

Review Problem Set II

- 1) Three balanced coins are tossed independently. One of the variables of interest is Y_1 , the number of heads. Let Y_2 denote the amount of money won on a side bet in the following manner. If the first head occurs on the first toss, you win \$1. If the first head occurs on toss 2 or on toss 3, you win \$2 or \$3, respectively. If no heads appear, you lose \$1.

- a) Find the joint probability function for Y_1 and Y_2 .
- b) Find $F(2,1)$.

- 2) The joint density function of Y_1 and Y_2 is given as:

$$f(y_1, y_2) = \begin{cases} 6y_1^2 y_2, & 0 \leq y_1 \leq y_2, y_1 + y_2 \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

- a) Find the marginal density function for Y_1 .
- b) Find the conditional density function of Y_2 given that $Y_1=y_1$.
- c) Find $P(Y_2 < 1.1 | Y_1 = 0.6)$.

- 3) Let Y_1 and Y_2 be independently exponentially distributed random variables, each with mean 1. Find $P(Y_1 > Y_2 | Y_1 < 2Y_2)$.

- 4) The joint density function of Y_1 and Y_2 is given as:

$$f(y_1, y_2) = \begin{cases} e^{-(y_1+y_2)}, & y_1 > 0, y_2 > 0 \\ 0, & \text{elsewhere} \end{cases}$$

- a) Show that Y_1 and Y_2 are independent.
- b) Find $E(Y_1+Y_2)$ and $V(Y_1+Y_2)$.
- c) Find $E(Y_1-Y_2)$ and $V(Y_1-Y_2)$.
- d) Find $P(Y_1-Y_2 > 3)$.

- 5) If c is any constant and Y is a random variable such that $E(Y)$ exists, show that $\text{Cov}(c, Y)=0$.

6) Show that $\text{Cov}(X,X)=\text{Var}(X)$.

7)

Suppose we have a bowl with 10 marbles - 2 red marbles, 3 green marbles, and 5 blue marbles. We randomly select 4 marbles from the bowl, with replacement. What is the probability of selecting 2 green marbles and 2 blue marbles?

8) Let the random variable Y possess a uniform distribution on the interval $(0, 1)$. Find the cumulative distribution function of the random variable $W=\sqrt{Y}$.

9)

Consider the random variable Y . Its distribution function is given by

$$F(Y) = \begin{cases} 0, & \text{if } y < 0 \\ y(1 - 0.25y), & \text{if } 0 \leq y \leq 2 \\ 1, & \text{if } y > 2 \end{cases}$$

a.) What is the density of Y ?

b.) Find the density of e^{-Y} .

10) The probability density function of Y is given by

$$f(y) = \begin{cases} \left(\frac{2y}{\theta}\right) e^{\frac{-y^2}{\theta}}, & y \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

Find the density function for $U=Y^2$.

Note that U is a strictly increasing function of Y for $y \geq 0$.

11) Let $Y_1, Y_2, \dots, Y_n$ be independent Poisson random variables with means $\lambda_1, \lambda_2, \dots, \lambda_n$, respectively. Find the probability distribution function of $\sum_{i=1}^n Y_i$.

12)

Let Y_1 and Y_2 have joint density function

$$f_{Y_1, Y_2}(y_1, y_2) = \begin{cases} 8y_1y_2, & 0 \leq y_1 < y_2 \leq 1, \\ 0, & \text{otherwise,} \end{cases}$$

and $U_1 = Y_1/Y_2$ and $U_2 = Y_2$

Derive the joint density function for (U_1, U_2) .

13)

An electronic system has one each of two different types of components in joint operation. Let Y_1 and Y_2 denote the random lengths of life of the components of type I and type II, respectively. The joint density function is given by

$$f(y_1, y_2) = \begin{cases} (1/8)y_1e^{-(y_1+y_2)/2}, & y_1 > 0, y_2 > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

(Measurements are in hundreds of hours.) Find $P(Y_1 > 1, Y_2 > 1)$.

14) For the following joint probability density function

$$f(y_1, y_2) = \begin{cases} 2, & 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1, 0 \leq y_1 + y_2 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

a Find $P(Y_1 \geq 1/2 | Y_2 \leq 1/4)$.

b Find $P(Y_1 \geq 1/2 | Y_2 = 1/4)$.

c. Are Y_1 and Y_2 are independent?

Note that $f_2(y_2) = \int_0^{1-y_2} 2dy_1 = 2(1 - y_2)$ if $0 \leq y_2 \leq 1$

15)

Suppose that Y_1 and Y_2 are independent χ^2 random variables with ν_1 and ν_2 degrees of freedom, respectively. Find

a $E(Y_1 + Y_2)$.

b $V(Y_1 + Y_2)$.

16) The joint density of Y_1 and Y_2 is given by

$$f(y_1, y_2) = \begin{cases} y_1 + y_2, & 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

Find $E(30Y_1 + 25Y_2)$ and $\text{Var}(30Y_1 + 25Y_2)$.

17)

The amount of flour used per day by a bakery is a random variable Y that has an exponential distribution with mean equal to 4 tons. The cost of the flour is proportional to $U = 3Y + 1$.

- a) Find the probability density function of U via the method of distribution functions.
- b) Find the probability density function of U via the method of transformations.

18) For the given joint density function, find the probability density function of $U = Y_1 - Y_2$.

$$f(y_1, y_2) = \begin{cases} e^{-y_1}, & 0 \leq y_2 \leq y_1 < \infty \\ 0, & \text{elsewhere} \end{cases}$$

19)

The joint probability density function of X and Y is given by

$$f(x, y) = \frac{6}{7} \left(x^2 + \frac{xy}{2} \right), \quad 0 < x < 1, \quad 0 < y < 2$$

- a) Verify that is indeed a joint density function.
- b) Compute the marginal density function of X .
- c) Find $P(X > Y)$.

20)

Assume that (X, Y) has the following joint probability function,

$$f(x, y) = \begin{cases} e^{-y}, & 0 < x < y < \infty \\ 0, & \text{elsewhere} \end{cases}$$

- a) Calculate $P(Y > 4 | X = 5)$.
- b) Calculate $E(Y | X = x)$.

21)

Let X be such that $\ln(X) \sim N(\mu, \sigma^2)$. Then prove that

a) $E(X) = e^{\mu + \frac{\sigma^2}{2}}$

b) $E(X^2) = e^{2\mu + 2\sigma^2}$

c) $V(X) = e^{2\mu + \sigma^2} (e^{\sigma^2} - 1)$

22)

Let the probability density function of X_1 and X_2 be given by

$$f(x_1, x_2) = \begin{cases} e^{-(x_1 + x_2)}, & \text{for } x_1 > 0, \quad x_2 > 0 \\ 0, & \text{otherwise} \end{cases}$$

Consider two random variables Y_1 and Y_2 are given by

$$Y_1 = X_1 + X_2$$

$$Y_2 = \frac{X_1}{X_1 + X_2}$$

Find the joint density of Y_1 and Y_2