

Econ 205

Probability – 1

Course Outline

- **Probability**

- Probability and Inference
- The Discrete Case
- Calculating the Probability of an Event
- Conditional Probability and the Independence of Events
- Laws of Probability
- The Law of Total Probability and Bayes' Rule
- Introduction to Random Variables and their distributions

- **Discrete Random Variables and Their Probability Distributions**

- Discrete Random Variable
- The Probability Distribution for Discrete Random Variable
- The Expected Value of Random Variable
- Moments and Moment-Generating Functions
- The Bernoulli Probability Distribution
- The Binomial Probability Distribution
- The Geometric Probability Distribution
- The Negative Binomial Probability Distribution
- The Hypergeometric Probability Distribution
- The Poisson Probability Distribution
- Probability-Generating Functions

- **Continuous Random Variables and Their Probability Densities**

- Continuous Random Variable
- The Probability Distribution for Continuous Random Variable
- The Expected Value for Continuous Random Variable
- Moments and Moment-Generating Functions
- The Uniform Probability Distribution
- The Normal Probability Distribution
- Exponential Probability Distribution
- Chi-Square Probability Distribution
- Tchebysheff's Theorem

Course Outline (cont'd)

- **Multivariate Probability Distributions**

- Bivariate and Multivariate Probability Distributions
- Independent Random Variables
- Marginal and Conditional Probability Distributions
- The Expected Value of a Function of Random Variables
- Special Theorems
- The Covariance of Two Random Variables
- The Expected Value and Variance of Linear Functions of Random Variables
- The Multinomial Probability Distribution
- The Bivariate Normal Distribution
- Conditional Expectations

- **Functions of Random Variables**

- Finding the Probability Distribution of a Function of Random Variables
- The Method of Distribution Functions
- The Method of Transformations
- The Method of Moment-Generating Functions
- Multivariable Transformations Using Jacobians

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Content of this presentation

- Statistics
- Probability
 - Probability and Inference
 - Sample Spaces and Events
 - Experiment, Sample Space, Event (Simple and Compound), Relations from Set Theory, Mutually Exclusive (Disjoint) events
 - Axioms, Interpretations and Properties of Probability
 - Axioms, More probability properties, Determining probabilities systematically, Equally likely outcomes
 - Counting Techniques
 - Product Rule for ordered pairs, tree diagrams, a more general product rule for k-tuples, permutations, combinations
- Examples

Statistics

- “A branch of mathematics dealing with the collection, analysis, interpretation, and presentation of masses of numerical data”
- “The discipline which provides methods for organizing and summarizing data and for drawing conclusions based on information contained in the data”
- “A theory of information with inference making as its objective”
- The objective of statistics is to make an inference about a population based on information contained in a sample from that population and to provide an associated measure of goodness for the inference.

Probability

- “Probability is a measure of one’s belief in the occurrence of a future event.”
- “Probability refers to the study of randomness and uncertainty.”
- “Probability can be associated with the concept of relative frequency.”
- The theory of probability provides methods for quantifying the chances or likelihoods associated with the various outcomes.

Probability and Inference

Probability and statistics both deal with questions involving populations and samples, but do so in an “inverse manner” to each other.

In a probability problem, properties of the population under study are assumed known (e.g., in a numerical population, some specified distribution of the population values may be assumed), and questions regarding a sample taken

from the population are posed and answered. In a statistics problem, characteristics of a sample are available to the experimenter, and this information enables the experimenter to draw conclusions about the population. The relationship between the two disciplines can be summarized by saying that probability reasons from the population to the sample (deductive reasoning), whereas inferential statistics reasons from the sample to the population (inductive reasoning). This is illustrated in Figure 1.2.

Before we can understand what a particular sample can tell us about the population, we should first understand the uncertainty associated with taking a sample from a given population. This is why we study probability before statistics.

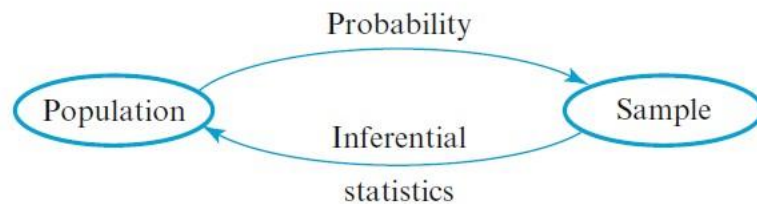


Figure 1.2 The relationship between probability and inferential statistics

Sample Spaces and Events:

An **experiment** is any action or process whose outcome is subject to uncertainty.

DEFINITION The **sample space** of an experiment, denoted by $\mathcal{S}$, is the set of all possible outcomes of that experiment.

DEFINITION An **event** is any collection (subset) of outcomes contained in the sample space $\mathcal{S}$. An event is said to be **simple** if it consists of exactly one outcome and **compound** if it consists of more than one outcome.

Some Relations From Set Theory:

DEFINITION

1. The **union** of two events A and B , denoted by $A \cup B$ and read “ A or B ,” is the event consisting of all outcomes that are *either in A or in B or in both events* (so that the union includes outcomes for which both A and B occur as well as outcomes for which exactly one occurs)—that is, all outcomes in at least one of the events.
2. The **intersection** of two events A and B , denoted by $A \cap B$ and read “ A and B ,” is the event consisting of all outcomes that are in *both A and B* .
3. The **complement** of an event A , denoted by A' , is the set of all outcomes in $\mathcal{S}$ that are not contained in A .

DEFINITION When A and B have no outcomes in common, they are said to be **disjoint** or **mutually exclusive** events. Mathematicians write this compactly as $A \cap B = \emptyset$ where $\emptyset$ denotes the event consisting of no outcomes whatsoever (the “null” or “empty” event).

Distributive Laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

DeMorgan's Laws

$$\overline{(A \cap B)} = \overline{A} \cup \overline{B} \quad \text{and} \quad \overline{(A \cup B)} = \overline{A} \cap \overline{B}$$

Remark: An alternative notation for the complement of a set is used here.

Axioms, Interpretations and Properties of Probability -1

AXIOM 1 For any event A , $P(A) \geq 0$.

AXIOM 2 $P(\mathcal{S}) = 1$.

AXIOM 3 If $A_1, A_2, A_3, \dots$ is an infinite collection of disjoint events, then

$$P(A_1 \cup A_2 \cup A_3 \cdots) = \sum_{i=1}^{\infty} P(A_i)$$

Proposition:

$P(\emptyset) = 0$ where $\emptyset$ is the null event. This in turn implies that the property contained in Axiom 3 is valid for a *finite* collection of events.

Proof First consider the infinite collection $A_1 = \emptyset, A_2 = \emptyset, A_3 = \emptyset, \dots$. Since $\emptyset \cap \emptyset = \emptyset$, the events in this collection are disjoint and $\cup A_i = \emptyset$. The third axiom then gives

$$P(\emptyset) = \sum P(\emptyset)$$

This can happen only if $P(\emptyset) = 0$.

Now suppose that $A_1, A_2, \dots, A_k$ are disjoint events, and append to these the infinite collection $A_{k+1} = \emptyset, A_{k+2} = \emptyset, A_{k+3} = \emptyset, \dots$. Again invoking the third axiom,

$$P\left(\bigcup_{i=1}^k A_i\right) = P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i) = \sum_{i=1}^k P(A_i)$$

as desired. ■

Proposition:

For any event A , $P(A) = 1 - P(A')$

Proof Since by definition of A' , $A \cup A' = \mathcal{S}$ while A and A' are disjoint, $1 = P(\mathcal{S}) = P(A \cup A') = P(A) + P(A')$, from which the desired result follows. ■

Proposition: For any event A , $P(A) \leq 1$.

Proposition: For any events A and B ,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Proof Note first that $A \cup B = A \cup (B \cap A')$. Because A and $(B \cap A')$ are disjoint, $P(A \cup B) = P(A) + P(B \cap A')$. But $B = (B \cap A) \cup (B \cap A')$ (the union of that part of B in A and that part of B not in A). Furthermore, $(B \cap A)$ and $(B \cap A')$ are disjoint, so that $P(B) = P(B \cap A) + P(B \cap A')$. Combining these results gives

$$\begin{aligned} P(A \cup B) &= P(A) + P(B \cap A') = P(A) + [P(B) - P(A \cap B)] \\ &= P(A) + P(B) - P(A \cap B) \end{aligned}$$

Proposition:

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) \\ &\quad - P(B \cap C) + P(A \cap B \cap C). \end{aligned}$$

Axioms, Interpretations and Properties of Probability-2

- Determining Probabilities Systematically

When the number of possible outcomes (simple events) is large, there will be many compound events. A simple way to determine probabilities for these events that avoids violating the axioms and derived properties is to first determine probabilities $P(E_i)$ for all simple events. These should satisfy $P(E_i) \geq 0$ and $\sum_{\text{all } i} P(E_i) = 1$. Then the probability of any compound event A is computed by adding together the $P(E_i)$'s for all E_i 's in A .

$$P(A) = \sum_{\text{all } E_i \text{ in } A} P(E_i)$$

- Equally Likely Outcomes

In many experiments consisting of N outcomes, it is reasonable to assign equal probabilities to all N simple events. These include such obvious examples as tossing a fair coin or fair die once or twice (or any fixed number of times), or selecting one or several cards from a well-shuffled deck of 52. With $p = P(E_i)$ for every i ,

$$1 = \sum_{i=1}^N P(E_i) = \sum_{i=1}^N p = p \cdot N \quad \text{so} \quad p = \frac{1}{N}$$

That is, if there are N possible outcomes, then the probability assigned to each is $1/N$.

Now consider an event A , with $N(A)$ denoting the number of outcomes contained in A . Then

$$P(A) = \sum_{E_i \text{ in } A} P(E_i) = \sum_{E_i \text{ in } A} \frac{1}{N} = \frac{N(A)}{N}$$

Counting Techniques-1

Product Rule for ordered pairs:

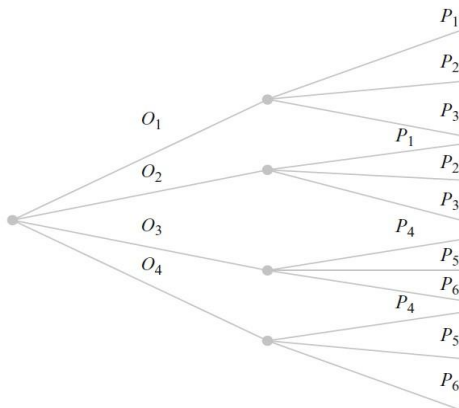
If the first element or object of an ordered pair can be selected in n_1 ways, and for each of these n_1 ways the second element of the pair can be selected in n_2 ways, then the number of pairs is $n_1 n_2$.

Product Rule for k-tuples:

Suppose a set consists of ordered collections of k elements (k -tuples) and that there are n_1 possible choices for the first element; for each choice of the first element, there are n_2 possible choices of the second element; ...; for each possible choice of the first $k - 1$ elements, there are n_k choices of the k th element. Then there are $n_1 n_2 \cdots n_k$ possible k -tuples.

Tree Diagrams:

In many counting and probability problems, a configuration called a *tree diagram* can be used to represent pictorially all the possibilities.



Permutations:

Any ordered sequence of k objects taken from a set of n distinct objects is called a **permutation** of size k of the objects. The number of permutations of size k that can be constructed from the n objects is denoted by $P_{k,n}$.

The number of permutations of size k is obtained immediately from the general product rule. The first element can be chosen in n ways, for each of these n ways the second element can be chosen in $n - 1$ ways, and so on; finally, for each way of choosing the first $k - 1$ elements, the k th element can be chosen in $n - (k - 1) = n - k + 1$ ways, so

$$P_{k,n} = n(n-1)(n-2) \cdots (n-k+2)(n-k+1)$$

Notation:

For any positive integer m , $m!$ is read “ m factorial” and is defined by $m! = m(m-1) \cdots (2)(1)$. Also, $0! = 1$.

$$P_{k,n} = \frac{n!}{(n-k)!}$$

Remark: An alternative notation used for $P_{k,n}$ is P_k^n .

Counting Techniques-2

Combinations:

Given a set of n distinct objects, any unordered subset of size k of the objects is called a **combination**. The number of combinations of size k that can be formed from n distinct objects will be denoted by $\binom{n}{k}$. (This notation is more common in probability than $C_{k,n}$, which would be analogous to notation for permutations.)

$$\binom{n}{k} = \frac{P_{k,n}}{k!} = \frac{n!}{k!(n-k)!}$$

EXAMPLES

Examples 1-2: Sample Spaces and Events

Example 1: Simple and compound events

Some events associated with a single toss of a balanced die are these:

- A: Observe an odd number.
- B: Observe a number less than 5.
- C: Observe a 2 or a 3.
- E_1 : Observe a 1.
- E_2 : Observe a 2.
- E_3 : Observe a 3.
- E_4 : Observe a 4.
- E_5 : Observe a 5.
- E_6 : Observe a 6.

You can see that there is a distinct difference among some of the events associated with the die-tossing experiment. For example, if you observe event A (an odd number), at the same time you will have observed E_1 , E_3 , or E_5 . Thus, event A , which can be decomposed into three other events, is called a *compound event*. In contrast, the events E_1 , E_2 , E_3 , E_4 , E_5 , and E_6 cannot be decomposed and are called *simple events*. A simple event can happen only in one way, whereas a compound event can happen in more than one distinct way.

Example 2: Simple and compound events

Two gas stations are located at a certain intersection. Each one has six gas pumps. Consider the experiment in which the number of pumps in use at a particular time of day is determined for each of the stations. An experimental outcome specifies how many pumps are in use at the first station and how many are in use at the second one. One possible outcome is (2, 2), another is (4, 1), and yet another is (1, 4). The 49 outcomes in $\mathcal{S}$ are displayed in the accompanying table. The sample space for the experiment in which a six-sided die is thrown twice results from deleting the 0 row and 0 column from the table, giving 36 outcomes.

First Station	Second Station						
	0	1	2	3	4	5	6
0	(0, 0)	(0, 1)	(0, 2)	(0, 3)	(0, 4)	(0, 5)	(0, 6)
1	(1, 0)	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 0)	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 0)	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 0)	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 0)	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 0)	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

When the number of pumps in use at each of two 6-pump gas stations is observed, there are 49 possible outcomes, so there are 49 simple events: $E_1 = \{(0, 0)\}$, $E_2 = \{(0, 1)\}$, $\dots$, $E_{49} = \{(6, 6)\}$. Examples of compound events are

- $A = \{(0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$ = the event that the number of pumps in use is the same for both stations
- $B = \{(0, 4), (1, 3), (2, 2), (3, 1), (4, 0)\}$ = the event that the total number of pumps in use is four
- $C = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ = the event that at most one pump is in use at each station

Example 3: Sample Spaces and Events, Axioms of Probability

Example 3: Infinite Sample Space, Verification of Axioms 1 and 2

If a new type-D flashlight battery has a voltage that is outside certain limits, that battery is characterized as a failure (F); if the battery has a voltage within the prescribed limits, it is a success (S). Suppose an experiment consists of testing each battery as it comes off an assembly line until we first observe a success. Although it may not be very likely, a possible outcome of this experiment is that the first 10 (or 100 or 1000 or ...) are F 's and the next one is an S . That is, for any positive integer n , we may have to examine n batteries before seeing the first S . The sample space is $\mathcal{S} = \{S, FS, FFS, FFFS, \dots\}$, which contains an infinite number of possible outcomes. The same abbreviated form of the sample space is appropriate for an experiment in which, starting at a specified time, the gender of each newborn infant is recorded until the birth of a male is observed.

The sample space for the battery examination experiment contains an infinite number of outcomes, so there are an infinite number of simple events. Compound events include

$A = \{S, FS, FFS\}$ = the event that at most three batteries are examined

$E = \{FS, FFFS, FFFFFS, \dots\}$ = the event that an even number of batteries are examined

The simple events are $E_1 = \{S\}$, $E_2 = \{FS\}$, $E_3 = \{FFS\}$, $E_4 = \{FFFS\}$, Suppose the probability of any particular battery being satisfactory is .99. Then it can be shown that $P(E_1) = .99$, $P(E_2) = (.01)(.99)$, $P(E_3) = (.01)^2(.99)$, ... is an assignment of probabilities to the simple events that satisfies the axioms. In particular, because the E_i 's are disjoint and $\mathcal{S} = E_1 \cup E_2 \cup E_3 \cup \dots$, it must be the case that

$$\begin{aligned} 1 &= P(\mathcal{S}) = P(E_1) + P(E_2) + P(E_3) + \dots \\ &= .99[1 + .01 + (.01)^2 + (.01)^3 + \dots] \end{aligned}$$

Here we have used the formula for the sum of a geometric series:

$$a + ar + ar^2 + ar^3 + \dots = \frac{a}{1 - r}$$

However, another legitimate (according to the axioms) probability assignment of the same "geometric" type is obtained by replacing .99 by any other number p between 0 and 1 (and .01 by $1-p$). ■

Examples 4-5: Axioms, Interpretations and Properties of Probability

Example 4: (Recall For any event A , $P(A) = 1 - P(A')$)

Consider a system of five identical components connected in series, as illustrated in Figure 2.3.

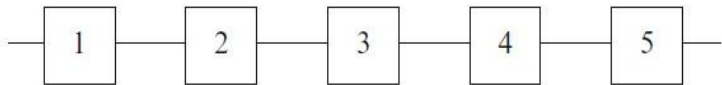


Figure 2.3 A system of five components connected in series

Denote a component that fails by F and one that doesn't fail by S (for success). Let A be the event that the *system* fails. For A to occur, at least one of the individual components must fail. Outcomes in A include $SSFSS$ (1, 2, 4, and 5 all work, but 3 does not), $FFSSS$, and so on. There are in fact 31 different outcomes in A . However, A' , the event that the system works, consists of the single outcome $SSSSS$. We will see in Section 2.5 that if 90% of all these components do not fail and different components fail independently of one another, then $P(A') = P(SSSSS) = .9^5 = .59$. Thus $P(A) = 1 - .59 = .41$; so among a large number of such systems, roughly 41% will fail. ■

Example 5: (Recall $P(A \cup B) = P(A) + P(B) - P(A \cap B)$)

In a certain residential suburb, 60% of all households get internet service from the local cable company, 80% get television service from that company, and 50% get both services from the company. If a household is randomly selected, what is the probability that it gets at least one of these two services from the company, and what is the probability that it gets exactly one of the services from the company?

With $A = \{\text{gets internet service from the cable company}\}$ and $B = \{\text{gets television service from the cable company}\}$, the given information implies that $P(A) = .6$, $P(B) = .8$, and $P(A \cap B) = .5$. The previous proposition then applies to give

$P(\text{gets at least one of these two services from the company})$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = .6 + .8 - .5 = .9$$

The event that a household gets only television service from the company can be written as $A' \cap B$ [(not internet) and television]. Now

$$.9 = P(A \cup B) = P(A) + P(A' \cap B) = .6 + P(A' \cap B)$$

from which $P(A' \cap B) = .3$. Similarly, $P(A \cap B') = P(A \cup B) - P(B) = .1$. This is all illustrated in Figure 2.5, from which we see that

$$P(\text{exactly one}) = P(A \cap B') + P(A' \cap B) = .1 + .3 = .4$$

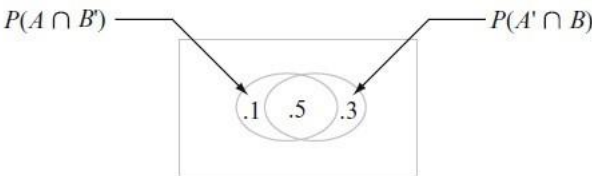


Figure 2.5 Probabilities for Example 2.14

Example 6: Axioms, Interpretations and Properties of Probability

Example 6: Determining probabilities systematically, Equally likely outcomes

Consider the problem of selecting two applicants for a job out of a group of five and imagine that the applicants vary in competence, 1 being the best, 2 second best, and so on, for 3, 4, and 5. These ratings are of course unknown to the employer. Define two events A and B as:

A : The employer selects the best and one of the two poorest applicants (applicants 1 and 4 or 1 and 5).

B : The employer selects at least one of the two best.

Find the probabilities of these events.

The steps are as follows:

1. The experiment involves randomly selecting two applicants out of five. Denote the selection of applicants 3 and 5 by $\{3, 5\}$.
2. The ten simple events, with $\{i, j\}$ denoting the selection of applicants i and j , are

$$\begin{aligned} E_1: \{1, 2\}, & \quad E_5: \{2, 3\}, & E_8: \{3, 4\}, & \quad E_{10}: \{4, 5\}. \\ E_2: \{1, 3\}, & \quad E_6: \{2, 4\}, & E_9: \{3, 5\}, & \\ E_3: \{1, 4\}, & \quad E_7: \{2, 5\}, & & \\ E_4: \{1, 5\}, & & & \end{aligned}$$

3. A random selection of two out of five gives each pair an equal chance for selection. Hence, we will assign each sample point the probability $1/10$. That is,

$$P(E_i) = 1/10 = .1, \quad i = 1, 2, \dots, 10.$$

4. Checking the sample points, we see that B occurs whenever $E_1, E_2, E_3, E_4, E_5, E_6$, or E_7 occurs. Hence, these sample points are included in B .

5. Finally, $P(B)$ is equal to the sum of the probabilities of the sample points in B , or

$$P(B) = \sum_{i=1}^7 P(E_i) = \sum_{i=1}^7 .1 = .7.$$

Similarly, we see that event $A = E_3 \cup E_4$ and that $P(A) = .1 + .1 = .2$. ■

Example 7: Axioms, Interpretations and Properties of Probability

Example 7: Determining probabilities systematically

The odds are two to one that, when A and B play tennis, A wins. Suppose that A and B play two matches. What is the probability that A wins at least one match?

1. The experiment consists of observing the winner (A or B) for each of two matches. Let AB denote the event that player A wins the first match and player B wins the second.
2. The sample space for the experiment consists of four sample points:

$$E_1: AA, \quad E_2: AB, \quad E_3: BA, \quad E_4: BB$$

3. Because A has a better chance of winning any match, it does not seem appropriate to assign equal probabilities to these sample points.
under certain conditions it is reasonable to make the following assignment of probabilities:

$$P(E_1) = 4/9, \quad P(E_2) = 2/9, \quad P(E_3) = 2/9, \quad P(E_4) = 1/9.$$

Notice that, even though the probabilities assigned to the simple events are not all equal, $P(E_i) \geq 0$, for $i = 1, 2, 3, 4$, and $\sum_S P(E_i) = 1$.

4. The event of interest is that A wins at least one game. Thus, if we denote the event of interest as C , it is easily seen that

$$C = E_1 \cup E_2 \cup E_3.$$

5. Finally,

$$P(C) = P(E_1) + P(E_2) + P(E_3) = 4/9 + 2/9 + 2/9 = 8/9. \quad \blacksquare$$

Example 8: Counting Techniques

Example 8: Product Rule for k-tuples

Consider an experiment that consists of recording the birthday for each of 20 randomly selected persons. Ignoring leap years and assuming that there are only 365 possible distinct birthdays, find the number of points in the sample space S for this experiment. If we assume that each of the possible sets of birthdays is equiprobable, what is the probability that each person in the 20 has a different birthday?

Number the days of the year $1, 2, \dots, 365$. A sample point for this experiment can be represented by an ordered sequence of 20 numbers, where the first number denotes the number of the day that is the first person's birthday, the second number denotes the number of the day that is the second person's birthday, and so on. We are concerned with the number of 20-tuples that can be formed, selecting a number representing one of the 365 days in the year from each of 20 sets. The sets are all identical, and each contains 365 elements. Repeated applications of the mn rule tell us there are $(365)^{20}$ such 20-tuples. Thus, the sample space S contains $N = (365)^{20}$ sample points. Although we could not feasibly list all the sample points, if we assume them to be equiprobable, $P(E_i) = 1/(365)^{20}$ for each simple event.

If we denote the event that each person has a different birthday by A , the probability of A can be calculated if we can determine n_a , the number of sample points in A . A sample point is in A if the corresponding 20-tuple is such that no two positions contain the same number. Thus, the set of numbers from which the first element in a 20-tuple in A can be selected contains 365 numbers, the set from which the second element can be selected contains 364 numbers (all but the one selected for the first element), the set from which the third can be selected contains 363 (all but the two selected for the first two elements), $\dots$, and the set from which the 20th element can be selected contains 346 elements (all but those selected for the first 19 elements). An extension of the mn rule yields

$$n_a = (365) \times (364) \times \dots \times (346).$$

Finally, we may determine that

$$P(A) = \frac{n_a}{N} = \frac{365 \times 364 \times \dots \times 346}{(365)^{20}} = .5886.$$



Remark: “mn-rule” is another name for “the product rule for k-tuples”.

Examples 9-10-11: Counting Techniques

Example 9: Permutations

Ten teaching assistants are available for grading papers in a particular course. The first exam consists of four questions, and the professor wishes to select a different assistant to grade each question (only one assistant per question). In how many ways can assistants be chosen to grade the exam? Here n = the number of assistants = 10 and k = the number of questions = 4. The number of different grading assignments is then $P_{4,10} = (10)(9)(8)(7) = 5040$.

Example 10: Permutations

The names of 3 employees are to be randomly drawn, without replacement, from a bowl containing the names of 30 employees of a small company. The person whose name is drawn first receives \$100, and the individuals whose names are drawn second and third receive \$50 and \$25, respectively. How many sample points are associated with this experiment?

Because the prizes awarded are different, the number of sample points is the number of ordered arrangements of $r = 3$ out of the possible $n = 30$ names. Thus, the number of sample points in S is

$$P_3^{30} = \frac{30!}{27!} = (30)(29)(28) = 24,360.$$



Example 11: Permutations

Prove the following theorem:

The number of ways of partitioning n distinct objects into k distinct groups containing $n_1, n_2, \dots, n_k$ objects, respectively, where each object appears in exactly one group and $\sum_{i=1}^k n_i = n$, is

$$N = \binom{n}{n_1 \ n_2 \ \dots \ n_k} = \frac{n!}{n_1! \ n_2! \ \dots \ n_k!}.$$

Proof:

N is the number of distinct arrangements of n objects in a row for a case in which rearrangement of the objects within a group does not count. For example, the letters a to l are arranged in three groups, where $n_1 = 3, n_2 = 4$, and $n_3 = 5$:

$$abc|defg|hijkl$$

is one such arrangement.

The number of distinct arrangements of the n objects, assuming all objects are distinct, is $P_n^n = n!$. Then P_n^n equals the number of ways of partitioning the n objects into k groups (ignoring order within groups) multiplied by the number of ways of ordering the $n_1, n_2, \dots, n_k$ elements within each group. This application of the extended mn rule gives

$$P_n^n = (N) \cdot (n_1! \ n_2! \ n_3! \ \dots \ n_k!),$$

where $n_i!$ is the number of distinct arrangements of the n_i objects in group i .

Solving for N , we have

$$N = \frac{n!}{n_1! \ n_2! \ \dots \ n_k!} \equiv \binom{n}{n_1 \ n_2 \ \dots \ n_k}.$$

Example 12: Counting Techniques

Example 12: Permutations, Combinations

Given n distinct objects and k fixed, which one is bigger:

$P_{k,n}$ or $C_{k,n}$?

The number of combinations of size k from a particular set is smaller than the number of permutations because, when order is disregarded, some of the permutations correspond to the same combination. Consider, for example, the set $\{A, B, C, D, E\}$ consisting of five elements. There are $5!/(5-3)! = 60$ permutations of size 3. There are six permutations of size 3 consisting of the elements A, B , and C because these three can be ordered $3 \cdot 2 \cdot 1 = 3! = 6$ ways: (A, B, C) , (A, C, B) , (B, A, C) , (B, C, A) , (C, A, B) , and (C, B, A) . These six permutations are equivalent to the single combination $\{A, B, C\}$. Similarly, for any other combination of size 3, there are $3!$ permutations, each obtained by ordering the three objects. Thus,

$$60 = P_{3,5} = \binom{5}{3} \cdot 3! \quad \text{so} \quad \binom{5}{3} = \frac{60}{3!} = 10$$

These ten combinations are

$\{A, B, C\}$ $\{A, B, D\}$ $\{A, B, E\}$ $\{A, C, D\}$ $\{A, C, E\}$ $\{A, D, E\}$ $\{B, C, D\}$
 $\{B, C, E\}$ $\{B, D, E\}$ $\{C, D, E\}$

When there are n distinct objects, any permutation of size k is obtained by ordering the k unordered objects of a combination in one of $k!$ ways, so the number of permutations is the product of $k!$ and the number of combinations. This gives

$$\binom{n}{k} = \frac{P_{k,n}}{k!} = \frac{n!}{k!(n-k)!}$$

Example 13: Combinations

Example 13: Counting Techniques

Example 13: Combinations

A bridge hand consists of any 13 cards selected from a 52-card deck without regard to order. There are $\binom{52}{13} = 52!/(13! \cdot 39!)$ different bridge hands, which works out to approximately 635 billion. Since there are 13 cards in each suit, the number of hands consisting entirely of clubs and/or spades (no red cards) is $\binom{26}{13} = 26!/(13! \cdot 13!) = 10,400,600$. One of these $\binom{26}{13}$ hands consists entirely of spades, and one consists entirely of clubs, so there are $[\binom{26}{13} - 2]$ hands that consist entirely of clubs and spades with both suits represented in the hand. Suppose a bridge hand is dealt from a well-shuffled deck (i.e., 13 cards are randomly selected from among the 52 possibilities) and let

$A = \{\text{the hand consists entirely of spades and clubs with both suits represented}\}$

$B = \{\text{the hand consists of exactly two suits}\}$

The $N = \binom{52}{13}$ possible outcomes are equally likely, so

$$P(A) = \frac{N(A)}{N} = \frac{\binom{26}{13} - 2}{\binom{52}{13}} = .0000164$$

Since there are $\binom{4}{2} = 6$ combinations consisting of two suits, of which spades and clubs is one such combination,

$$P(B) = \frac{N(B)}{N} = \frac{6 \left[\binom{26}{13} - 2 \right]}{\binom{52}{13}} = .0000983$$

That is, a hand consisting entirely of cards from exactly two of the four suits will occur roughly once in every 10,000 hands. If you play bridge only once a month, it is likely that you will never be dealt such a hand. ■

References

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- Wackerly, D.D., Mendenhall, W. and Scheaffer, R.L., (2008) Mathematical Statistics With Applications, 7th Edition, Pacific Grove: Duxbury.