x)$

If F is diff. then apply total derivative formula to both sides.

$x = x$
$y = f(x)$ $\quad F_x \cdot \frac{dx}{dx} + F_y \cdot \frac{dy}{dx} = 0$

so $f'(x) = -F_x / F_y$

Ex. Consider $F(x,y) = y^3 - x$

then $F(x,y) = 0$ defines $y = f(x)$ $(= x^{1/3})$

then $f'(x) = \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-1}{3y^2} \left(= \frac{1}{3x^{2/3}}\right)$

At $(0,0)$ $f'(0)$ cannot defined by $-F_x / F_x$

since $F_y = 0$ at $(0,0)$

⍺ So if $F_y|_{P_0} = 0$ we cannot solve y as a differentiable func.

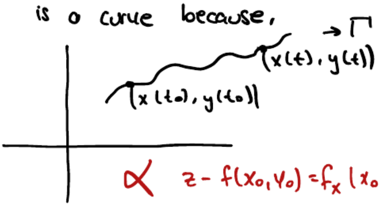
but sometimes, just as a cont. func.

TANGENT PLANES

Recall that in plane,

$\{x(t), y(t) : t \in I, \text{some interval}\} = \Gamma$

is a curve because,



α $z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

Now if $x(t), y(t)$ have derivatives, then Γ has tangent lines say at $t = t_0$, we have the tangent line

$y = mx + n$

so $\left.\frac{dy}{dx}\right|_{t_0} = m$ Then, $\frac{d}{dt} y(t) = \frac{dy}{dx} \cdot \frac{dx(t)}{dt}$

$m = \frac{y'(t)}{x'(t)}$ since $\vec{v} = (1, m) \parallel$ tangent line

then $\vec{v} = \left(1, \frac{y'(t)}{x'(t)}\right) = \frac{1}{x'(t)} (x'(t), y'(t))$

we conclude that $(x'(t), y'(t)) \parallel$ tangent line

α the tangent line to a curve $(x(t), y(t), z(t))$ in space is defined to be the line parallel to $(x'(t), y'(t), z'(t))|_{t_0}$ passing through $P_0 = (x(t_0), y(t_0), z(t_0))$

α Consider a differentiable function $f(x,y)$ given by the implicit formulation $F(x,y,z)=0$

Consider any curve on this surface passing through $P_0=(x_0,y_0,z_0)$

Then Γ satisfies the eqn. of the surface

So $F(x(t),y(t),z(t))=0$

taking derivative $\frac{d}{dt}$

we get $\left(F_1 . x'(t) + F_2 . y'(t) + F_3 . z'(t)\right)\Big|_{P_0} = \frac{d\,0}{dt} = 0$

$$(F_1,F_2,F_3)\big|_{P_0} \cdot \underbrace{(x'(t),y'(t),z'(t))\big|_{P_0}}_{\parallel \text{ to the tangent line to } \Gamma}$$

So $(F_1,F_2,F_3)\big|_{P_0}$ (which is indep. of Γ's)

$\perp$ tangent lines to all Γ's on the surface thru. P_0.

CONCLUSION:

The tangent lines drawn to any curve (thru P_0) on the surface $F(x,y,z)=0$ lie on one plane with normal vector $\vec{N}=(F_1,F_2,F_3)\big|_{P_0}$ and thru P_0.

DEFN: This plane is called the tangent plane to $F(x,y,z)=0$ at $P_0=(x_0,y_0,z_0)$

the eqn is: $\Big[\underbrace{(x,y,z)}_{=P} - \underbrace{(x_0,y_0,z_0)}_{=P_0}\Big] \cdot (F_1,F_2,F_3) = 0$

Remarks:

1) To define the tangent plane we assumed F is diff. at P_0. (Actually, a geometric defn. of tangent planes says that: tangent plane at P_0 exists $\iff$ F is diff. at P_0)

2) If the surface is given explicitly, $z = f(x,y)$ then obviously $F(x,y,z) = f(x,y) - z = 0$ is an implicit form. So, f is diff $\Rightarrow$ the eqn of the tangent line is

$$(P-P_0)\cdot(\underset{F_1}{f_x}, \underset{F_2}{f_y}, \underset{F_3}{-1}) = 0$$

normal vector to the tangent plane

3) One shows (F_x, F_y, F_z) by ∇F, called the gradient of F. (If $f(x,y): \mathbb{R}^3 \to \mathbb{R}$ then $\nabla F = (f_x, f_y)$)
Thus, if F is diff then $\nabla F|_{P_0}$ is a normal vector to the plane.

Ex. Let us find the eqn. of the tangent plane drawn to $x^2+y^2+z^2=4$ at $P_0 = (0, \sqrt{2}, \sqrt{2})$
($F(x,y,z) = x^2+y^2+z^2-4$ is diff)
A normal is $\vec{N} = \nabla F(P_0) = (2x, 2y, 2z)|_{P_0} = (0, 2\sqrt{2}, 2\sqrt{2})$
So the eqn:

$$((x,y,z) - (0,\sqrt{2},\sqrt{2}))\cdot(0, 2\sqrt{2}, 2\sqrt{2}) = 0$$

$$0.x + (y-\sqrt{2}).2\sqrt{2} + (z-\sqrt{2})2\sqrt{2} = 0$$

$$= z+y = 2\sqrt{2}$$

2) Find the eqn of the tangent line drawn to $x^2+y^2=1$ at $P_0 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$F(x,y) = x^2+y^2-1=0$

$(x,y)-\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \cdot \nabla F|_{P_0} = 0 \rightarrow x+y=\sqrt{2}$

Ex. Find the tangent line to the curve of intersection of the surfaces at $P_0 = (1,1,1)$

$\Pi_1 = x^2+y^2+2z^2=4$

$\Pi_2 = z = e^{x-y}$

(a curve in space is the intersection of two surfaces)

$\Gamma \rightarrow$ curve

$t \rightarrow$ tangent line at P_0

Γ lies on Π_1 so its tangent lines lies on the tangent plane drawn to Π_1 at P_0.

So, $t \perp \vec{N}_1$

$\downarrow$

normal to the tangent plane to Π_1

Γ lies on Π_2 as well. So, $t \perp \vec{N}_2$

So we can take $\vec{V}_t$ $(\parallel t)$ as $\vec{V}_t = N_1 \times N_2$

$N_1 = (2x, 2y, 4z)|_{(1,1,1)} = (2,2,4)$

$N_2 = (e^{x-y}, -e^{x-y}, -1)_{(1,1,1)} = (1,-1,-1)$

$\vec{V}_t = \begin{vmatrix} i & j & k \\ 2 & 2 & 4 \end{vmatrix} = (2,6,-4)$

$|1 \quad -1 \quad -1|$

So eqn of t: $P = (1,1,1) + t(2,6,-4)$

Alternatively: t lies on both tangent planes at P_0.
So find these tangent planes and intersect them.

Ex. Let us find the points on the tangent planes parallel to the xz-plane where the surface is

$$x^2 + 4y^2 + 16z^2 - 2xy = 12$$

since parallelity of two planes means their normals are //, we get

$$\vec{N} = \nabla(x^2 + 4y^2 + 16z^2 - 2xy - 12) \mathbin{/\!/} j$$

a normal to xz plane

So, $\nabla F = (2x - 2y, 8y - 2x, 32z) = \lambda(0, 1, 0)$

$$\left.\begin{array}{l} 2x - 2y = 0 \\ 8y - 2x = \lambda \\ 32z = 0 \end{array}\right\}$$

$x = y$
→ we are not interested
$z = 0$

So $P = (x, x, 0)$ this point must be on the surface.

So substituting $(x, x, 0)$ in the surface

eqn: $x^2 + 4x^2 + 0 - 2x^2 = 12 \quad x = \pm 2$

$P_1 = (2, 2, 0) \quad P_2 = (-2, -2, 0)$

we expect 2 points because the surface is a (rotated) ellipsoid,

DIRECTIONAL DERIVATIVES

bu sağlanırsa diff

$D_{\vec{u}} f(x_0,y_0) = \nabla f(x_0,y_0) \cdot \frac{\vec{u}}{|\vec{u}|}$ birim vektör → Youtube

$$\lim_{t\to 0} \frac{f(x_0+tv_1,\ y_0+tv_2) - f(x_0,y_0)}{t\,\|\vec{v}\|} \rightarrow D_{\vec{v}} f(P_0)$$

$$D_{\vec{v}} f(P_0) = \lim_{t\to 0} \frac{f(x_0+t\cos\alpha,\ y_0+t\sin\alpha) - f(x_0,y_0)}{t}$$

$\vec{v} = (\cos\alpha, \sin\alpha)$

Remark: If f is diff at P_0 then all directional derivatives exist and

$D_v f(P_0) = \nabla f|_{P_0} \cdot \vec{u}$ $\quad \vec{u} = \left(\frac{\vec{v}}{|\vec{v}|}\right)$

Ex. Let $f(x,y) = x^2+2y^2$

Find $D_{\vec{i}+\vec{j}} f(1,1)$ $\quad$ f is diff everywhere

$$= \nabla f(1,1) \cdot \frac{\vec{i}+\vec{j}}{\sqrt{2}} = (2x,4y)\big|_{(1,1)} \cdot \frac{(1,1)}{\sqrt{2}} = \frac{2+4}{\sqrt{2}}$$

Ex. Find the rate of change of $f(x,y) = 2xy + x^2$ if p moves from $P_0 = (1,2)$ towards $P_1(1,1)$

$D_{\vec{v}} f(1,2)$ is wanted and $\vec{v} = P_0P_1 = P_1 - P_0 = (0,-1)$

since f is polyn. so it is diff everywhere.

$D_{(0,-1)} f(1,2) = \nabla f(1,2) \cdot \underline{(0,-1)}$ ↳ unit vector

$$(2y+2x,\ 2x)\big|_{(1,2)}$$
$$= (6,2)(0,-1) = -2$$

Ex. Assuming f is diff at P_0 and given

$D_{i+j}\, f(P_0) = 1$

$\|\nabla f(P_0)\| = \sqrt{2}$

find all possible values of $f_x(P_0), f_y(P_0)$

since f is diff at P_0,

$$D_{i+j}\, f(P_0) = (f_x(P_0),\ f_y(P_0)) \cdot \frac{(1,1)}{\sqrt{2}} = 1 \leftarrow \text{given}$$

So, $f_x(P_0) + f_y(P_0) = \sqrt{2}$ → karesini al

$f_x^2(P_0) + f_y^2(P_0) = 2$ → given

So $f_x(P_0) \cdot f_y(P_0) = 0$

So if $f_x(P_0) = 0$ then $f_y(P_0) = \sqrt{2}$

if $f_y(P_0) = 0$ then $f_x(P_0) = \sqrt{2}$

Ex. Assuming the only force is gravity a ball is put at P_0 on the surface $z = f(x,y) = xy - 2x^2y$

a) If $P_0(0,1,0)$ then in which direction will the ball move? (obviously in the direction where the slope is steepest downwards.)

$$D_{\vec{v}}\, f(P_0) = \nabla f(P_0) \cdot (\cos\alpha, \sin\alpha)$$

(dik)

$\vec{v} = (\cos\alpha, \sin\alpha)$

$$\nabla f(P_0) = (1, 0)$$

So, $D_v f(P_0) = 1.\cos\alpha + 0.\sin\alpha = \cos\alpha$

This is minimum, if $\alpha = \pi$ So $\vec{v} = -\vec{i}$ (so west in compass)

b) Same, $P_0(1/2, 0, 0)$.

$D_v f(P_0) = (0,0) \cdot (\cos\alpha, \sin\alpha) = 0$ for all directions.

So tangent plane is horizontal and ball does not move.

Remark: If f is diff at P_0 then it is easy to calculate the direction $\vec{v}$ s.t. $D_v f(P_0)$ is min or max

Because

$$D_{\vec{v}} f(P_0) = \underbrace{(f_x, f_y)\big|_{P_0}}_{\nabla f} \cdot \underbrace{(\cos\alpha, \sin\alpha)}_{\text{unit vector}} = \|\nabla f(P_0)\| \, \|(\cos\alpha, \sin\alpha)\| \cdot \cos\beta$$

β = smaller angle between $\nabla f(P_0)$ and $\vec{v}$.

$= \|\nabla f(P_0)\| \cdot 1 \cdot \cos\beta$ (minimize / maximize when α changes, β changes)

if $\beta = \pi$ $\cos\beta$ is min

if $\beta = 0$ $\cos\beta$ is max

Conclusion: $D_v f(P_0)$ is min if $\vec{v}$ makes an angle π with $\nabla f(P_0)$, that is $\vec{v} = -\nabla f(P_0)$, is max if $\vec{v} = \nabla f(P_0) = (f_x, f_y)\big|_{P_0}$

↓ most negative direction ↑ most positive direction

Ex. $f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

Find $D_{(\cos\alpha, \sin\alpha)} f(0,0) = \lim_{t\to 0} \dfrac{f(0+t\cos\alpha, 0+t\sin\alpha) - \overset{\shortparallel}{f(0,0)}}{t}$

$$\lim_{t\to 0} \frac{t\cos\alpha\,(t\sin\alpha)^2}{t^2\cdot\cos^2\alpha + t^4\cdot\sin^4\alpha} \cdot \frac{1}{t} = \begin{cases} \lim_{t\to 0} \dfrac{0}{t^4 \cdot \sin^4\alpha} = 0 & \text{if } \cos\alpha = 0 \\ \lim_{t\to 0} \dfrac{\cos\alpha \cdot \sin^2\alpha}{\cos^2\alpha + t^2\cdot\sin\alpha} & \text{if } \cos\alpha \neq 0 \\ = \dfrac{\sin^2\alpha}{\cos\alpha} & \end{cases}$$

So $D_v f(0,0)$ exists in all directions.

But is not differentiable at $(0,0)$, because if it were, f would be cont. at $(0,0)$ but we showed earlier that $\lim_{(x,y)\to(0,0)} (x,y)$ = d.n.e.

Just check: $\underset{\overset{\shortparallel}{(f_x, f_y)|_{(0,0)}}}{\nabla f(0,0)} \cdot (\cos\alpha, \sin\alpha) = (0,0)\cdot(\cos\alpha, \sin\alpha) = 0 \neq D_{(\cos,\sin)} f(0,0)$

$\cos \neq 0$
$\sin \neq 0$

$f_x(0,0) = \lim_{x\to 0} \dfrac{f(x,0) - f(0,0)}{x-0} = 0$ $\qquad f_y(0,0) = 0$

f diff değilse dot product kullanılmaz.

Last fancy example:

A factory produces $Q = 200$ items and sells each item for the price $P = 10$.

Profit function: $f(P,Q) = QP - \dfrac{Q^2}{4P} - \dfrac{P^4}{8}$

since P, Q cannot be changed immediately, small changes are possible. What should be done?

a)

$f(10, 200) = -60k$

(D f(10,200) is most positive)

since f is diff at P_0, this means, $\vec{v} = \nabla f(P_0)$

$$\left.\begin{aligned} f_1 &= \frac{\partial f}{\partial P} = Q + \frac{Q^2}{4P^2} - \frac{4P^3}{8} \\ f_2 &= \frac{\partial f}{\partial Q} = P - \frac{2Q}{4P} \end{aligned}\right\} \text{ at } P_0$$

so $\nabla f(P_0) = (-200, 0) = 200 \cdot (-\vec{i}\,)$

P must move in $-\vec{i}$ direction i.e. P must be lowered

Say $P: 10 \searrow 9$ $f(9,200) = -131$

b) What is the ultimate goal? (maximize f in the long run)

Find critic points.

It happens that $P_0\,(6,72)$ is a crit. point and it gives max f

$f(6,72) = 54$

Extreme Values of $f: \mathbb{R}^2 \to \mathbb{R}$

bounded ama cont. değilse abs. max, min olmayabilir.

if we have closed interval, then $f: \mathbb{R} \to \mathbb{R}$ has extreme values.

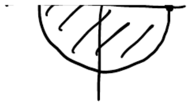
Defn (unofficial): A subset $S \subset \mathbb{R}^3$ is called closed if it contains its boundary.

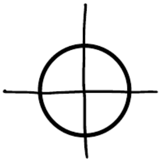
closed

not bounded

not closed

bounded





closed

bounded

Extreme Value Theorem

Let $D \subset \mathbb{R}^2$ be closed and bounded. If $f: \mathbb{R}^2 \to \mathbb{R}$ is cont. on D then f has a max and a min values on D.

i) checking all critical points in (a,b)

$f'(x_0)=0 \to$ local max, min $\qquad \to \mathbb{R} \to \mathbb{R}$

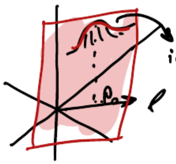
ii) checking end-pts.

A crit. pt. of $\mathbb{R}^2 \to \mathbb{R}$ is a point such that the tangent plane is horizontal. (f is diff)

This happens when $\vec{k}$ is $\perp$ to the tangent plane, that is,

$f_x(P_0), f_y(P_0)=0$ or $\nabla f(P_0), \nabla f(P_0)=(0,0)$

observation: If f is diff at P_0 and $f(P_0)$ is a l.max then,



intersection curve

$D_{\vec{v}} f(P_0)=0 \quad$ So $\nabla f(P_0) \cdot \frac{\vec{v}}{\|\vec{v}\|}=0$

Say $\vec{v}=(\cos\alpha, \sin\alpha)$ then

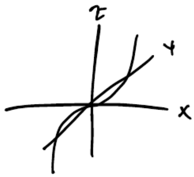
$f_x(P_0).\cos\alpha + f_y(P_0)\sin\alpha = 0$

$\forall \alpha \underset{\underset{\alpha=\frac{\pi}{2},\alpha=0}{\uparrow}}{\Longrightarrow} f_x(P_0) = f_y(P_0) = 0$ so P_0 is crit. pt.

So we conclude: If f is diff, any ℓ max, min pt. is crit. pt.

$(\nabla f(P_0) = (0,0))$

But a crit. pt. need not be a ℓ max or ℓ. min point.



$z = x^3 \rightarrow P_0 \neq \ell$ max, min

tangent plane is horizontal

saddle

both ℓ.max and ℓ.min

Let f have cont. 2^{nd} order partial derivatives.
(So f, f_x, f_y are all diff) then f has a ℓ.max and min at P_0
$\forall$ V direction, the curve on $z = f(x,y)$ vertically above this direction also has a ℓ.max. and min

2^{nd} derivative $\rightarrow D_{\vec{v}}(D_{\vec{u}} f)(P_0)$

$D_v f = \nabla f \cdot (\cos\alpha, \sin\alpha) = f_x \cos\alpha, f_y \sin\alpha$

$\nabla (f_x \cos\alpha, f_y \sin\alpha) \cdot (\cos\alpha, \sin\alpha) \Big|_{P_0}$

Clairaults thm $= f_{xy} = f_{yx}$

$\cos^2\alpha \, f_{xx} + 2f_{xy} \sin\alpha\cos\alpha + f_{yy} \sin^2\alpha \Big|_{P_0}$

we want this to be (-) for all α.

if $\cos\alpha = 0$ then this is $f_{yy}(P_0)$

if not; $= \cos^2\alpha \left(f_{xx} + 2f_{xy} \cdot \tan\alpha + f_{yy} \tan^2\alpha \right) \Big|_{P_0}$

Conclusion:

If the discriminant < 0

and $f_{yy}(P_0) < 0$

then the second derivative are all < 0

so in all direction we have a local max

so $f(x,y)$ has a local max

discriminant: $4 f_{xy}(P_0)^2 - f_{xx}(P_0) f_{yy}(P_0) < 0$

hessian = -discriminant

Theorem: Let $f: \mathbb{R}^2 \to \mathbb{R}$ have second order cont. partial derivatives then,

if $\nabla f(P_0) = (0,0)$ (i.e. a crit. pt.) then

1) Hessian $(P_0) > 0$ and $f_{yy}(P_0) > 0 \Rightarrow f$ has a ℓ. min

2) Hessian $(P_0) > 0$ and $f_{yy}(P_0) < 0 \Rightarrow f$ has a ℓ. max

3) " < 0 then f has a saddle

4) " $= 0$ then no info at P_0.

Remark:

the case $\alpha = \frac{\pi}{2}$

f_{xx} and f_{yy} have same sign

so it does not matter $f_{xx}(P_0)$ or $f_{yy}(P_0)$

Ex. Find and classify all critical points.

$f(x,y) = (x-1)^2 + 2y^2$ (cont and all partials exist)

$f_x = 0 \quad x = 1$

$f_y = 0 \quad y = 0$

$P_0 = (1,0)$

$f_{xx} = 2 \quad f_{yy} = 4 \quad f_{xy} = 0$

Hessian $= 2.4 - 0^2 = > 0$

$f_{xx}(P_0) = 2 > 0$

} l. min at P_0

Ex. Aynı devam

$f(x,y) = 2x^3 - 3x^2y + y^3 - 6y^2 + 9y$

poly. + all order partials + cont → theorem applicable

$f_x \rightarrow 6x^2 - 6xy = 0$ obtain form like $AB=0$

$f_y \rightarrow -3x^2 + 3y^2 - 12y + 9 = 0$ put in 2

By 1: $6x\,(x-y) = 0$ $\quad x = 0 \quad y = x$

$y^2 - 4y + 3 = 0 \quad y = 3/4$

$y=1 \quad y=3 \qquad P_3=(3/4,\ 3/4)$

$P_1=(0,1) \quad P_2=(0,3)$

Apply the theorem:

$P_1=(0,1)$: $f_{xx}=12x-6y\,\big|_{P_0}=-6$

$f_{yy}=6y-12\,\big|_{P_0}=-6$

$f_{xy}=-6x\,\big|_{P_0}=0$

$$\left.\begin{array}{l}\text{Hessian}=36>0\\ f_{yy}(P_0)=-6<0\end{array}\right\}\ \text{l. max at } P_1$$

at $P_2=(0,3)$: Hessian <0 saddle pt.

at $P_3=(3/4,\ 3/4)$: Hessian <0 saddle pt.

Ex. $f(x,y)=x^4+y^4-2x^2+4xy-2y^2$

$f_x=4x^3-4x+4y=0$

$f_y=4y^3+4x-4y$

$1+2 \rightarrow 4(x^3+y^3)=0$ so $y=-x$ $\qquad 1\rightarrow x^3=2x \qquad x=\pm\sqrt{2}$

$P_1=(0,0)$
$P_2=(\sqrt{2},-\sqrt{2})$
$P_3=(-\sqrt{2},\sqrt{2})$

$f_{xx} = 12x^2-4$

$f_{yy} = 12y^2-4$

$f_{xy} = 4$

$\left.\begin{array}{l}\text{Hessian} = 20.20-4^2>0 \\ f_{xx}(P_2) = 20>0\end{array}\right\}$ local min

$\left.\begin{array}{l}\text{Hessian} = 20^2-4^2>0 \\ f_{xx}(P_3) = 12.2-4>0\end{array}\right\}$ local min

Hessian $=0$ at P_1

no info

look the function itself $f(x,y) = x^4+y^4-2(x-y)^2$

As $(x,y)\to(0,0)$ on $y=x$, $f(x,y) = 2x^4 \geq 0$

and $f(P_1) = f(0,0) = 0$

So on $y=x$, f has a local min at $(0,0)$

Consider $(x,y)\to(0,0)$ on x-axis $(y=0)$

$f(x,0) = x^2(x^2-2)$

if $(x,0)$ is near enough to $(0,0)$ $(x^2-2)<0$

and $f(0,0)=0$

So $f(0,0)$ is the biggest value

for nearby points on x-axis. As $(x,y)\to(0,0)$ on x-axis

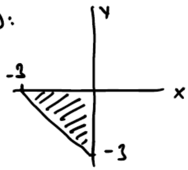
f has a l.max at $(0,0)$

Conclusion: So P_1 is a saddle point.

EXTREMA, RESTRICTIONS ?

Ex. $f(x,y) = x^2 + y^2 - xy + x + y$

and D:

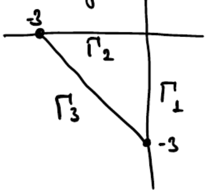


since D is bounded and closed and f is cont. everywhere, f has min and max on D by the extreme value theorem.

They can occur, at interior of D: (without boundaries). then it is l.max or l.min, so it will be a crit. pt. Let us find these pts.

$$\left.\begin{aligned} f_x &= 2x - y + 1 = 0 \\ f_y &= 2y - x + 1 = 0 \end{aligned}\right\}$$ we get $P_0(-1,-1) \in D$? Yes

boundary of D:



on Γ_1 : $x=0$, $-3 \le y \le 0$

$f(x,y) = y^2 + y$

crit. pts: $\frac{d}{dy}(y^2+y) = 2y+1 = 0$

So; $P_1(0,-1/2)$

end pts.: $y=-3$, $y=0$ so $P_2 = (0,-3)$

$P_3 = (0,0)$

on Γ_2 : $y=0$, $-3 \le x \le 0$

$f(x,y) = x^2 + x$

So $P_4 = (-3,0)$ $P_5 = (-1/2, 0)$

on Γ_3 : $y = -x-3$, $-3 \le x \le 0$

$f(x,y) = 3(x^2+3x+2)$ ← by substitution

$\frac{d}{dx} 3(x^2+3x+2) = 3(2x+3) = 0$

$x = -\frac{3}{2}$ $y = -\frac{3}{2}$

$P_6 = (-3/2, -3/2)$

end pts. give $(-3,0), (0,-3)$ again

then; $f(P_0) = -1$ (abs) min $f(P_2) = f(P_4) = 6$ ⇐ (abs) max

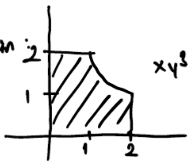
$f(P_1) = f(P_5) = -1/4$ $f(P_3) = 0$ $f(P_6) = -3/4$

Remark: The basic idea of solution in the above example was, solve x or y on the boundary curves and substitute in f. thus make it a func. of one variable.

Q: If the boundary curve is given implicitly and we cannot solve for x or for y, what then?

Example:

Domain:



$xy^3 + yx^3 = 10$ $f(x,y) = x+y$

lagrange

LAGRANGE'S METHOD → Γ_2

Question: Given $z = f(x,y)$ find its min / max values on top of a curve (in plane) $h(x,y) = 0$ → Γ.

(If the curve has points where no tangent line can be drawn, we check them separately.)

$z'(t) = \frac{d}{dt} f(x(t), y(t)) = f_x \, x'(t) + f_y \cdot y'(t)$

$h'(t) = h_x \cdot x'(t) + h_y \cdot y'(t) = 0$

so $\nabla h \cdot \underbrace{(x'(t), y'(t))}_{// \ell} = 0$ so $\nabla h \perp V \ell$

So at P_0 on Γ_1, $D_{\vec{v}_\ell} f(P_0) = 0$ must hold

$\nabla f(P_0) \cdot \frac{\vec{V_\ell}}{\|V_\ell\|} = 0$ So $\nabla f(P_0) \perp \vec{V_\ell}$

$f_x(P_0)\, x'(t) + f_y(P_0) \cdot y'(t) = 0$ So $\nabla f(P_0) \perp \underbrace{(x'(t), y'(t))}_{// \ell \, // \vec{V_\ell}}$

So at P_0 the vectors, $\nabla f(P_0)$ and $\nabla h(P_0)$ are both $\perp \vec{V_\ell}$.

(all vectors considered in xy-plane)

$\nabla h(P_0) \, // \, \nabla f(P_0)$

Method:

$\nabla h(P_0) + \lambda \nabla f(P_0) = (0,0)$ unknowns : x_0, y_0, λ

$h(P_0) = 0$ $\lambda \rightarrow$ lagrange multiplier we have three eqns.

Remark: Generalizations

$w = f(x,y,z)$, $\left.\begin{array}{l} g(x,y,z) = 0 \\ h(x,y,z) = 0 \end{array}\right\}$ a curve in space
intersection of two surfaces

$v_t \perp \nabla g$, $v_t \perp \nabla h$

$D_{v_t} f(P_0) = 0 = \nabla f(P_0) \cdot \dfrac{v_t}{\|v_t\|}$ So $\nabla f(P_0) \perp v_t$

α we get at P_0,

∇h, ∇g, $\nabla f \perp v_t$ at P_0

so $\nabla h, \nabla g, \nabla f$ are // to the plane with normal v_t.

So $\nabla f = -\lambda \nabla g - \mu \nabla h$ (at P_0)

hence $\nabla f(P_0) + \lambda \nabla g(P_0) + \mu \nabla h(P_0) = (0,0,0)$

$h(P_0) = 0$

$g(P_0) = 0$

5 unknowns

5 scalar eqns

Ex. Find min/max of $f(x,y) = x+y+2$ on $\overbrace{x^2 + 2y^2 = 1}^{}$
constraint

$h(x,y) = x^2 + 2y^2 - 1 = 0$

$\nabla f + \lambda \nabla h\big|_{P_0} = (0,0)$

$h(P_0) = 0$

$f_x + h_x = 0 \qquad 1 + 2\lambda x = 0$ ①

$f_y + h_y = 0 \qquad 1 + 4\lambda y = 0$ ②

$x^2 + 2y^2 = 1$ ③

first observation $x, y, \lambda \neq 0$

$\lambda = \frac{-1}{2x}$ ① → ② $1 - 2\frac{y}{x} = 0$

$x = 2y$ → put ③

or ① - ② $= 2\lambda(x - 2y) = 0$

$\lambda = 0$ ✗ $\qquad x = 2y$

$4y^2 + 2y^2 = 1$

$y = \pm \frac{1}{\sqrt{6}}$

$P_1 \to$ max $\qquad$ put in f $\qquad P_1 = \left(\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

$P_2 \to$ min $\qquad P_2 = \left(\frac{-2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right)$

the func. f is cont. on the closed and bounded curve

Ex. Find the shortest distance from $x^2 + y^2 + z^2 = 1$ to the point $P_0 = (1, 1, 3)$

The question is to find min of

$d(P_0, P) = \sqrt{(x-1)^2 + (y-1)^2 + (z-3)^2}$

on the sphere

$x^2 + y^2 + z^2 = 1$ (constraint)

kendi min'ken karesi de min'dir. , for $x \geq 0$

So consider instead,

$d(P_0, P)^2 = (x-1)^2 + (y-1)^2 + (z-3)^2 = f(x,y,z)$

$x^2 + y^2 + z^2 = 1$

Applying lagrange method:

$$\left.\begin{array}{l} \nabla f + \lambda \nabla h = (0,0,0) \\ h(P) = 0 \end{array}\right\} \text{4 eqns, 4 unknown}$$

$2(x-1) + 2\lambda x = 0$ ① $x(2+2\lambda) - 2 = 0$ $x, y, z \neq 0$

$2(y-1) + 2\lambda y = 0$ ② $y(2+2\lambda) - 2 = 0$

$2(z-3) + 2\lambda z = 0$ ③ $z(2+2\lambda) - 6 = 0$

$x^2 + y^2 + z^2 = 1$ ④

① - ② $2(x-y)(1+\lambda) = 0$

$x = y$ $\lambda = -1$ + contradicts ①

$3 \cdot$ ② - ③ $3y(2+2\lambda) - z(2+2\lambda) = 0$

$2 \cdot (3y - z)(2+2\lambda) = 0$

$z = 3y$

$y^2 + y^2 + 9y^2 = 1$ $y = \pm \frac{1}{\sqrt{11}}$ $P_1 = \left(\frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{3}{\sqrt{11}}\right)$

$P_2 = \left(\frac{-1}{\sqrt{11}}, \frac{-1}{\sqrt{11}}, \frac{-3}{\sqrt{11}}\right)$

put in f

Ex. Find min/max of $f(x,y,z) = xy + yz$
subject to the constraints $x^2+y^2=2 \quad yz=2$

say $g(x,y,z) = x^2+y^2-2$
$h(x,y,z) = yz-2$

$\nabla f + \lambda \nabla g + \mu \nabla h = (0,0,0)$
$x^2+y^2=2$
$yz=2$

1 $y + 2x\lambda = 0$
2 $(x+z) + 2y\lambda + \mu z = 0$
3 $y + \mu y = 0$
4 $x^2+y^2=2$
5 $yz=2$

from 3 $(\mu+1)y = 0$
$\boxed{\mu=-1}$ $\to y=0$ contradicts 5

1 $y + 2\lambda x = 0$
2 $x + 2\lambda y = 0$
if $x=0$ $y=0 \Rightarrow \alpha$ $x\neq 0$ $y\neq 0$

1 $.x \Rightarrow xy + 2\lambda x^2$
2 $.y \Rightarrow xy + 2\lambda y^2$
$\lambda \neq 0$ $\boxed{x=y}$ $\boxed{x=-y}$

So if $y=x$ by 4

$x=\pm 1$ $P_1(1,1,2)$ $P_2(-1,-1,-2)$
5

if $y=-x$
$P_3(1,-1,-2)$ $P_4(-1,1,2)$

Since $f(x,y) = xy + yz$

$f(P_1) = 1+2 = 3 = f(P_2) \rightarrow \max$

$f(P_3) = -1+2 = 1 = f(P_4) \rightarrow \min$

NOTE: $yz = 2 \rightarrow$ yi f'de yerine yazabiliriz

Ex. Find min / max of $f(x,y) = xy$

subject to $x^3 + y^3 = 1$

$\hookrightarrow h(x,y) = x^3 + y^3 - 1$

$\nabla f + \lambda \nabla h = (0,0)$

① $y + 3x^2\lambda = 0$

② $x + 3y^2\lambda = 0$

③ $x^3 + y^3 = 1$

if $x = 0$ by ① $y = 0$ contra ③ $\rightarrow y \neq 0$

$y^2$① $- x^2$②

$y^3 - x^3 = 0$

$y = x$ by ③ $\rightarrow x = \frac{1}{\sqrt[3]{2}}$

$P_0\left(\frac{1}{\sqrt[3]{2}}, \frac{1}{\sqrt[3]{2}}\right)$

Better question, do min / max values exist?

$x = n$

$y = (1-n^3)^{1/3}$

$f(x,y) = n.(1-n^3)^{1/3} \Rightarrow$ there is no min.

$n \to \infty \rightarrow -\infty$

max $\rightarrow x, y < 0$ then $x^3 + y^3 = 1$ is violated.

So, $xy \geq 0$. this says

$0 \le x \le 1$
$0 \le y \le 1$, xy cannot go to ∞ so there is a max

CONCLUSION: $f(P_0)$ is the max.

Ex. Find min / max of $f(x,y,z) = xyz$
subject to $\underbrace{x+y+z=1}_{h}$

$\nabla f = yz, xz, xy$ So $yz + \lambda = 0$ ①
$\nabla h = 1, 1, 1$ $xz + \lambda = 0$ ②
$xy + \lambda = 0$ ③
$x+y+z = 1$ ④

① - ② $z(y-x) = 0$

$z = 0 \to \lambda = 0 \to x.y = 0$

$x = 0 \to y = 1$

$y = 0 \to x = 1$

$y = x$

② - ③ gives
$x(z-y) = 0$
$x = 0 \to y = 0,\ z = 1$
$z = y$
$x = y = z = \frac{1}{3}$

$(0,1,0)$ $(1,0,0)$ $(0,0,1)$ $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

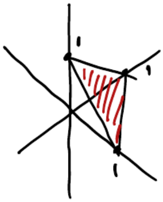
put in f 0 or $\frac{1}{3^3}$

are these min / max values?
NO, because there is no min / max

take $\underset{x}{n} + \underset{y}{n} + \underset{z}{1-2n} = 1$ $\quad xyz = n^2(1-2n) \to -\infty$
no min

take $-n - n + 1 + 2n = 1$ $\quad xyz = n^2(1+2n) \to \infty$
no max

Ex. Same as previous, but $x, y, z \geq 0$



if one of x, y, z is $= 0$
then $xyz = 0$ gives min value
(on the edges of triangle)

Since this triangle is closed and bdd and $f(x,y,z) = xyz$ is cont. it has min and max. So, $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ gives the max.

Ex. Let $x_1, x_2, \dots, x_n \geq 0$, $x_1 + x_2 + \dots + x_n = A$ $\quad (A > 0)$
Find min of $f(x_1, x_2, \cdots, x_n) = x_1 . x_2 \dots x_n$
(we assume we know that lagrange's method works)

$\nabla f + \lambda \nabla h = \underset{n\text{-fold}}{(0, 0, \dots, 0)}$

$x_1 + x_2 \dots x_n = A$

$\nabla f = (f_{x_1}, f_{x_2}, \dots f_{x_n})$

f_{x_1} $\quad x_2 x_3 \dots x_n + \lambda . 1 = 0$

$x_1 \dots x_3 \dots x_n + \lambda . 1 = 0$

$x_1 x_2 \dots x_{n-1} + \lambda . 1 = 0$

If any of $x_i = 0$
then $x_1 x_2 x_3 \dots x_n = 0$
and we get min

① - ② : $x_3 x_4 \dots x_n (x_2 - x_1) = 0$ $\quad x_1 = x_2$

$x_1 = x_2 = x_3 \cdots = x_n$

So, $x_1 = \frac{A}{n} = x_2 = x_3$

So $P_0 = \left(\frac{A}{n}, \frac{A}{n}, \dots, \frac{A}{n}\right)$

Hence $f(x_1, x_2, \dots, x_n) = x_1 . x_2 \dots x_n \le f\left(\frac{A}{n}, \frac{A}{n}, \dots\right) = \frac{A^n}{n^n}$

take n-th roots :

$(x_1 x_2 \dots x_n)^{1/n} \le \frac{A}{n} = \frac{x_1 + x_2 + \dots + x_n}{n}$

Ex. A rectangular box with sides parallel to coordinate axes is put in the pyramid with vertices at $(0,0,1)$, $(0,2,0)$, $(1,0,0)$ and $(0,0,0)$.

Find the dimensions s.t. its volume largest.

$x + \frac{y}{2} + z = 1 \Rightarrow 1\, y \le 2$ 

$f(x,y,z) = xyz$ maximize volume

if one of x,y,z is $=0$ volume $=0$ $\Rightarrow$ min.

$\nabla f = (yz, xz, xy)$ So $yz + \lambda = 0$

$\nabla h = (1, \frac{1}{2}, 1)$ $\quad xz + \lambda/2 = 0$

$xy + \lambda = 0$

$x + \frac{y}{2} + z = 1$

① - ③ $y(z - x) = 0$

$y \neq 0$ $\quad z = x$

2. ② - ③ $x(2z - y) = 0$

$2z = y$

$z = \frac{1}{3} \rightarrow P_0 = \left(\frac{1}{3}, \frac{2}{3}, \frac{1}{3}\right)$

gives max

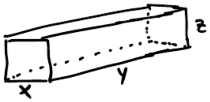
Ex. A rectangular box is to be manufactured s.t. the materials used cost:

a for bottom per unit square

b for sides

c for top

If this box is have a volume $V_0 > 0$, find the dimension so that cost is min.



cost $= xy.a + 2(xz + xy).b + xy.c$

minimize cost, subject to $xyz = V_0$

$h(x) = xyz - V_0$

$\nabla f = (ya + 2zb + yc,\ xa + 2zb + cx,\ 2(x+y)b)$

$\nabla h = (yz,\ xz,\ xy)$

$x, y, z \neq 0$

$(a+c)\,y + 2zb + \lambda yz = 0$

$(a+c)\,x + 2zb + \lambda xz = 0$

$2(x+y)b + \lambda xy = 0$

$xyz = V_0$

$x(1) - y(2) = 2bz(x-y) = 0$

$b \neq 0$, $z \neq 0$ $\quad \underline{x = y}$

$y(2) - z(3)$ $\quad \underline{z = \frac{a+c}{2b}\, y}$

$$y \cdot y \cdot \left(\frac{a+c}{2b}\right) y = V_0$$

$$P_0 \begin{cases} \text{So } \ y = \left(\frac{2b}{a+c} V_0\right)^{1/3} = x \\ z = \frac{a+c}{2b}\left(\frac{2b}{a+c} V_0\right)^{1/3} \end{cases}$$

Ex. Find the (abs) max / min of $f(x,y) = x^2+y^2-2x-4y+6$ on $D = \{(x,y) : x^2+y^2 \le 9\}$

(this is not a purely lagrange problem since inside of the disc should be considered)

since f is diff (cont.) D is closed and bounded, f has min/max on D.

In the interior of D (without boundary circle) they would be crit. pt. so we check $\nabla f = (0,0)$

$$\left.\begin{aligned} f_x &= 2x-2=0 \\ f_y &= 2y-4=0 \end{aligned}\right\} P_0 = (1,2)$$

On the boundary:

find max/min of f s.t. $x^2+y^2=9$ (this is lagrange)

$$\nabla f + \lambda \nabla h = (0,0) : \quad \begin{aligned} &2x-2+\lambda 2x = 0 \\ &2y-4+\lambda 2y = 0 \\ &x^2+y^2=9 \end{aligned}$$

we observe that if $x=0$ then we get contr. to ①
if $y=0$ " " " " " ②
So $x,y \neq 0$

y ① $-$ x ② : $-2y+4x=0$ $\quad y=2x$

$x^2+4x^2=9 \quad x = \pm\sqrt{\frac{9}{5}}$ $\quad P_1 = \left(\frac{3}{\sqrt{5}}, \frac{6}{\sqrt{5}}\right)$

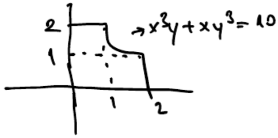
$P_2 = \left(\frac{-3}{\sqrt{5}}, \frac{-6}{\sqrt{5}}\right)$

$f(P_0) = 1$ ← min

$f(P_1) \approx 1.8$

$f(P_2) \approx 15 + 10 \cdot \frac{3}{\sqrt{5}}$ ← max

Exercise: max / min of $f(x,y) = x+y$ on D



Double Integrals

$$\iint 1\, dxdy \to \text{Alan}$$

$$\iint f(x,y)\, dxdy \to \text{hacim}$$

Ex. Let $f(x,y) = x^2+y^2$ $\quad R = [0,2] \times [3,5]$

Properties:

$$\iint_D 1\, dA = \text{area } (D)$$

if $f(x,y) \geq 0$ on D then

$\iint f(x,y)\, dA$ is the volume of D

$f(x,y)$ is odd , $f(-(x,y)) = -f(x,y)$

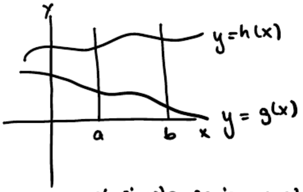
Remarks:

α $\displaystyle\int_{g(x)}^{h(x)} f(x,y)\, dy$ is a partial integral since x is constat

$\propto$ An iterated integral may have no geometric meaning

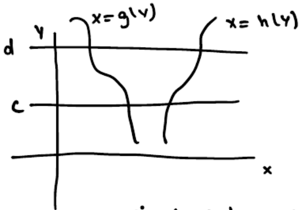
$\propto$ Iterated integrals are "easy" to calculate

ITERATION THEOREM



$$\iint_{D_1} f(x,y)\,dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx$$

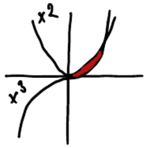
$\propto$ simple region wrt y , x is constant



$$\iint_{D_2} f(x,y)\,dA = \int_c^d \int_{g(y)}^{h(y)} f(x,y)\,dx\,dy$$

$\propto$ simple region wrt x , y is constant

Ex. Let D:



looking at y-direction (increasing y)

we have $y : y = x^3 \to y = x^2$

$x : 0 \to 1$

$$\iint_D f(x,y)\,dA = \int_0^1 \int_{x^3}^{x^2} f(x,y)\,dy\,dx$$

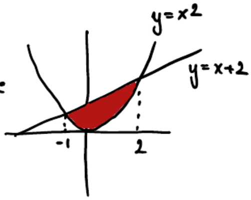
D is also simple wrt x-first

$\rightarrow$ $x: \; x=\sqrt{y} \rightarrow x=y^{1/3}$
$y: \; 0 \rightarrow 1$

$$\iint_D f(x,y)\,dA = \int_0^1 \int_{\sqrt{y}}^{y^{1/3}} f(x,y)\,dx\,dy$$

Ex.

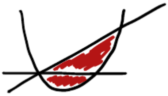
Let D be



wrt y-first $\uparrow$ $\quad y: \; y=x^2 \rightarrow y=x+2$
$x: \; -1 \rightarrow 2$

$$\iint f(x,y)\,dA = \int_{-1}^{2} \int_{x^2}^{x+2} f(x,y)\,dy\,dx$$

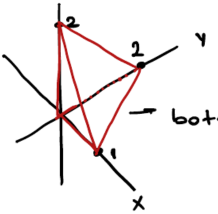
or D is not simple wrt x-first
to do that we should divide D into 2 parts.



$$\iint f(x,y)\,dA = \int_1^4 \int_{y-2}^{\sqrt{y}} f(x,y)\,dx\,dy + \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)\,dx\,dy$$

Ex. Find the volume of tetrahedron (pyramid) bounded by the coordinate planes and the plane $2x+y+z=2$

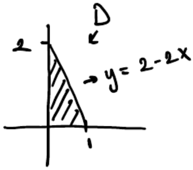
z



→ bottom is D (base)

$z = 2-2x-y \geq 0$ on D

so, Volume: $\iint\limits_D (2-2x-y)\, dA$



$$\int_0^1 \int_{y=0}^{2-2x} 2-2x-y \; dy\, dx$$

OR:

$$\int_0^2 \int_0^{\frac{2-y}{2}} (2-2x-y)\, dx\, dy$$

Finding volume

$z = x^2+y^2$ paraboloid

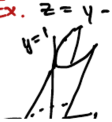
bounded by $y=x^2$, $y=1$, $z=-1$, $z=2$

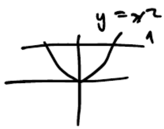
$$V = \iint\limits_D (2-(-1))\, dA$$

D: 

$$\int_{-1}^{1} \int_{x^2}^{1} 3 \; dy\, dx$$

Ex. $z = y - x^2$ $y=1$ $z=0$



D: 

C4/

$$\iint_D y-x^2\, dA = \int_{-1}^{1}\int_{x^2}^{1} y-x^2\, dy\, dx$$

MEAN VALUE THM.

$$f(P_0) = \frac{\iint_D f\, dA}{\text{area}(D)}$$

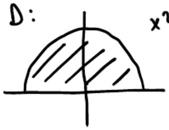
$\downarrow$
average value

f is cont
D is closed, bounded, connected

centroid: $\bar{x}, \bar{y}$ $\quad \bar{x} = \dfrac{\iint_D x\, dA}{\text{area}(D)} \quad \bar{y} = \dfrac{\iint_D y\, dA}{\text{area}(D)}$

Ex.

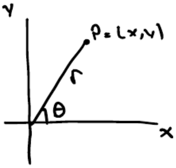
D:



$$\bar{x} = \frac{\int_{-R}^{R}\int_{0}^{\sqrt{R^2-x^2}} x\, dy\, dx}{\pi R^2/2} = 0 \quad \text{because it's odd}$$

$$\bar{y} = \frac{\int_{-R}^{R}\int_{0}^{\sqrt{R^2-x^2}} y\, dy\, dx}{\pi R^2/2} = \frac{2}{3}\frac{R^3}{\pi R^2}2 = \frac{4}{3\pi}R \quad \text{centroid} = \left(0, \frac{4R}{3\pi}\right)$$

POLAR COORDINATES



$x = r\cos\theta$

$y = r\sin\theta$

$x^2 + y^2 = r^2$

ex. $r = a.\cos\alpha$

$r.r = r.a.\cos\alpha$

$x^2 + y^2 = ax$

$\left(x - \frac{a}{2}\right)^2 + y^2 = \frac{a^2}{4}$

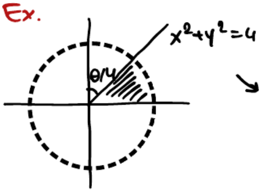
$\left(\frac{a}{2}, 0\right)$ merkezli $\frac{a}{2}$ yarıçaplı çember

r (−) ise $|r|$ şeklinde bul, orijine göre yansıt

$$\iint_D f(x,y)\,dA = \lim \sum \sum f(P_{ij}^*) \cdot r \cdot \underbrace{\Delta r\, \Delta\theta}_{\text{area of rectangle}}$$

$\in S_{ij}$ $\quad$ $r\theta$-plane

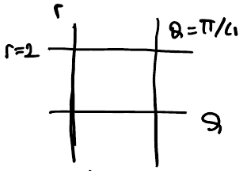
Ex.



boundaries

$r = 2$

$\theta = 0 \to \pi/4$



$r=0$ is also a boundary

$$\text{Area of } D_{xy} = \iint_{D_{xy}} 1\, dA_{xy} = \iint_{D_{r\theta}} 1 \underbrace{r\, dA_{r,\theta}} = \int_0^{\pi/4}\int_0^{2} r\, dr\, d\theta$$

! extra r coming

$$= \frac{1}{2}\cdot 4\cdot\frac{\pi}{4} = \frac{\pi}{2}$$

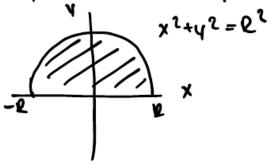
Simple regions in polars.

$y \to r$

$x \to \theta$

Ex.

express in iterated polars



$$I = \iint_{D=D_{xy}} f(x,y)\, dA_{xy}$$

$$I = \iint_{D_{r\theta}} f(r\cos\theta, r\sin\theta).r\, dA_{r\theta}$$

wrt r-first

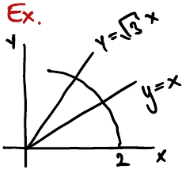
$0 \le r \le R$

$0 \le \theta \le \pi$

$$I = \int_0^{\pi}\int_0^{R} f(r\cos\theta, r\sin\theta).r.\, dr\, d\theta$$

since it is rectangle $$= \int_0^{R}\int_0^{\pi} f(r\cos\theta, r\sin\theta).r.\, d\theta\, dr$$

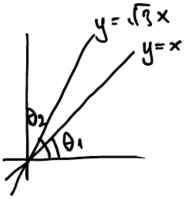
Ex.



$$\iint_{D_{xy}} f\, dA_{xy} = \iint_{D_{r\theta}} f(r\cos\theta, r\sin\theta).r.\, dA_{r\theta}$$

1

$0 \le r \le 2$



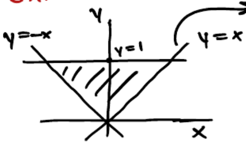
$$I = \int_{\pi/4}^{\pi/3} \int_0^2 f(r\cos\theta, r\sin\theta) . r . dr\, d\theta$$

tersi de okey

$\theta_1 = \pi/4$

$\theta_2 = \pi/3$

Ex.



$y = r.\sin\theta$ so $r = \csc\theta$
$y = 1$

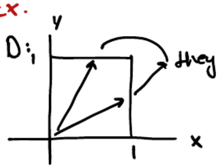
$$I = \iint_{D_{xy}} f\, dA_{xy} = \int_{\pi/4}^{3\pi/4} \int_0^{\csc\theta} f(\ldots)\, r . dr . d\theta$$

$\theta_1 = \pi/4$

$\theta_2 = 3\pi/4$

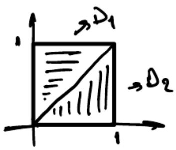
Ex.

D_1:



they are different so not simple wrt r-first

($y=1$ means $r=\csc\theta$)

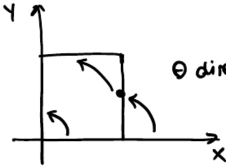


$$\iint_{D_1} f\, dA = \int_{\pi/4}^{\pi/2} \int_0^{\csc\theta} f \quad r\, dr\, d\theta$$

($x=1$ means $r = \sec\theta$)

$$\iint_{D_2} f\,dA = \int_0^{\pi/4}\int_0^{\sec\theta} f \quad r\,dr\,d\theta$$

Ex.



θ direction

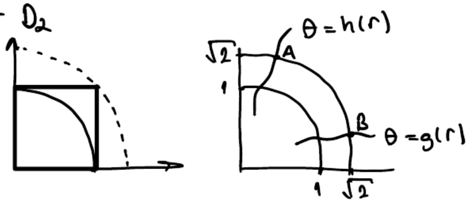
is not simple wrt θ-first

In D_1, $\theta: 0 \to \pi/2$

$r: 0 \to 1$

$$\iint_{D_1} f\,dA = \int_0^1 \int_0^{\pi/2} f \quad r.d\theta.dr$$

For D_2

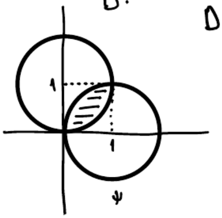


A=B

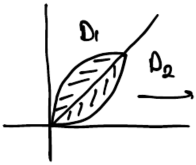
$$\iint_{D_2} f\,dA = \int_1^{\sqrt{2}} \int_{\substack{x=1 \\ r=\sec\theta}}^{\substack{y=1 \\ r=\csc\theta}} f \quad r\,d\theta\,dr$$

Ex.

D:



D is not simple wrt r first



$(x-1)^2+y^2=1$

$x^2-2x+y^2=0$

$r^2-2r\cos\theta=0$

$\boxed{r=2\cos\theta}$ $\quad r\neq 0$

$$\iint_{D_1} f\, dA = \int_{\pi/4}^{\pi/2}\int_0^{2\cos\theta} f \; r\,dr\,d\theta$$

$(y-1)^2+x^2=1$

$r^2-2r\sin\theta=0$ $\quad r\neq 0$

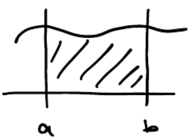
$\boxed{r=2\sin\theta}$

$$\iint_{D_2} f\, dA \quad \int_0^{\pi/4}\int_0^{2\sin\theta} f \; r\,dr\,d\theta$$

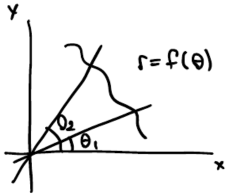
wrt θ-first simple

$$\iint_D f\, dA = \int_0^{\sqrt{2}}\int_{2\sin\theta}^{2\cos\theta}$$

Remark:



$$\text{Area} = \int_a^b f(x)\,dx$$



wrt r first

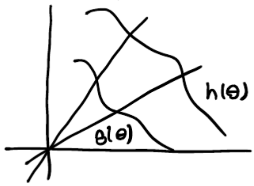
$$\text{Area} = \iint_D 1\, dA = \int_{\theta_1}^{\theta_2} \int_0^{f(\theta)} 1\, r\, dr\, d\theta$$

$$\int_{\theta_1}^{\theta_2} \frac{f^2(\theta)}{2}\, d\theta$$

Ex. Area of a circle with radius R

$$\int_0^{2\pi} \frac{R^2}{2}\, d\theta = \frac{R^2}{2} \cdot 2\pi$$

α In polars if $0 \leq g(\theta) \leq h(\theta)$



$$\text{Area} = \int_{\theta_1}^{\theta_2} \frac{1}{2}\left[h^2(\theta) - g^2(\theta)\right] d\theta$$

Recall

$$V = \iint_D (h(x,y) - g(x,y))\, dA$$

Ex.

Volume of a sphere of radius R.

upper hemisphere $\rightarrow x^2+y^2+z^2=\ell^2$

$$V = 2 \cdot \iint_D \sqrt{\ell^2-x^2-y^2}\, dA$$

$$= 2 \cdot \int_0^{2\pi} \int_0^{\ell} \sqrt{\ell^2-r^2}\, r\, dr\, d\theta = \frac{4\pi}{3} \cdot \ell^3$$

Ex. Find the volume bounded by

$z = 1+x^2+y^2$, $x^2+y^2=4$, $z=0$

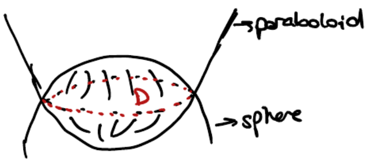
$\downarrow$ paraboloid $\quad \downarrow$ silindir

$$V = \iint_D (1+x^2+y^2)\, dA = \int_0^{2\pi} \int_0^{2} (1+r^2)\, r\, dr\, d\theta$$

Ex. Solid bounded by

$x^2+y^2+z^2=2a^2$ from above and $\quad a>0$

$x^2+y^2=az$ from below $\quad V=?$



$$\iint_D \left(\sqrt{2a^2-x^2-y^2} - \frac{x^2+y^2}{a} \right) dA$$

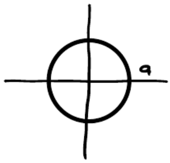
$$\underbrace{x^2+y^2}_{az}+z^2=2a^2$$

$$(z-a)(z+2a)=0$$

$$\boxed{z=a} \quad \cancel{z=-2a}$$

$x^2+y^2=az=a^2$

So D:



$$V=\int_0^{2\pi}\int_0^{a}\left(\sqrt{2a^2-r^2}-\frac{r^2}{a}\right).r.dr.d\theta$$

General Change Of Variable → formülü var

Triple Integrals

$$\iiint_R f(x,y,z)\,dV$$

R → hacim

dV → hacim
→ $dxdydz$

$f \to 1$ ise hacim hesaplar

$ax+by+cz+d=0$ → düzlem

$x^2+y^2+z^2=r^2$ → küre

değişebilir

$x=\sqrt{16-y^2-z^2}$ → yarım küre

$z=\sqrt{x^2+y^2}$ → koni

$z^2=\sqrt{x^2+y^2}$ → çift taraflı koni

$x^2+y^2=r^2$ → silindir

$y = x^2 + z^2 \rightarrow$ paraboloid

Properties

$$\iiint_S f \, dV = F, \quad \iiint_S g \, dV = G$$

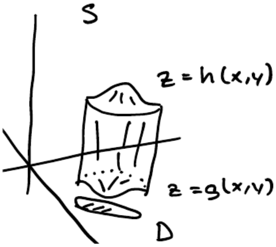
1) $\iiint_S c.f + g \, dV = c.F + G$

2) if $f \leq g$ on S then $\iiint_S f \, dV \leq \iiint_S g \, dV$

3) if $S = S_1 \cup S_2$, $S_1 \cap S_2 = \emptyset$

$$\iiint_S f \, dV = \iiint_{S_1} f \, dV + \iiint_{S_2} f \, dV$$

4) $\iiint_S 1 \, dV =$ volum of S



$$\iiint_S f \, dV = \iint_D \left(\int_{g(x,y)}^{h(x,y)} f(x,y,z) \, dz \right) dA$$

simple wrt z-first

\

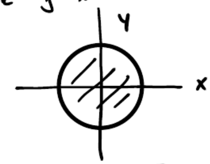
Ex. Express $\iiint_S f\,dV$ in iterated integrals.

1) S be the sphere $x^2+y^2+z^2=R^2$

it is simple wrt all of x,y,z (we take z)

So $$\iiint_S f\,dV = \iint_D \int_{-\sqrt{R^2-y^2-x^2}}^{\sqrt{R^2-y^2-x^2}} f(x,y,z)\,dz\,dA$$

where D:

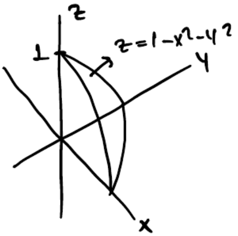


So $$\iiint_S f\,dV = \int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \int_{-\sqrt{R^2-x^2-y^2}}^{\sqrt{R^2-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx$$

OR IN POLARS

$$\iiint_S f\,dV = \int_0^{2\pi} \int_0^{R} \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} f(r.\cos\theta, r.\sin\theta, z)\,dz\; r.dr.d\theta$$

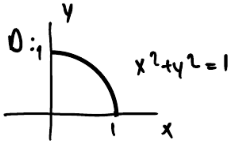
Ex. 2) S is bounded by $z = 1-x^2-y^2$ and coordinate planes in 1^{st}.



simple wrt all, we take z

So $\iiint\limits_S f\,dV = \iint\limits_D \left(\int_0^{1-x^2-y^2} f(x,y,z)\,dz \right) dA$

D is what you see when you look from the top.



$\iiint\limits_S f\,dV = \int_0^{\pi/4}\int_0^1\int_0^{1-r^2} f(r.\cos\theta, r.\sin\theta, z)\,dz\,dr\,d\theta$

Using polars:

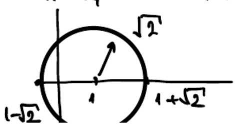
Ex 3) S be given by $z=x^2+y^2$, $z=2x+1$

simple wrt z-first

$\iiint\limits_S f\,dV = \iint\limits_D \int_{x^2+y^2}^{2x+1} f(x,y,z)\,dz.dA$

D is determined by the intersection of paraboloid and the plane.

$x^2+y^2 = z = 2x+1$ so $(x-1)^2+y^2=2$



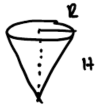
it is simple both wrt x,y first

So $$\iiint_S f\, dV = \int_{1-\sqrt{2}}^{1+\sqrt{2}} \int_{-\sqrt{2-(x-1)^2}}^{\sqrt{2-(x-1)^2}} \int_{x^2+y^2}^{2x+1} f(x,y,z)\, dz\, dy\, dx$$

IN POLARS: $x = r.\cos\theta + 1$ $\quad r^2 = 2$

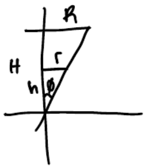
$y = r.\sin\theta$ $\quad r = \sqrt{2}$

4) Find the volume of a cone

S:  $V = \iiint_S 1\, dV$

simple wrt z-first

$$V_0 = \iint_D \left(\int_{\text{"cone"}}^{H} 1\, dz \right) dA$$



$\frac{R}{H} = \frac{r}{h} = \tan\phi$ $\qquad r = \underset{\substack{\downarrow \\ z}}{h} \cdot \tan\phi$

$$z = \frac{H}{R} \cdot \tan\phi = \sqrt{x^2+y^2}\, \frac{H}{R}$$

$$\left(z^2 = \frac{H}{R}(x^2+y^2) \right)$$

D:  So $V = \iint_D \int_{\frac{H}{R}\sqrt{x^2+y^2}}^{H} 1\, dz\, dA$

In polars: $\displaystyle\int_0^{2\pi}\int_0^{R}\int_{\frac{H}{R}\cdot r}^{H} 1\,dz\cdot r\,dr\,d\theta$

$$H - \frac{H}{R}\cdot r \rightarrow \left.\frac{Hr^2}{2} - \frac{H}{R}\cdot\frac{r^3}{3}\right|_0^R \cdot 2\pi$$

Koninin hacmi $= \dfrac{H\cdot R^2\cdot\pi}{3}$

Ex. Find the volume of the solid bounded by $x^2+y^2=a^2$

$x=a$, $y=a$ $\quad a>0$

$z=0$, $z=x+y$

simple wrt z-first $\quad$ not simple wrt r

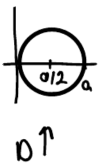
$$V=\iint_D \int_0^{x+y} 1\,dz\,dA$$

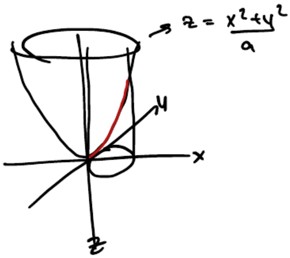
$$= \int_0^a \int_{\sqrt{a^2-x^2}}^{a} (x+y)\,dy\,dx$$

Ex. Find the volume of the solid bdd by $x^2+y^2=ax$

$x^2+y^2=az$

$z=0 \quad a>0$





S:

$$V: \iint_D \int_0^{\frac{x^2+y^2}{a}} 1\, dz\, dA$$

In polars: $\displaystyle\int_{-\pi/2}^{\pi/2} \int_0^{\text{circle}} \frac{x^2+y^2}{a} r.dr.d\theta$ circle: $x^2+y^2=ax$

$r^2 = a\, r\cos\theta$

$r = a.\cos\theta$

$$= \int_{-\pi/2}^{\pi/2} \int_0^{a.\cos\theta} \frac{r^2}{a} \cdot r.dr.d\theta$$

A better idea: use modified polars:

$x = r.\cos\theta + \frac{a}{2}$ then the circle is $\left(x-\frac{a}{2}\right)^2 + y^2 = \frac{a^2}{4}$

$y = r.\sin\theta$

so $r = a/2$ Jacobian is still r.

$$\iint \frac{x^2+y^2}{a} r\, dr\, d\theta$$

$$\int_0^{2\pi} \int_0^{a/2} \frac{1}{a} \cdot \left[\left(r.\cos\theta + \frac{a}{2}\right)^2 + r^2\sin^2\theta\right] r.dr d\theta$$

$r^2 + \frac{a^2}{4} + \underbrace{\frac{2ar}{2} .\cos\theta}_{\downarrow 0 \to 2\pi \; 0}$

$$\int_0^{2\pi} \int_0^{a/2} \frac{1}{a} \cdot \left(r^2 + \frac{a^2}{4}\right) r\, dr d\theta$$

$$= \frac{3\pi a^3}{32}$$

Triple Integrals in cylindrical coordinates

$\mathbb{R}^3 \quad (x,y,z) \longrightarrow (r, \theta, z)$

$x = r.\cos\theta$

$y = r.\sin\theta$

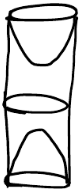
$z = z$

$x^2+y^2 = r^2$

$dxdydz = r\,dzdrd\theta$

$\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2}\int_{z_1}^{z_2}$ x,y yerine $r\cos\theta, r\sin\theta$

Ex S is bounded by $x^2+y^2=9 \quad z = 1+x^2+y^2 \quad z=-1-x^2-y^2$



$$\iiint_S f(x,y,z)\,dV = \int_0^{2\pi}\int_0^3\int_{-1-r^2}^{1+r^2} f(r\cos\theta, r\sin\theta, z)\, r\,dzdrd\theta$$

Ex. S is bounded from below by $z = \dfrac{x^2+y^2}{3}$ and by $x^2+y^2+z^2=4$ from above.

$$\iiint_S f\,dV = \iint_D \int_{r^2/3}^{\sqrt{4-r^2}} f(r\cos\theta, r\sin\theta, z)\, r\,dzdrd\theta$$

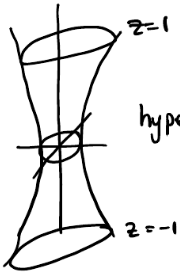
D is intersection curve

$r^2 + z^2 = 4$ $\quad z = 1$ ✓ $\rightarrow D: x^2+y^2=3$

$\overset{\shortparallel}{3z}$ $\quad z = -4$ ✗ $\quad r = \sqrt{3}$

$$\iiint_S f\, dV = \int_0^{2\pi} \int_0^{\sqrt{3}} \int_{r^2/3}^{\sqrt{4-r^2}} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

Ex. S is bounded by $x^2+y^2 = z^2+1$, $z=-1$, $z=1$



S is not a simple solid in z.
we should decompose it.
içinden silindir çıkararak
kalanı da ikiye bölerek.

$$\iiint_S f\, dV = \underbrace{\int_0^{2\pi}\int_0^1\int_{-1}^1 f\, r\, dz\, dr\, d\theta}_{\text{silindir}} + \underbrace{\int_0^{2\pi}\int_1^{\sqrt{2}}\int_{\sqrt{r^2-1}}^1 f\, r\, dz\, dr\, d\theta}_{\text{üst}} + \int_0^{2\pi}\int_1^{\sqrt{2}}\int_{-1}^{-\sqrt{r^2-1}} f\, r\, dz\, dr\, d\theta$$

D: 

NOTE: S is simple wrt r first

thus $r: 0 \rightarrow \sqrt{1+z^2}$

$$\iiint_S f\, dV = \int_0^{2\pi}\int_{-1}^{1}\int_0^{\sqrt{1+z^2}} f\, r\, dr\, dz\, d\theta$$

Ex. Sketch S if $\iiint_S f\, dV = \int_0^2 \int_0^{\sqrt{2x-x^2}} \int_0^a f\, dz\,dy\,dx$

$y = \sqrt{2x - x^2}$

$y^2 + x^2 = 2x$

$y^2 + (x-1)^2 = 1 \rightarrow$ yarım silindir

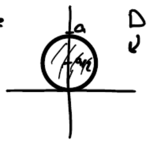
Ex. Sketch S $\int_0^{\pi} \int_0^{a\sin\theta} \int_0^{\sqrt{a^2 - r^2}} f\, r\, dz\, dr\, d\theta$

$r = a \sin\theta$

$r^2 = a . r\sin\theta$

$x^2 + y^2 = ay$ $\rightarrow$ circle

D:



$z^2 + y^2 + x^2 = a^2$ sphere

Spherical Coordinates

verilen hacim küreselse

$(x, y, z) \rightarrow (\rho, \theta, \phi)$

ρ: orijine uzaklık

θ: izdüşümü ile x ekseni arası açı

ϕ: angle from positive z axis to the point

$\rho = \sqrt{x^2+y^2+z^2}$

$0 \le \theta \le 2\pi$

$0 \le \phi \le \pi$

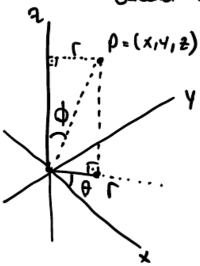
$dV = \underbrace{\rho^2 \sin\phi}_{\text{Jacobian}} \, d\rho \, d\theta \, d\phi$

$x = \rho \cdot \sin\phi \cdot \cos\theta \leftarrow r = \rho \cdot \sin\phi$

$y = \rho \cdot \sin\phi \cdot \sin\theta$ $\quad x = r \cdot \cos\theta$

$z = \rho \cdot \cos\phi$ $\quad y = r \sin\theta$

$x^2+y^2+z^2 = \rho^2$



$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \det \begin{bmatrix} x_\rho & x_\phi & x_\theta \\ y_\rho & y_\phi & y_\theta \\ z_\rho & z_\phi & z_\theta \end{bmatrix}$$

$\downarrow$

$= \rho^2 \sin\phi \ge 0$

Ex. Find the spherical coordinates of $P=(1,1,1)$

$\rho = \sqrt{1+1+1} = \sqrt{3}$

$\cos\phi = \frac{z}{\rho} = \frac{1}{\sqrt{3}}$ $\quad \phi = \arccos\frac{1}{\sqrt{3}}$

$y = r.\sin\theta$ so $\sin\theta = \frac{y}{r} = \frac{1}{\sqrt{2}}$ So $\theta = \pi/4, 3\pi/4$

Ex. Express the cone $x^2+y^2 = \frac{z^2}{3}$ in spherical coordinates.

$\underbrace{x^2+y^2}_{r^2 = (\rho \sin\phi)^2} = \frac{z^2}{3} \rightarrow (\rho\cos\phi)^2$

$\cancel{\rho^2} \sin^2\phi = \frac{1}{3} \cancel{\rho^2} \cos^2\phi$

$|\tan\phi| = \frac{1}{\sqrt{3}}$

$\phi = \pi/6, \pi - \pi/6$

Ex. Express in iterated spherical coordinates.

Let S be sphere of radius R (with interior part)

$$\iiint_{S_{xyz}} f(x,y,z)\, dV_{xyz} = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\theta)\, \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

Ex. Same, let S be part of the solid sphere of Radius R lying in the first octant.

$$\iiint_S f\, dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{R} f\, \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

Ex. Let S be bounded by the paraboloid $z = x^2+y^2$, $z=1$.

not simple wrt ρ first, decompose

S_1

$$\int_0^{2\pi}\int_0^{\overset{\pi/4}{\underset{\uparrow}{\text{cone}}}}\int_0^{\overset{\sec\phi}{\underset{\uparrow}{1=z}}} f\, \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$z=1$ means : $\rho\cos\phi = 1$ $\rho = \sec\phi$

the cone: $\underset{\overset{\shortparallel}{1}}{z} = x^2+y^2$ $r=1$

$$\frac{r}{z} = \tan\phi$$

so $\phi = \pi/4$

(S2)

$$+ \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^{\text{paraboloid}} f \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$z = x^2+y^2$ means $\overbrace{\rho\cos\phi}^{z} = r^2 = \overbrace{\rho^2\sin^2\phi}^{r^2}$

$$\rho = \frac{\cos\phi}{\sin^2\phi}$$

Ex. $\displaystyle\iiint_S \frac{dV}{(x^2+y^2+z^2)^{5/2}}$ $\quad S = \{(x,y,z) : 1 \le x^2+y^2+z^2 \le 4\}$

$\rho : 1 \to 2$ $\quad\displaystyle\int_0^{2\pi}\int_0^{\pi}\int_1^{2} \frac{\rho^2 \sin\phi}{(\rho^2)^{5/2}} \, d\rho \, d\phi \, d\theta$

Ex. $\displaystyle I = \iiint_S (x^2+y^2+z^2) \, dV$ $\quad S = \{(x,y,z) : x^2+y^2+z^2 \le z\}$

$x^2+y^2+\left(z-\frac{1}{2}\right)^2 \le \frac{1}{4}$ $\quad$ it is a shifted sphere $\quad$ simple wrt ρ-first

it's center $(0,0,1/2)$ $z=\frac{1}{2}$

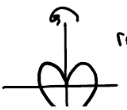
$$I = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{\cos\phi} \rho^2 . \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$\underbrace{x^2+y^2+z^2}_{} = \underset{\shortparallel}{z}$

$\rho^2 = z = \rho . \cos\phi$

so $\rho = \cos\phi$

Ex. Find the volume of $\rho = a(1-\cos\phi)$



rotate around z-axis

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^{a(1-\cos\phi)} 1\, \rho^2 \cdot \sin\phi \; d\rho\, d\phi\, d\theta$$

Ex. Find the volume of the solid bounded by the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad a,b,c > 0$$

küreye benzer ama a,b,c var

Volume $= \iiint_S 1\, dV$ we use modified spherical coordinates

$$\left.\begin{aligned} x &= \rho \cdot \sin\phi \cos\theta \cdot a \\ y &= \rho \cdot \sin\phi \cdot \sin\theta \cdot b \\ z &= \rho \cdot \cos\phi \cdot c \end{aligned}\right\} \text{ böylece elips küre olur ve } \rho = 1$$

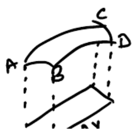
In this change of variable

the Jab. det. $\left| \dfrac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} \right| = abc \cdot \rho^2 \cdot \sin\phi$

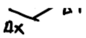
$$V = \int_0^{2\pi} \int_0^{\pi} \int_0^{1} 1 \cdot abc\, \rho^2 \cdot \sin\phi\, d\rho\, d\phi\, d\theta = \frac{4}{3}\pi abc$$

APPLICATIONS

Surface Area



$$\vec{AB} = \Delta x \cdot \hat{i} + 0 \cdot \hat{j} + f_x\, \Delta x\, \hat{k}$$
$$\vec{AC} = 0 \cdot \hat{i} + \Delta y \cdot \hat{j} + f_y \cdot \Delta y \cdot \hat{k}$$



So $\vec{AB} \approx \Delta x \, (1, 0, f_x)$

$\vec{AC} \approx \Delta y \, (0, 1, f_y)$

$\Delta S = \text{area} \approx \| AB \times AC \|$

$$\text{Area} = \begin{vmatrix} i & j & k \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} \cdot \Delta x . \Delta y = \| (-f_x, -f_y, 1) \| \, \Delta x . \Delta y$$

$$= \sqrt{f_x^2 + f_y^2 + 1} \; \Delta x . \Delta y$$

$$\text{Surface Area} = \lim \sum \sum \sqrt{f_x^2 + f_y^2 + 1} \, . \Delta x . \Delta y$$

$$= \iint_D \sqrt{f_x^2 + f_y^2 + 1} \, dA$$

Ex. Find the area of the surface of a sphere of radius R.
Consider just the upper part.

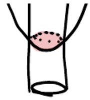
$$z = \sqrt{R^2 - x^2 - y^2} = f(x,y)$$

$$\text{Area} = 2 . \iint_D \sqrt{f_x^2 + f_y^2 + 1} \, dA \qquad = \frac{R}{\sqrt{R^2 - x^2 - y^2}}$$

Using Polars $\Rightarrow \displaystyle\int_0^{2\pi} \int_0^{R} \frac{R}{\sqrt{R^2 - x^2 - y^2}} \, r . dr . d\theta = 4\pi R^2$

Ex. Find the area of the surface of the paraboloid $z = x^2 + y^2$ lying inside the cylinder $x^2 + y^2 = a^2$

$$\text{Area} = \iint_D \sqrt{f_x^2 + f_y^2 + 1}\, dA$$



$z = f(x,y) = x^2+y^2$

$D =$ 

$$\text{Area} = \iint_D \sqrt{(2x)^2 + (2y)^2 + 1}\, dA$$

use polars: $\displaystyle\int_0^{2\pi}\int_0^{a} \sqrt{4r^2+1}\; r\, dr\, d\theta = \frac{(4a^2+1)^{3/2}}{6}$

Center of mass

moment: $\displaystyle M_{xy} = \iiint_V z \cdot \rho(x,y,z)\, dV$ m.d

$\displaystyle M_{yz} = \iiint_V \overset{(x-x_0)}{x} \cdot \rho(x,y,z)\, dV$

$\displaystyle M_{xz} = \iiint_V y \cdot \rho(x,y,z)\, dV$

mass: $\displaystyle m = \iiint_V \rho(x,y,z)\, dV$

center of mass $\rightarrow (\bar{x}, \bar{y}, \bar{z})$

$$\bar{x} = \frac{M_{yz}}{m} \qquad \bar{y} = \frac{M_{xz}}{m} \qquad \bar{z} = \frac{M_{xy}}{m}$$

LINE INTEGRALS

arclength → $\int_a^b \sqrt{1+[f'(x)]^2}\, dx$

$\int_C 1\, ds = \text{arclength } (C)$

yay uzunluğuna göre

$\int_C f(x,y)\, ds \qquad ds = \sqrt{x'(t)^2 + y'(t)^2}\, dt$

If f is cont and C is smooth integral exist

$\text{mass} = \int_C \rho\,(x,y)\, ds$

Ex. Çemberin çevresi (radius = R)

$(R.\cos t, R.\sin t) \quad t \in [0, 2\pi]$ → parametrization

$L = \int_C 1\, ds = \int_0^{2\pi} 1 \underbrace{\sqrt{x'(t)^2 + y'(t)^2}}_{R}\, dt = 2\pi R$

Ex. $\int_C (x+y^2)\, ds$ C is the unit circle.

$C: (\cos t, \sin t) \ , \ t \in [0, 2\pi]$

$$\int_0^{2\pi} (\cos t + \sin^2 t) \underbrace{\sqrt{x'(t)^2 + y'(t)^2}}_{1} \, dt$$

$$0 + \int_0^{2\pi} \frac{1-\cos(2t)}{2} \, dt = \pi$$

Ex. $\int_C xy \, ds$ $\quad$ C is the line segment from $(0,0)$ to $(1,2)$

so $\rightarrow$ $y = 2x$

$(x, 2x)$, $x \in [0,1]$

$$\int_0^1 x \cdot 2x \sqrt{1^2+2^2} \, dx = 2\sqrt{5} \cdot \frac{1}{3}$$

Ex. Find the length of the arc of $x^{2/3} + y^{2/3} = 4$ from $(3^{3/2}, 1)$ to $(8,0)$ in the first quadrant.

$(x^{1/3})^2 + (y^{1/3})^2 = 4$

$x^{1/3} = \cos t \cdot 2$ $\qquad$ $x = 8 \cdot \cos^3 t$

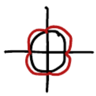
$y^{1/3} = \sin t \cdot 2$ $\qquad$ $y = 8 \cdot \sin^3 t$

$t \in [0, 2\pi]$

! $x^{2p} + y^{2p} = 1$

if $p=1$, circle

if $p>1$,



if $p = 1/2$

if $p<1$,

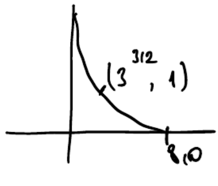
$$\sqrt{x'(t)^2+y'(t)^2} = \sqrt{(24\cos^2 t \cdot (-\sin t))^2 + (24\sin^2 t \cdot \cos t)^2}$$

$$= 24\,|\cos t|\,|\sin t|\sqrt{\cos^2 t + \sin^2 t} = 24\sin t \cdot \cos t$$

$$\int_0^{\pi/6} 24\sin t \cdot \cos t \, dt$$

$12\sin 2t$

$$-\frac{12}{2}\cos 2t \Big|_0^{\pi/6} = 3$$



$y = 1$ so $\sin t = \frac{1}{2}$

$t = \pi/6$

Parametrization

Doğru parçası $x = (1-t)x_0 + t.x_1$

$y = (1-t)y_0 + t.y_1$ $\quad 0 \le t \le 1$ (her zaman)

$z = (1-t)z_0 + t.z_1$

Çember:

$x^2+y^2=r^2$ $\circlearrowleft$ $x = r.\cos t$ $\quad t \in [0, 2\pi]$

$y = r.\sin t$

Elips:

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $\quad x = a.\cos t$ $\quad 0 \le t \le 2\pi$

$y = b.\sin t$

$\circlearrowleft$ $\quad\quad$ $\circlearrowright$

$x = a.\cos t$

$y = -b.\sin t$

x,y,z eksenlere göre line integral

$$\int_C f(x,y)\,dx = \int_a^b f(h(t), g(t)) \cdot h'(t)\,dt$$

$x = h(t)$

$y = g(t)$

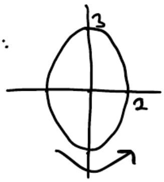
$a \leq t \leq b$

$$\int_C f(x,y)\,dy = \int_a^b f(h(t), g(t)) \cdot g'(t) \cdot dt$$

Ex. $\int_C xy\,dx$, C: part of the ellipse

$\frac{x^2}{4} + \frac{y^2}{9} = 1$, from $(-2,0)$ to $(2,0)$

C:



$(\underset{x}{2\cos t}, \underset{y}{3\sin t})$ $t \in [\pi, 2\pi]$

$$\int_\pi^{2\pi} (2\cos t)(3\sin t) \cdot \underbrace{-2\sin t\,dt}_{dx} \qquad -12 \cdot \frac{\sin^3 t}{3}\Big|_\pi^{2\pi} = 0$$

Ex. C is the first quadrant unit circle , CCW

Calculate $\int_C xy\,dx$, $\int_C xy\,dy$

$C: (\cos t, \sin t) \quad t \in [0, \pi/2]$

$$\int_0^{\pi/2} \cos t . \sin t \, (-\sin t) \, dt \leftarrow x$$

$$\text{"} \quad (\cos t) \, dt \leftarrow y$$

NOTATIONS

$$\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy = \int_C P(x,y)\,dx + Q(x,y)\,dy$$

$$\Delta x, \Delta y = \underbrace{\frac{(x'(t), y'(t))}{\sqrt{x'(t)^2 + y'(t)^2}}}_{\vec{n}} . \Delta s$$

So $\displaystyle\int_C P\,dx + Q\,dy = \int_C (P,Q) \cdot \vec{n} \, ds \qquad (P,Q) = \vec{F}$

$\vec{F}(x,y) = (P(x,y), Q(x,y))$

Ex. $\displaystyle\int_C (x+y)\,dx$, C: the right half of the unit circle.

① $C: (\cos t, \sin t) \quad t \in [-\pi/2, \pi/2]$

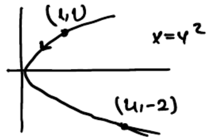
$$= \int_{-\pi/2}^{\pi/2} (\cos t + \sin t)(-\sin t)\, dt$$

$$\cos t \sin t \boxed{-\sin^2 t} = \frac{\cos(2t) - 1}{2}$$

② $C : (\cos t, -\sin t) \quad t \in [-\pi/2, \pi/2]$

$\boxed{\sin^2 t}$

Ex. $I = \int_C x\,dx + y\,dy$ $\quad C:$



C: trivial parametrization using $x = f(y)$

we get $(\underset{x}{y^2}, y)$, $y \in [-2, 1]$ → opposite direction $-C$

Theorem: $\int_C P\,dx + Q\,dy = -\int_{-C} P\,dx + Q\,dy$

$$\int_{-C} x\,dx + y\,dy = \int_{-2}^{1} y^2 (2y\,dy) + y\,dy = -9$$

So the answer = 9

NOTE on work :

Work = F.d

If F is not constant $W = \int_C \vec{F} \cdot \vec{n}\, ds$

$$\vec{F} = m . \vec{a}$$

$$\frac{d\,\vec{v}(t)}{dt} = \frac{d^2}{dt^2} \underbrace{(x(t), y(t))}_{\text{position}}$$

$$\int_C m \cdot \frac{d}{dt}\vec{v}(t) \cdot \underbrace{\frac{(x'(t), y'(t))}{\sqrt{x'(t)^2 + y'(t)^2}}}_{\vec{n}} \cdot ds$$

$$\downarrow$$

$$m \cdot \frac{d\vec{v}(t)}{dt} \cdot \frac{\vec{v}(t)}{\cancel{\|\vec{v}(t)\|}} \cdot \cancel{\|\vec{v}(t)\|} \rightarrow$$

$$v'(t) \cdot v(t) + v(t) \cdot v'(t) = \frac{d}{dt} \|\vec{v}(t)\|^2$$

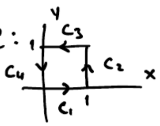
Ex. $\int_C y\,dx + x\,dy$

a) C: unit circle ccw

$(\cos t, \sin t)$ $t \in [0, 2\pi]$

$$\int_0^{2\pi} [\sin t(-\sin t) + \cos t \cdot \cos t]\,dt = 0$$

b) C :

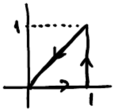


$$\int_{C_1} 0 \cdot dx + x \cdot 0 = 0 \qquad \int_{C_2} \rightarrow \int_0^1 y \cdot 0 + 1 \cdot dy = 1$$

$$\int_{C_3} = -\int_{-C_3} 1 \cdot dx + x \cdot 0 = -\int_0^1 dx = -1$$

$$\int_{C_4} y \cdot 0 + 0 \cdot dy = 0$$

$$0 + 1 - 1 + 0 = 0$$

c) C:  Show that $\int_C y\,dx + x\,dy = 0$

it is not coincidence

GREEN'S THM: Under certain conditions, if C is closed and R is the region bdd by C, then

$$\oint_C P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dA$$

in our ex. $P(x,y) = y$ so $Q_x = P_y = 1$

$Q(x,y) = x$

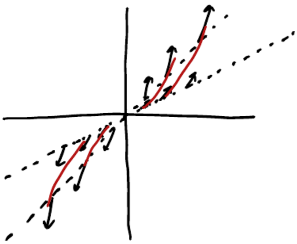
so it is 0 for any closed C.

Scaler and Vector Field

gravitional field $= -G.M.\dfrac{(x,y,z)}{(x^2+y^2+z^2)^{3/2}}$

$\vec{E} = K.Q.$ 〃

Ex. Let $\vec{F}(x,y) = (x, 2y)$ find the field lines



$\vec{v}(t) \parallel \vec{F}$

So $(x'(t), y'(t)) = \lambda\,(x(t), 2y(t))$ $\vec{F}(x,y)$

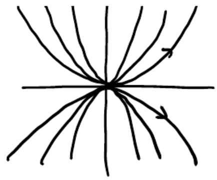
Thus $\dfrac{x'(t)}{x(t)} = \lambda = \dfrac{y'(t)}{2y(t)}$

Integrating

$\ln|x(t)| = \frac{1}{2}\ln|y(t)| + c$

$x = e^c . |y(t)|^{1/2}$

so $y = \pm e^{-c} . x^2$ → parabolas $y = c_1 . x^2$



CONSERVATIVE FIELDS

if $\vec{F} = \nabla f$ → some function

$\vec{F}(x,y) = (P(x,y), Q(x,y)) = \nabla \phi = (\phi_x, \phi_y)$

Remarks:

$P = \phi_x$, $Q = \phi_y$

$C: (x(t), y(t)$, $t \in [a,b]$

then $\int_C P\,dx + Q\,dy = \int_a^b \phi_x(x,y)\,\underset{x'(t)}{dx} + \phi(x,y)\,\underset{y'(t)}{dy}$

$\vec{F}$ is conservative

$\int \vec{F}\,d\vec{r} = \int_a^b \frac{d}{dt}\phi(x(t),y(t))\,dt \underset{FTC}{=} \phi(x(b),y(b)) - \phi(x(a),y(a))$

$= \Delta Q$ change in potential function

Ex. Given that $\vec{F}$ is conservative, find a potential for it.

a) $\vec{F}(x,y) = (x+y^2, 2xy)$

$\phi_x = x+y^2 \quad \phi(x,y) = \frac{x^2}{2} + y^2x + c(y)$ so say $c(y)$ → can be a function of y

$\phi_y = 2xy \quad \phi(x,y) = xy^2 + c(x)$

so $\phi(x,y) = \frac{x^2}{2} + 2xy + c$ · potentials are not unique

Ex. $\vec{F} = (P, Q, R) = (x+yz, xz+y, xy+z)$

$\phi_x = P = x+yz \rightarrow \frac{x^2}{2} + xyz + c(y,z)$

$\phi_y = Q = xz+y \rightarrow xyz + \frac{y^2}{2} + C(x,z)$

$\phi_z = R = xy+z \rightarrow xyz + \frac{z^2}{2} + c(x,y)$

$$\phi = xyz + \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} + c$$

GREEN'S THEOREM

path independent

$$\int_C P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dA$$

$F = (P, Q)$

i) C kapalı bir eğri

ii) ccw yön

iii) $Q_x - P_y \neq 0$

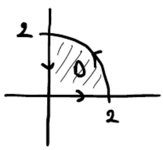
α $$\oint_C \vec{F}\,d\vec{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$

$$\oint_C \vec{F}\,d\vec{r} = \underset{\text{to}}{\phi(P_0)} - \underset{\text{from}}{\phi(P_1)}$$

Ex. $\oint (x-y^3)\,dx + (y^3+x^3)\,dy$, where C is the positively oriented boundary of the quarter-disk

$D: 0 \le x^2+y^2 \le 4,\ x \ge 0,\ y \ge 0$

$$= \iint_D (Q_x - P_y)\,dA$$



$$\iint_D 3x^2 + 3y^2 = 3\int_0^{\pi/2}\int_0^2 r^2 . r . dr . d\theta = 6\pi$$

Ex. Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

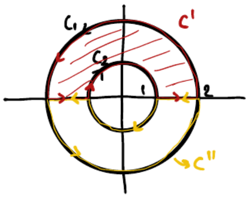
$$A(D) = \iint_D 1\, dA = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C (x\,dy - y\,dx)$$

$x = a\cos\theta$

$y = b\sin\theta$ $\quad 0 \leq \theta \leq 2\pi$ $\quad A(D) = \int_C x\,dy = \int_0^{2\pi} a\cos\theta\, b\cos\theta\, d\theta$

$$= \frac{ab}{2}\int_0^{2\pi} (\cos 2\theta + 1)\, d\theta = ab\pi$$

Ex. $\int_C x . e^{-2x}\, dx + (x^4 + 2x^2y^2)\, dy$



$C = C_1 \cup C_2$

$$I_1 = \iint_{D'} (Q_x - P_y)\, dA = \int_{C'} P\,dx + Q\,dy$$

$$I_2 = \iint_{D''} (Q_x - P_y)\, dA = \int_{C''} P\,dx + Q\,dy$$

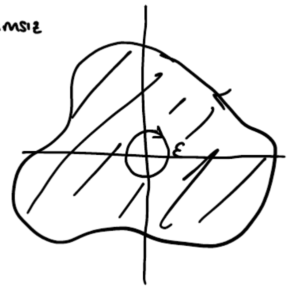
$$I_1 = \iint_D 4x\,(x^2+y^2)\, dA = 4\int_0^{2\pi}\int_1^2 \cos\theta\, r^4\, dr\, d\theta$$

Ex. $F(x,y) = \dfrac{-y\vec{i} + x\vec{j}}{x^2+y^2}$

for every positively oriented simple closed curve that encloses the origin

show that $\displaystyle\int_C F \cdot dr = 2\pi$

$F(x,y) = \left\langle \dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2} \right\rangle \nearrow (0,0)$ tanımsız



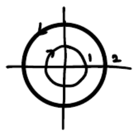
$\dfrac{\partial Q}{\partial x} = \dfrac{y^2-x^2}{(x^2+y^2)^2} \qquad \dfrac{\partial P}{\partial y} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$

$Q_x = P_y$

$\int \quad \int \quad \int$

$\iiint \quad \iint$

Ex. Evaluate $\oint y^3 dx - x^3 dy$ where C



$= -3 \displaystyle\iint_D (x^2+y^2)\, dA = -3 \int_0^{2\pi} \int_1^2 r^2\, r\, dr\, d\theta$

Ex. Area of a disk, radius = a

$$A = \frac{1}{2} \oint_C x\,dy - y\,dx \qquad \begin{array}{l} x = a\cos t \\ y = a\sin t \end{array} \quad t \in [0, 2\pi]$$

$$\frac{1}{2} \int_0^{2\pi} a\cos t \; a\cos t \, dt - \int_0^{2\pi} a\sin t \, (-a\sin t)\, dt$$

$$a^2(\underbrace{c^2 + s^2}_{1}) \rightarrow \frac{1}{2}\int_0^{2\pi} a^2\,dt = \pi a^2$$