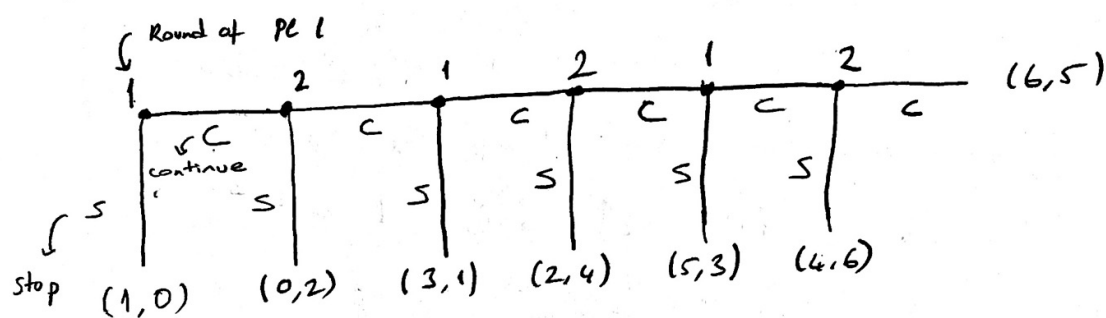


Q1: Centipede Game : an extensive form game in which two players alternately get a chance to take the larger share of a slowly increasing money stash. It is arranged so that if player passes the stash to his opponent who then takes the stash, the player receives a smaller amount, than if he taken the pot. The centipede game concludes as soon as a player takes the stash, with that player getting the larger portion and the other player getting the smaller portion. The game has a pre defined total number of rounds, which are known to each player in advance.



The backward induction eqm of the game is (SSS, SSS), i.e. each player plays S whenever it is her turn to move.

Q2: a) The strategic game and its equilibria.

The corresponding strategic game is

	$U_2 U_3$	$U_2 D_3$	$D_2 U_3$	$D_2 D_3$
$U_1 U_4$	7, 5	7, 5	4, 6	4, 6
$U_1 D_4$	8, 7**	8, 7	4, 6	4, 6
$D_1 U_4$	6, 5	9, 5*	6, 5*	9, 5*
$D_1 D_4$	6, 5	9, 5**	6, 5*	9, 5*

There are seven pure Nash equilibria, marked with (one or two) stars.
The subgame perfect equilibria are marked with two stars.

b) Iterated elimination and subgame perfect equilibria:

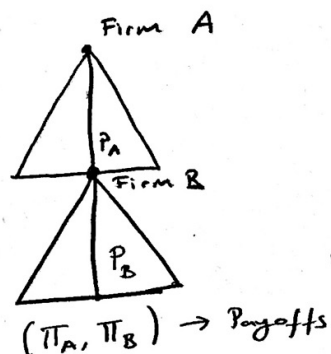
- Consider the subgame perfect equilibrium $(U_1 D_4, U_2 U_3)$
- In the subgame after history (U_1, U_2) , player 1 prefers D_4 over U_4 : $U_1 U_4$ can be eliminated as it is weakly dominated by $U_1 D_4$.
- Knowing that player 1 prefers D_4 over U_4 , in the subgame after history U_1 , now prefers: U_2 over D_2 . So now $D_2 U_3$ weakly dominates $D_2 U_4$ and $U_2 D_3$ weakly dominates $D_2 D_3$.
- In the remaining game, no more actions are weakly dominated so the elimination process stops.
- Notice that both subgame perfect equilibria have survived the process.

Iterated elimination and Nash equilibria:

- Nash equilibria in which player 2 chooses D_2 are always eliminated.

Q3: $N = \{\text{Firm A, Firm B}\}$

$$MC = 0$$



- Begin to solve the problem from the last decision node:

$$\Pi_B = (1 - P_B + \alpha P_A) \cdot P_B$$

$$\frac{\partial \Pi_B}{\partial P_B} = 1 - 2P_B + \alpha P_A = 0$$

$$P_B^* = \frac{1 + \alpha P_A}{2}$$

is the reaction function of firm B, against the decision of firm A. Firm A predicts choice of firm B in the second period and thus includes in its profit function.

$$\Pi_A = (1 - P_A + \alpha P_B) P_A = \left(1 - P_A + \alpha \left(\frac{1 + \alpha P_A}{2}\right)\right) P_A$$

$$\Pi_A = \left(\frac{2 - 2P_A + \alpha + \alpha^2 P_A}{2}\right) P_A$$

$$\frac{\partial \Pi_A}{\partial P_A} = 2 - 4P_A + \alpha + 2\alpha^2 P_A = 0$$

$$\Rightarrow 2 + \alpha = P_A (4 - 2\alpha^2) \Rightarrow P_A^* = \frac{(2 + \alpha)}{2(2 - \alpha^2)}$$

If we plug P_A^* in its place in the reaction function of firm B:

$$P_B^* = \frac{4 - \alpha^2 + 2\alpha}{4(2 - \alpha^2)}$$

$$SPNE = \left\{ \frac{2 + \alpha}{2(2 - \alpha^2)}, \frac{4 - \alpha^2 + 2\alpha}{4(2 - \alpha^2)} \right\}$$

Q4: $c(q) = 2q$

$P(q) = 9 - q$

a) Price and quantity?

$\max_q \pi = (9 - q) \cdot q - 2q$

FOC: $\frac{\partial \pi}{\partial q} = 9 - 2q - 2 = 7 - 2q$

$7 - 2q = 0 \Rightarrow q_m^* = \frac{7}{2}$ and $P_m^* = 9 - \frac{7}{2} = \frac{11}{2}$

b) $\pi_m^* = \left(\frac{11}{2} - 2\right) \cdot \frac{7}{2} = \left(\frac{7}{2}\right)^2 = \frac{49}{4}$

c) There are two firms - what is Cournot eqm?

$\max_i \pi_i = (9 - q_i - q_j) \cdot q_i - 2q_i$

FOC: $\frac{\partial \pi_i}{\partial q_i} = 9 - 2q_i - q_j - 2 = 0$

$q_i = \frac{7 - q_j}{2}$ and $q_j = \frac{7 - q_i}{2}$ (insert)

$q_1 = \frac{7 - \left(\frac{7 - q_1}{2}\right)}{2}$

$4q_1 = 7 + q_1 \Rightarrow q_1 = \frac{7}{3}$ and $q_2 = \frac{7}{3}$

$Q_{\text{total}} = \frac{14}{3}$ and $P = 9 - \frac{14}{3} = \frac{13}{3}$

d) Stackelberg eqm?

Reaction function of Firm 2 is $q_2^*(q_1) = \frac{7 - q_1}{2}$

Firm 1 max. profit taking into account the reaction function of firm 2.

$\max_{q_1} \pi_1 = \left(9 - q_1 - \frac{(7 - q_1)}{2}\right) q_1 - 2q_1$

FOC: $9 - 2q_1 - \frac{7}{2} + q_1 - 2 = 0 \Rightarrow q_1^* = \frac{7}{2}$ and $q_2^* = \frac{7 - q_1^*}{2} = \frac{7}{4}$

So the first mover produces twice as much. The total market

output is $Q^* = \frac{7}{2} + \frac{7}{4} = \frac{21}{4}$, larger than in Cournot.

e) Profit in each case:

$$\pi_c = \left(9 - \frac{14}{3}\right) \frac{7}{3} - \frac{14}{3} = \frac{91}{9} - \frac{14}{3} = \frac{49}{9} ; \text{ Profits are equal in Cournot case.}$$

— Profits in Stackelberg differ. The profit of the leader is

$$\pi_s^1 = \left(9 - \frac{21}{4}\right) \frac{7}{2} - \frac{14}{2} = \frac{105}{8} - \frac{14}{2} = \frac{49}{8} > \pi_c$$

The profit of the follower is

$$\pi_s^2 = \left(9 - \frac{21}{4}\right) \frac{7}{4} - \frac{14}{4} = \frac{105}{16} - \frac{14}{4} = \frac{49}{16} < \pi_c$$

f) Consumers prefer the market with the highest total output (and by consequence the lowest price), since this maximizes Consumer Surplus. Then they will prefer the Stackelberg duopoly.

Q5: This game has one proper (final) subgame in which firm 2 and 3 simultaneously choose their quantities, and an improper (initial) subgame - the game itself - in which firm 1 chooses its quantity given its beliefs about the outcome of the competition between firm 2 and 3 in the final subgame.

$$\begin{cases} MC = m \\ P = A - Q \\ Q = q_1 + q_2 + q_3 \end{cases}$$

• Final subgame: Firm i 's payoff is given by: $(A - Q - m) \cdot q_i$ where $Q = q_i + q_j + q_1$, $i, j = 2, 3$. Firm i 's best response function is;

$$\begin{aligned} \Pi_i &= (A - Q - m) q_i \\ &= (A - q_i - q_j - q_1 - m) q_i \end{aligned}$$

• For firm 2: $\Pi_2 = (A - q_1 - q_2 - q_3 - m) q_2$

$$\frac{\partial \Pi_2}{\partial q_2} = A - q_1 - 2q_2 - q_3 - m = 0$$

$$q_2 = \frac{A - q_1 - q_3 - m}{2}$$

↓

$$BR_2(q_1, q_3)$$

$$q_3 = \frac{A - q_1 - q_2 - m}{2}$$

↓

$$BR_3(q_1, q_2)$$

$$A + NE \quad q_2 = q_3 \Rightarrow q_2 = \frac{A - q_1 - q_2 - m}{2} \Rightarrow \frac{3q_2}{2} = \frac{A - q_1 - m}{2}$$

$$\Rightarrow q_2 = q_3 = \frac{A - q_1 - m}{3}$$

Firm 2 and firm 3 best response functions cross at the point:

$$\left(\frac{A - m - q_1}{3}, \frac{A - m - q_1}{3} \right)$$

• Initial subgame: Firm 1's payoff is given by: $(A - Q - m) \cdot q_1$, where $Q = q_i + q_j + q_1$, $i, j = 2, 3$.

$$\Pi_1 = (A - q_1 - q_2 - q_3 - m) \cdot q_1$$

$$= \left(A - q_1 - \frac{A - m - q_1}{3} - \frac{A - m - q_1}{3} - m \right) \cdot q_1$$

$$\pi_1 = \left(A - q_1 - \frac{2(A - m - q_1)}{3} - m \right) q_1 = \left(\frac{A - q_1 - m}{3} \right) \cdot q_1$$

$$\frac{\partial \pi_1}{\partial q_1} = A - 2q_1 - m - 0 \Rightarrow q_1 = \frac{A - m}{2}$$

The unique SPNE of the game

$$\left(\frac{A - m}{2} ; \frac{A - m - q_1}{3} ; \frac{A - m - q_1}{3} \right)$$

↓

The unique subgame perfect outcome is

$$\left(\frac{A - m}{2} ; \frac{A - m}{6} ; \frac{A - m}{6} \right)$$

Q6: At the last stage, firm 3 takes as given q_1 and q_2 and chooses q_3 that maximizes its profit; which is:

Third stage: $\pi_3 = p q_3 - c q_3 = [120 - (q_1 + q_2 + q_3)] q_3 - c q_3$

$$\frac{\partial \pi_3}{\partial q_3} : 120 - q_1 - q_2 - 2q_3 - c = 0$$

$$BR_3(q_1, q_2) = q_3 = \frac{120 - q_1 - q_2 - c}{2}$$

Second stage: Let us solve the second stage. Firm 2 takes q_1 and $q_3(q_1, q_2)$ as given and chooses q_2 that maximizes its profit

$$\pi_2 = \left[120 - (q_1 + q_2 + \frac{120 - q_1 - q_2 - c}{2}) \right] \cdot q_2 - c q_2$$

$$\pi_2 = \left(\frac{120 - q_1 - q_2 - c}{2} \right) \cdot q_2$$

$$\frac{\partial \pi_2}{\partial q_2} = 120 - q_1 - 2q_2 - c = 0 \Rightarrow q_2 = BR_2(q_1, q_3) = \frac{120 - q_1 - c}{2}$$

First stage: Let us now solve the first stage. Firm 1 takes $q_2(q_1)$ and $q_3(q_1, q_2(q_1))$ as given and chooses q_1 that maximizes its profit:

$$\pi_1 = \left[120 - \left(q_1 - \frac{120 - q_1 - c}{2} - \frac{120 - q_1 - (120 - q_1 - c)/2 - c}{2} - c \right) \right] \cdot q_1$$

$$\pi_1 = \left(\frac{120 - q_1 - c}{2} - \frac{120 - q_1 - c}{4} \right) q_1 = \left(\frac{120 - q_1 - c}{4} \right) q_1$$

$$\frac{\partial \pi_1}{\partial q_1} : 120 - 2q_1 - c = 0$$

$$q_1 = \frac{120 - c}{2}$$

$$\text{So, } q_2 = \frac{120 - c}{4} \text{ and } q_3 = \frac{120 - c}{8}$$

$$p = 120 - Q = 120 - 60 + \frac{c}{2} - 30 + \frac{c}{4} - 15 + \frac{c}{8}$$

$$p = 15 + \frac{7c}{8} \Rightarrow \text{SPNE} = \left\{ \left(60 - \frac{c}{2}, 30 - \frac{c}{4}, 15 - \frac{c}{8} \right) \right\}$$