

Econ 205

Transformations of Random Variables – 1

Course Outline

- **Probability**

- Probability and Inference
- The Discrete Case
- Calculating the Probability of an Event
- Conditional Probability and the Independence of Events
- Laws of Probability
- The Law of Total Probability and Bayes' Rule
- Introduction to Random Variables and their distributions

- **Discrete Random Variables and Their Probability Distributions**

- Discrete Random Variable
- The Probability Distribution for Discrete Random Variable
- The Expected Value of Random Variable
- Moments and Moment-Generating Functions
- The Bernoulli Probability Distribution
- The Binomial Probability Distribution
- The Geometric Probability Distribution
- The Negative Binomial Probability Distribution
- The Hypergeometric Probability Distribution
- The Poisson Probability Distribution
- Probability-Generating Functions

- **Continuous Random Variables and Their Probability Densities**

- Continuous Random Variable
- The Probability Distribution for Continuous Random Variable
- The Expected Value for Continuous Random Variable
- Moments and Moment-Generating Functions
- The Uniform Probability Distribution
- The Normal Probability Distribution
- Exponential Probability Distribution
- Chi-Square Probability Distribution
- Tchebysheff's Theorem

Course Outline (cont'd)

- **Multivariate Probability Distributions**

- Bivariate and Multivariate Probability Distributions
- Independent Random Variables
- Marginal and Conditional Probability Distributions
- The Expected Value of a Function of Random Variables
- Special Theorems
- The Covariance of Two Random Variables
- The Expected Value and Variance of Linear Functions of Random Variables
- The Multinomial Probability Distribution
- The Bivariate Normal Distribution
- Conditional Expectations

- **Functions of Random Variables**

- Finding the Probability Distribution of a Function of Random Variables
- The Method of Distribution Functions
- The Method of Transformations
- The Method of Moment-Generating Functions
- Multivariable Transformations Using Jacobians

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Content of this presentation

- Finding the Probability Distribution of a Function of Random Variables
- Finding the Joint Density Function of Two Functions of Random Variables
- Summary of the four methods
- Obtaining the Transformation Method using the Distribution Function Method
- Examples

Finding the Probability Distribution of a Function of Random Variables

Consider random variables $Y_1, Y_2, \dots, Y_n$ and a function $U(Y_1, Y_2, \dots, Y_n)$ denoted as U . We will introduce three methods for finding **$f_U(U)$, the probability density function of U :**

- The Distribution Function Method
- The Transformation Method
- The Moment-Generating Function Method

The Transformation Method is obtained from the Distribution Function Method.

Finding the Joint Density Function of Two Functions of Random Variables

Suppose that Y_1 and Y_2 are continuous random variables with joint density function $f_{Y_1, Y_2}(Y_1, Y_2)$. The fourth method presented in the next slide is for finding **the joint density function $f_{U_1, U_2}(U_1, U_2)$** of two functions of random variables $U_1(Y_1, Y_2)$ and $U_2(Y_1, Y_2)$ using $f_{Y_1, Y_2}(Y_1, Y_2)$ if certain assumptions are satisfied. This fourth method is:

- The Bivariate Transformation Method

Summary of the four methods

Summary of the Distribution Function Method

Let U be a function of the random variables $Y_1, Y_2, \dots, Y_n$.

1. Find the region $U = u$ in the $(y_1, y_2, \dots, y_n)$ space.
2. Find the region $U \leq u$.
3. Find $F_U(u) = P(U \leq u)$ by integrating $f(y_1, y_2, \dots, y_n)$ over the region $U \leq u$.
4. Find the density function $f_U(u)$ by differentiating $F_U(u)$. Thus, $f_U(u) = dF_U(u)/du$.

Summary of the Transformation Method

Let $U = h(Y)$, where $h(y)$ is either an increasing or decreasing function of y for all y such that $f_Y(y) > 0$.

1. Find the inverse function, $y = h^{-1}(u)$.
2. Evaluate $\frac{dh^{-1}}{du} = \frac{d[h^{-1}(u)]}{du}$.
3. Find $f_U(u)$ by

$$f_U(u) = f_Y[h^{-1}(u)] \left| \frac{dh^{-1}}{du} \right|.$$

Summary of the Moment-Generating Function Method

Let U be a function of the random variables $Y_1, Y_2, \dots, Y_n$.

1. Find the moment-generating function for U , $m_U(t)$.
2. Compare $m_U(t)$ with other well-known moment-generating functions. If $m_U(t) = m_V(t)$ for all values of t , Theorem 6.1 implies that U and V have identical distributions.

THEOREM 6.1

Let $m_X(t)$ and $m_Y(t)$ denote the moment-generating functions of random variables X and Y , respectively. If both moment-generating functions exist and $m_X(t) = m_Y(t)$ for all values of t , then X and Y have the same probability distribution.

The Bivariate Transformation Method

Suppose that Y_1 and Y_2 are continuous random variables with joint density function $f_{Y_1, Y_2}(y_1, y_2)$ and that for all (y_1, y_2) , such that $f_{Y_1, Y_2}(y_1, y_2) > 0$,

$$u_1 = h_1(y_1, y_2) \quad \text{and} \quad u_2 = h_2(y_1, y_2)$$

is a one-to-one transformation from (y_1, y_2) to (u_1, u_2) with inverse

$$y_1 = h_1^{-1}(u_1, u_2) \quad \text{and} \quad y_2 = h_2^{-1}(u_1, u_2).$$

If $h_1^{-1}(u_1, u_2)$ and $h_2^{-1}(u_1, u_2)$ have continuous partial derivatives with respect to u_1 and u_2 and Jacobian

$$J = \det \begin{bmatrix} \frac{\partial h_1^{-1}}{\partial u_1} & \frac{\partial h_1^{-1}}{\partial u_2} \\ \frac{\partial h_2^{-1}}{\partial u_1} & \frac{\partial h_2^{-1}}{\partial u_2} \end{bmatrix} = \frac{\partial h_1^{-1}}{\partial u_1} \frac{\partial h_2^{-1}}{\partial u_2} - \frac{\partial h_2^{-1}}{\partial u_1} \frac{\partial h_1^{-1}}{\partial u_2} \neq 0,$$

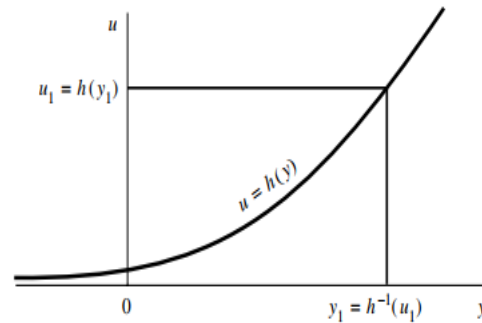
then the joint density of U_1 and U_2 is

$$f_{U_1, U_2}(u_1, u_2) = f_{Y_1, Y_2}(h_1^{-1}(u_1, u_2), h_2^{-1}(u_1, u_2)) |J|,$$

where $|J|$ is the absolute value of the Jacobian.

Obtaining the Transformation Method using the Distribution Function Method

FIGURE 6.8
An increasing function



Suppose that $h(y)$ is an increasing function of y and that $U = h(Y)$, where Y has density function $f_Y(y)$. Then $h^{-1}(u)$ is an increasing function of u : If $u_1 < u_2$, then $h^{-1}(u_1) = y_1 < y_2 = h^{-1}(u_2)$. We see from Figure 6.8 that the set of points y such that $h(y) \leq u_1$ is precisely the same as the set of points y such that $y \leq h^{-1}(u_1)$. Therefore (see Figure 6.8),

$$P(U \leq u) = P[h(Y) \leq u] = P[h^{-1}[h(Y)] \leq h^{-1}(u)] = P[Y \leq h^{-1}(u)]$$

or

$$F_U(u) = F_Y[h^{-1}(u)].$$

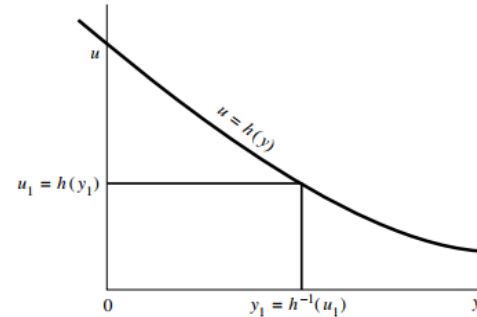
Then differentiating with respect to u , we have

$$f_U(u) = \frac{dF_U(u)}{du} = \frac{dF_Y[h^{-1}(u)]}{du} = f_Y(h^{-1}(u)) \frac{d[h^{-1}(u)]}{du}.$$

To simplify notation, we will write dh^{-1}/du instead of $d[h^{-1}(u)]/du$ and

$$f_U(u) = f_Y[h^{-1}(u)] \frac{dh^{-1}}{du}.$$

FIGURE 6.9
A decreasing function



If $h(y)$ is a decreasing function of y , then $h^{-1}(u)$ is a decreasing function of u . That is, if $u_1 < u_2$, then $h^{-1}(u_1) = y_1 > y_2 = h^{-1}(u_2)$. Also, as in Figure 6.9, the set of points y such that $h(y) \leq u_1$ is the same as the set of points such that $y \geq h^{-1}(u_1)$. It follows that, for $U = h(Y)$, as shown in Figure 6.9,

$$P(U \leq u) = P[Y \geq h^{-1}(u)] \quad \text{or} \quad F_U(u) = 1 - F_Y[h^{-1}(u)].$$

If we differentiate with respect to u , we obtain

$$f_U(u) = -f_Y[h^{-1}(u)] \frac{d[h^{-1}(u)]}{du}.$$

So we have:

Let Y have probability density function $f_Y(y)$. If $h(y)$ is either increasing or decreasing for all y such that $f_Y(y) > 0$, then $U = h(Y)$ has density function

$$f_U(u) = f_Y[h^{-1}(u)] \left| \frac{dh^{-1}}{du} \right|, \quad \text{where} \quad \frac{dh^{-1}}{du} = \frac{d[h^{-1}(u)]}{du}.$$

EXAMPLES

Example 1: The Distribution Function Method

A univariate example:

EXAMPLE

A process for refining sugar yields up to 1 ton of pure sugar per day, but the actual amount produced, Y , is a random variable because of machine breakdowns and other slowdowns. Suppose that Y has density function given by

$$f(y) = \begin{cases} 2y, & 0 \leq y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

The company is paid at the rate of \$300 per ton for the refined sugar, but it also has a fixed overhead cost of \$100 per day. Thus the daily profit, in hundreds of dollars, is $U = 3Y - 1$. Find the probability density function for U .

Solution To employ the distribution function approach, we must find

$$F_U(u) = P(U \leq u) = P(3Y - 1 \leq u) = P\left(Y \leq \frac{u+1}{3}\right).$$

If $u < -1$, then $(u+1)/3 < 0$ and, therefore, $F_U(u) = P(Y \leq (u+1)/3) = 0$. Also, if $u > 2$, then $(u+1)/3 > 1$ and $F_U(u) = P(Y \leq (u+1)/3) = 1$. However, if $-1 \leq u \leq 2$, the probability can be written as an integral of $f(y)$, and

$$P\left(Y \leq \frac{u+1}{3}\right) = \int_{-\infty}^{(u+1)/3} f(y)dy = \int_0^{(u+1)/3} 2y \, dy = \left(\frac{u+1}{3}\right)^2.$$

(Notice that, as Y ranges from 0 to 1, U ranges from -1 to 2.) Thus, the distribution function of the random variable U is given by

$$F_U(u) = \begin{cases} 0, & u < -1, \\ \left(\frac{u+1}{3}\right)^2, & -1 \leq u \leq 2, \\ 1, & u > 2, \end{cases}$$

and the density function for U is

$$f_U(u) = \frac{dF_U(u)}{du} = \begin{cases} (2/9)(u+1), & -1 \leq u < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

■

Example 2: The Distribution Function Method

A bivariate example:

EXAMPLE

Consider the random variables Y_1 (the proportional amount of gasoline stocked at the beginning of a week) and Y_2 (the proportional amount of gasoline sold during the week). The joint density function of Y_1 and Y_2 is given by

$$f(y_1, y_2) = \begin{cases} 3y_1, & 0 \leq y_2 \leq y_1 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the probability density function for $U = Y_1 - Y_2$, the proportional amount of gasoline remaining at the end of the week. Use the density function of U to find $E(U)$.

FIGURE 6.1
Region over which
 $f(y_1, y_2)$ is positive

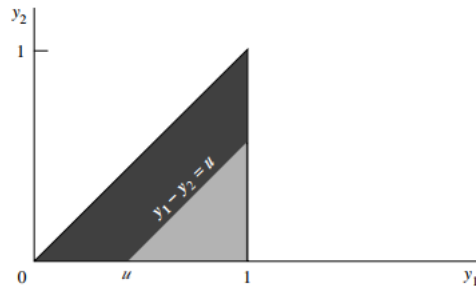
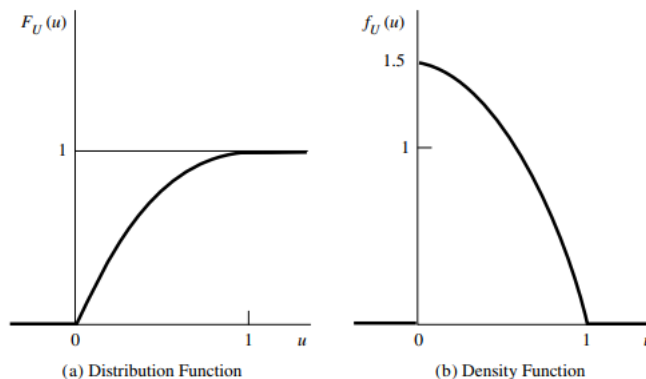


FIGURE 6.2
Distribution and
density functions



Solution The region over which $f(y_1, y_2)$ is not zero is sketched in Figure 6.1. Also shown there is the line $y_1 - y_2 = u$, for a value of u between 0 and 1. Notice that any point (y_1, y_2) such that $y_1 - y_2 \leq u$ lies above the line $y_1 - y_2 = u$.

If $u < 0$, the line $y_1 - y_2 = u$ has intercept $-u < 0$ and $F_U(u) = P(Y_1 - Y_2 \leq u) = 0$. When $u > 1$, the line $y_1 - y_2 = u$ has intercept $-u < -1$ and $F_U(u) = 1$. For $0 \leq u \leq 1$, $F_U(u) = P(Y_1 - Y_2 \leq u)$ is the integral over the dark shaded region above the line $y_1 - y_2 = u$. Because it is easier to integrate over the lower triangular region, we can write, for $0 \leq u \leq 1$,

$$\begin{aligned} F_U(u) &= P(U \leq u) = 1 - P(U \geq u) \\ &= 1 - \int_u^1 \int_0^{y_1-u} 3y_1 dy_2 dy_1 \\ &= 1 - \int_u^1 3y_1(y_1 - u) dy_1 \\ &= 1 - 3 \left[\frac{y_1^3}{3} - \frac{u y_1^2}{2} \right]_u^1 \\ &= 1 - \left[1 - \frac{3}{2}(u) + \frac{u^3}{2} \right] \\ &= \frac{1}{2}(3u - u^3). \end{aligned}$$

Summarizing,

$$F_U(u) = \begin{cases} 0, & u < 0, \\ (3u - u^3)/2, & 0 \leq u \leq 1, \\ 1, & u > 1. \end{cases}$$

A graph of $F_U(u)$ is given in Figure 6.2(a).

It follows that

$$f_U(u) = \frac{dF_U(u)}{du} = \begin{cases} 3(1 - u^2)/2, & 0 \leq u \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

The density function $f_U(u)$ is graphed in Figure 6.2(b).

We can use this derived density function to find $E(U)$

$$E(U) = \int_0^1 u \left(\frac{3}{2} \right) (1 - u^2) du = \frac{3}{2} \left[\frac{u^2}{2} - \frac{u^4}{4} \right]_0^1 = \frac{3}{8}$$

Example 3: The Distribution Function Method

A bivariate example:

EXAMPLE Let (Y_1, Y_2) denote a random sample of size $n = 2$ from the uniform distribution on the interval $(0, 1)$. Find the probability density function for $U = Y_1 + Y_2$.

Solution The density function for each Y_i is

$$f(y) = \begin{cases} 1, & 0 \leq y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

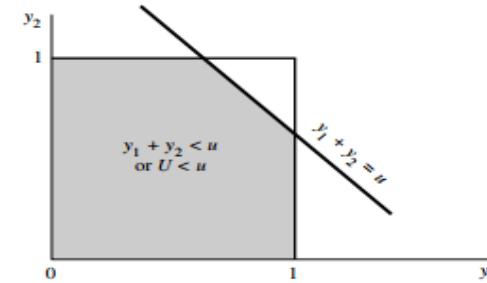
Therefore, because we have a random sample, Y_1 and Y_2 are independent, and

$$f(y_1, y_2) = f(y_1)f(y_2) = \begin{cases} 1, & 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

The random variables Y_1 and Y_2 have nonzero density over the unit square, as shown in Figure 6.3. We wish to find $F_U(u) = P(U \leq u)$. The first step is to find the points (y_1, y_2) that imply $y_1 + y_2 \leq u$. The easiest way to find this region is to locate the points that divide the regions $U \leq u$ and $U > u$. These points lie on the line $y_1 + y_2 = u$.

Graphing this relationship in Figure 6.3 and arbitrarily selecting y_2 as the dependent variable, we find that the line possesses a slope equal to -1 and a y_2 intercept equal to u . The points associated with $U < u$ are either above or below the line and can be determined by testing points on either side of the line. Suppose that $u = 1.5$.

FIGURE 6.3
The region of integration for Example



Let $y_1 = y_2 = 1/4$; then $y_1 + y_2 = 1/4 + 1/4 = 1/2$ and (y_1, y_2) satisfies the inequality $y_1 + y_2 < u$. Therefore, $y_1 = y_2 = 1/4$ falls in the shaded region below the line. Similarly, all points such that $y_1 + y_2 < u$ lie below the line $y_1 + y_2 = u$. Thus,

$$F_U(u) = P(U \leq u) = P(Y_1 + Y_2 \leq u) = \iint_{y_1 + y_2 \leq u} f(y_1, y_2) dy_1 dy_2.$$

If $u < 0$,

$$F_U(u) = P(U \leq u) = \iint_{y_1 + y_2 \leq u} f(y_1, y_2) dy_1 dy_2 = \iint_{y_1 + y_2 \leq u} 0 dy_1 dy_2 = 0$$

and for $u > 2$,

$$F_U(u) = P(U \leq u) = \iint_{y_1 + y_2 \leq u} f(y_1, y_2) dy_1 dy_2 = \int_0^1 \int_0^1 (1) dy_1 dy_2 = 1.$$

For $0 \leq u \leq 2$, the limits of integration depend upon the particular value of u (where u is the y_2 intercept of the line $y_1 + y_2 = u$). Thus, the mathematical expression for $F_U(u)$ changes depending on whether $0 \leq u \leq 1$ or $1 < u \leq 2$.

If $0 \leq u \leq 1$, the region $y_1 + y_2 \leq u$ is the shaded area in Figure 6.4. Then for $0 \leq u \leq 1$, we have

$$\begin{aligned} F_U(u) &= \iint_{y_1 + y_2 \leq u} f(y_1, y_2) dy_1 dy_2 = \int_0^u \int_0^{u-y_2} (1) dy_1 dy_2 = \int_0^u (u - y_2) dy_2 \\ &= \left(uy_2 - \frac{y_2^2}{2} \right) \Big|_0^u = u^2 - \frac{u^2}{2} = \frac{u^2}{2}. \end{aligned}$$

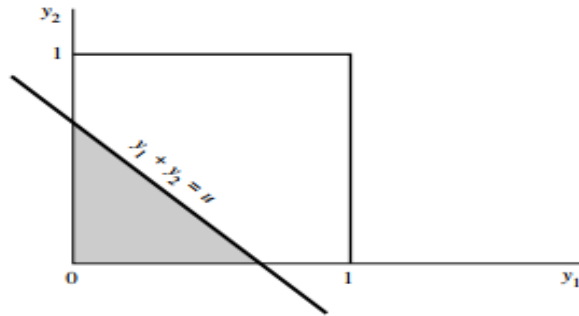
The solution, $F_U(u)$, $0 \leq u \leq 1$, could have been acquired directly by using elementary geometry. The bivariate density $f(y_1, y_2) = 1$ is uniform over the unit square, $0 \leq y_1 \leq 1$, $0 \leq y_2 \leq 1$. Hence, $F_U(u)$ is the volume of a solid with height equal to $f(y_1, y_2) = 1$ and a triangular cross section, as shown in Figure 6.4. Hence,

$$F_U(u) = (\text{area of triangle}) \cdot (\text{height}) = \frac{u^2}{2}(1) = \frac{u^2}{2}.$$

The distribution function can be acquired in a similar manner when u is defined over the interval $1 < u \leq 2$. Although the geometric solution is easier, we will obtain

Example 3 (cont'd): The Distribution Function Method

FIGURE 6.4
The region
 $y_1 + y_2 \leq u$ for
 $0 \leq u \leq 1$



$F_U(u)$ directly by integration. The region $y_1 + y_2 \leq u$, $1 \leq u \leq 2$ is the shaded area indicated in Figure 6.5.

The complement of the event $U \leq u$ is the event that (Y_1, Y_2) falls in the region A of Figure 6.5. Then for $1 < u \leq 2$,

$$\begin{aligned} F_U(u) &= 1 - \int_A f(y_1, y_2) dy_1 dy_2 \\ &= 1 - \int_{u-1}^1 \int_{u-y_2}^1 (1) dy_1 dy_2 = 1 - \int_{u-1}^1 \left(y_1 \right)_{u-y_2}^1 dy_2 \\ &= 1 - \int_{u-1}^1 (1 - u + y_2) dy_2 = 1 - \left[(1 - u)y_2 + \frac{y_2^2}{2} \right]_{u-1}^1 \\ &= (-u^2/2) + 2u - 1. \end{aligned}$$

To summarize,

$$F_U(u) = \begin{cases} 0, & u < 0, \\ u^2/2, & 0 \leq u \leq 1, \\ (-u^2/2) + 2u - 1, & 1 < u \leq 2, \\ 1, & u > 2. \end{cases}$$

The distribution function for U is shown in Figure 6.6(a).

FIGURE 6.5
The region
 $y_1 + y_2 \leq u$,
 $1 < u \leq 2$

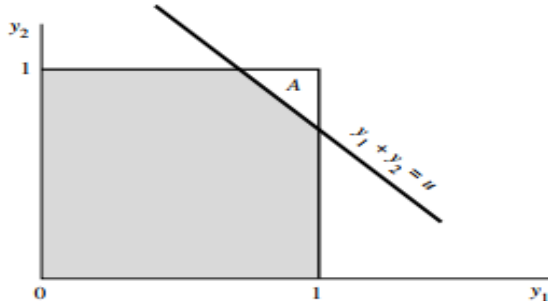
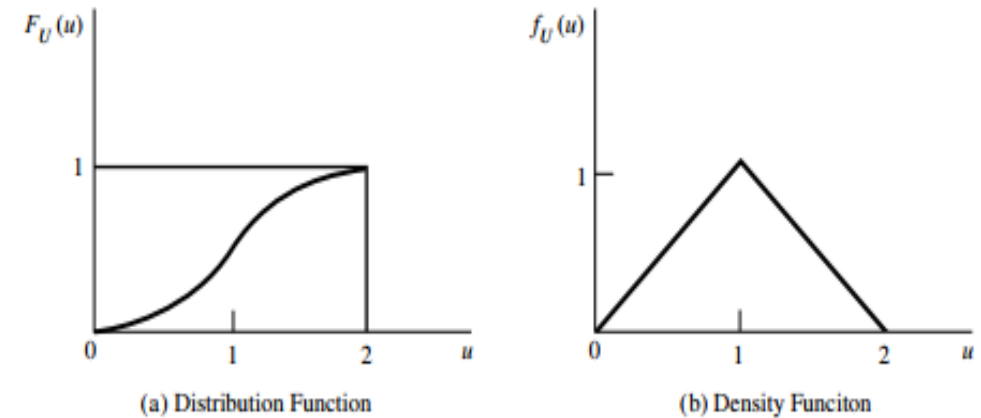


FIGURE 6.6
Distribution and
density functions
for Example



The density function $f_U(u)$ can be obtained by differentiating $F_U(u)$. Thus,

$$f_U(u) = \frac{dF_U(u)}{du} \begin{cases} \frac{d}{du}(0) = 0, & u < 0, \\ \frac{d}{du}(u^2/2) = u, & 0 \leq u \leq 1, \\ \frac{d}{du}[(-u^2/2) + 2u - 1] = 2 - u, & 1 < u \leq 2, \\ \frac{d}{du}(1) = 0, & u > 2, \end{cases}$$

or, more simply,

$$f_U(u) = \begin{cases} u, & 0 \leq u \leq 1, \\ 2 - u, & 1 < u \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

A graph of $f_U(u)$ is shown in Figure 6.6(b). ■

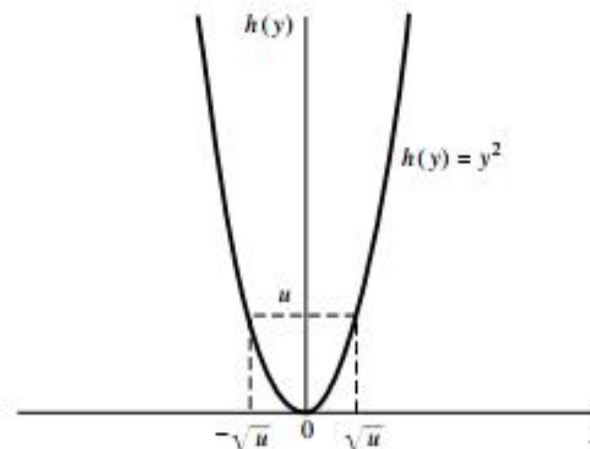
Example 4: The Distribution Function Method

A univariate example:

we will consider the case $U = h(Y) = Y^2$, where Y is a continuous random variable with distribution function $F_Y(y)$ and density function $f_Y(y)$. If $u \leq 0$, $F_U(u) = P(U \leq u) = P(Y^2 \leq u) = 0$ and for $u > 0$ (see Figure 6.7),

$$\begin{aligned} F_U(u) &= P(U \leq u) = P(Y^2 \leq u) \\ &= P(-\sqrt{u} \leq Y \leq \sqrt{u}) \\ &= \int_{-\sqrt{u}}^{\sqrt{u}} f(y) dy = F_Y(\sqrt{u}) - F_Y(-\sqrt{u}). \end{aligned}$$

FIGURE 6.7
The function
 $h(y) = y^2$



In general,

$$F_U(u) = \begin{cases} F_Y(\sqrt{u}) - F_Y(-\sqrt{u}), & u > 0, \\ 0, & \text{otherwise.} \end{cases}$$

On differentiating with respect to u , we see that

$$f_U(u) = \begin{cases} f_Y(\sqrt{u}) \left(\frac{1}{2\sqrt{u}} \right) + f_Y(-\sqrt{u}) \left(\frac{1}{2\sqrt{u}} \right), & u > 0, \\ 0, & \text{otherwise,} \end{cases}$$

or, more simply,

$$f_U(u) = \begin{cases} \frac{1}{2\sqrt{u}} [f_Y(\sqrt{u}) + f_Y(-\sqrt{u})], & u > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Examples 5 and 6: The Transformation Method

The transformation method can also be used in multivariate situations. The following example illustrates the bivariate case.

EXAMPLE 5 Let Y have the probability density function given by

$$f_Y(y) = \begin{cases} 2y, & 0 \leq y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the density function of $U = -4Y + 3$.

Solution In this example, the set of values of y such that $f_Y(y) > 0$ are the values $0 < y \leq 1$. The function of interest, $h(y) = -4y + 3$, is decreasing for all y , and hence for all $0 < y \leq 1$, if $u = -4y + 3$, then

$$y = h^{-1}(u) = \frac{3-u}{4} \quad \text{and} \quad \frac{dh^{-1}}{du} = -\frac{1}{4}.$$

Notice that $h^{-1}(u)$ is a decreasing function of u and that $dh^{-1}/du < 0$. Thus,

$$f_U(u) = f_Y[h^{-1}(u)] \left| \frac{dh^{-1}}{du} \right| = \begin{cases} 2 \left(\frac{3-u}{4} \right) \left| -\frac{1}{4} \right|, & 0 \leq \frac{3-u}{4} \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Finally, some simple algebra gives

$$f_U(u) = \begin{cases} \frac{3-u}{8}, & -1 \leq u \leq 3, \\ 0, & \text{elsewhere.} \end{cases} \quad \blacksquare$$

EXAMPLE 6

Let Y_1 and Y_2 have a joint density function given by

$$f(y_1, y_2) = \begin{cases} e^{-(y_1+y_2)}, & 0 \leq y_1, 0 \leq y_2, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the density function for $U = Y_1 + Y_2$.

Solution This problem must be solved in two stages: First, we will find the joint density of Y_1 and U ; second, we will find the marginal density of U . The approach is to let Y_1 be fixed at a value $y_1 \geq 0$. Then $U = y_1 + Y_2$, and we can consider the one-dimensional transformation problem in which $U = h(Y_2) = y_1 + Y_2$. Letting $g(y_1, u)$ denote the joint density of Y_1 and U , we have, with $y_2 = u - y_1 = h^{-1}(u)$,

$$g(y_1, u) = \begin{cases} f[y_1, h^{-1}(u)] \left| \frac{dh^{-1}}{du} \right| = e^{-(y_1+u-y_1)}(1), & 0 \leq y_1, 0 \leq u - y_1, \\ 0, & \text{elsewhere.} \end{cases}$$

Simplifying, we obtain

$$g(y_1, u) = \begin{cases} e^{-u}, & 0 \leq y_1 \leq u, \\ 0, & \text{elsewhere.} \end{cases}$$

(Notice that $Y_1 \leq U$.) The marginal density of U is then given by

$$\begin{aligned} f_U(u) &= \int_{-\infty}^{\infty} g(y_1, u) dy_1 \\ &= \begin{cases} \int_0^u e^{-u} dy_1 = ue^{-u}, & 0 \leq u, \\ 0, & \text{elsewhere.} \end{cases} \quad \blacksquare \end{aligned}$$

Example 7: The Moment-Generating Function Method

EXAMPLE

The number of customer arrivals at a checkout counter in a given interval of time possesses approximately a Poisson probability distribution (see Section 3.8). If Y_1 denotes the time until the first arrival, Y_2 denotes the time between the first and second arrival, $\dots$, and Y_n denotes the time between the $(n - 1)$ st and n th arrival, then it can be shown that $Y_1, Y_2, \dots, Y_n$ are independent random variables, with the density function for Y_i given by

$$f_{Y_i}(y_i) = \begin{cases} \frac{1}{\theta} e^{-y_i/\theta}, & y_i > 0, \\ 0, & \text{otherwise.} \end{cases}$$

[Because the Y_i , for $i = 1, 2, \dots, n$, are exponentially distributed, it follows that $E(Y_i) = \theta$; that is, θ is the average time between arrivals.] Find the probability density function for the waiting time from the opening of the counter until the n th customer arrives. (If $Y_1, Y_2, \dots$ denote successive interarrival times, we want the density function of $U = Y_1 + Y_2 + \dots + Y_n$.)

Solution To use Theorem 6.2, we must first know $m_{Y_i}(t)$, $i = 1, 2, \dots, n$. Because each of the Y_i 's is exponentially distributed with mean θ , $m_{Y_i}(t) = (1 - \theta t)^{-1}$ and, by Theorem 6.2,

$$\begin{aligned} m_U(t) &= m_{Y_1}(t) \times m_{Y_2}(t) \times \dots \times m_{Y_n}(t) \\ &= (1 - \theta t)^{-1} \times (1 - \theta t)^{-1} \times \dots \times (1 - \theta t)^{-1} = (1 - \theta t)^{-n}. \end{aligned}$$

This is the moment-generating function of a gamma-distributed random variable with $\alpha = n$ and $\beta = \theta$. Theorem 6.1 implies that U actually has this gamma distribution and therefore that

$$f_U(u) = \begin{cases} \frac{1}{\Gamma(n)\theta^n} (u^{n-1} e^{-u/\theta}), & u > 0, \\ 0, & \text{elsewhere.} \end{cases} \quad \blacksquare$$

Recall Theorem 5.9 from earlier:

THEOREM 5.9

Let Y_1 and Y_2 be independent random variables and $g(Y_1)$ and $h(Y_2)$ be functions of only Y_1 and Y_2 , respectively. Then

$$E[g(Y_1)h(Y_2)] = E[g(Y_1)]E[h(Y_2)],$$

provided that the expectations exist.

Using Theorem 5.9 we can prove the following theorem and use it in this example:

THEOREM 6.2

Let $Y_1, Y_2, \dots, Y_n$ be independent random variables with moment-generating functions $m_{Y_1}(t), m_{Y_2}(t), \dots, m_{Y_n}(t)$, respectively. If $U = Y_1 + Y_2 + \dots + Y_n$, then

$$m_U(t) = m_{Y_1}(t) \times m_{Y_2}(t) \times \dots \times m_{Y_n}(t).$$

Proof

We know that, because the random variables $Y_1, Y_2, \dots, Y_n$ are independent (see Theorem 5.9),

$$\begin{aligned} m_U(t) &= E[e^{t(Y_1 + \dots + Y_n)}] = E(e^{tY_1} e^{tY_2} \dots e^{tY_n}) \\ &= E(e^{tY_1}) \times E(e^{tY_2}) \times \dots \times E(e^{tY_n}). \end{aligned}$$

Thus, by the definition of moment-generating functions,

$$m_U(t) = m_{Y_1}(t) \times m_{Y_2}(t) \times \dots \times m_{Y_n}(t).$$

Example 8: The Bivariate Transformation Method

Consider as an example a system having a component that can be replaced just once before the system itself expires. Let X_1 denote the lifetime of the original component and X_2 the lifetime of the replacement component.

Continuing with the component lifetime situation, suppose that X_1 and X_2 are independent, each having an exponential distribution with parameter λ . Let's determine the joint pdf of

$$Y_1 = u_1(X_1, X_2) = X_1 + X_2 \text{ and } Y_2 = u_2(X_1, X_2) = \frac{X_1}{X_1 + X_2}$$

We have already inverted this transformation:

$$x_1 = v_1(y_1, y_2) = y_1 y_2 \quad x_2 = v_2(y_1, y_2) = y_1(1 - y_2)$$

The image of the transformation, i.e. the set of (y_1, y_2) pairs with positive density, is $0 < y_1$ and $0 < y_2 < 1$. The four relevant partial derivatives are

$$\frac{\partial v_1}{\partial y_1} = y_2 \quad \frac{\partial v_1}{\partial y_2} = y_1 \quad \frac{\partial v_2}{\partial y_1} = 1 - y_2 \quad \frac{\partial v_2}{\partial y_2} = -y_1$$

from which the Jacobian is $-y_1 y_2 - y_1(1 - y_2) = -y_1$

Since the joint pdf of X_1 and X_2 is

$$f(x_1, x_2) = \lambda e^{-\lambda x_1} \cdot \lambda e^{-\lambda x_2} = \lambda^2 e^{-\lambda(x_1 + x_2)} \quad x_1 > 0, x_2 > 0$$

we have

$$g(y_1, y_2) = \lambda^2 e^{-\lambda y_1} \cdot y_1 = \lambda^2 y_1 e^{-\lambda y_1} \cdot 1 \quad 0 < y_1, 0 < y_2 < 1$$

The joint pdf thus factors into two parts. The first part is a gamma pdf with parameters $\alpha = 2$ and $\beta = 1/\lambda$, and the second part is a uniform pdf on $(0, 1)$.

Since the pdf factors and the region of positive density is rectangular, we have demonstrated that

1. The distribution of system lifetime $X_1 + X_2$ is $\text{gamma}(\alpha = 2, \beta = 1/\lambda)$
2. The distribution of the proportion of system lifetime during which the original component functions is uniform on $(0, 1)$
3. $Y_1 = X_1 + X_2$ and $Y_2 = X_1/(X_1 + X_2)$ are independent of each other ■

Recall:

A continuous random variable X is said to have a **gamma distribution** if the pdf of X is

$$f(x; \alpha, \beta) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

where the parameters α and β satisfy $\alpha > 0, \beta > 0$. The **standard gamma distribution** has $\beta = 1$.

Example 9: The Bivariate Transformation Method

Consider a rectangular coordinate system with a horizontal x_1 axis and a vertical x_2 axis as shown in Figure 5.7(a). First a point (X_1, X_2) is randomly selected, where the joint pdf of X_1, X_2 is

$$f(x_1, x_2) = \begin{cases} x_1 + x_2 & 0 < x_1 < 1, \quad 0 < x_2 < 1 \\ 0 & \text{otherwise} \end{cases}$$

Then a rectangle with vertices $(0, 0)$, $(X_1, 0)$, $(0, X_2)$, and (X_1, X_2) is formed. What is the distribution of X_1X_2 , the area of this rectangle? To answer this question, let

$$Y_1 = X_1X_2 \quad Y_2 = X_2$$

so

$$y_1 = u_1(x_1, x_2) = x_1x_2 \quad y_2 = u_2(x_1, x_2) = x_2$$

Then

$$x_1 = v_1(y_1, y_2) = \frac{y_1}{y_2} \quad x_2 = v_2(y_1, y_2) = y_2$$

Notice that because $x_2 (= y_2)$ is between 0 and 1 and y_1 is the product of the two x_i 's, it must be the case that $0 < y_1 < y_2$. The region of positive density for the new variables is then

$$T = \{(y_1, y_2) : 0 < y_1 < y_2, 0 < y_2 < 1\}$$

which is the triangular region shown in Figure 5.7(b).

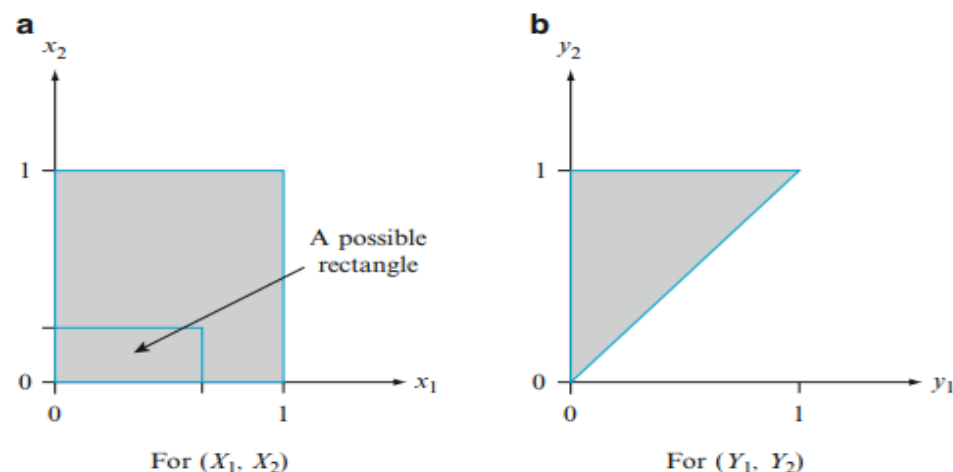


Figure 5.7 Regions of positive density for Example 5.26

Since $\partial v_2 / \partial y_1 = 0$, the product of the two off-diagonal elements in the matrix M will be 0, so only the two diagonal elements contribute to the Jacobian:

$$M = \begin{pmatrix} \frac{1}{y_2} & ? \\ 0 & 1 \end{pmatrix} \quad |\det(M)| = \frac{1}{y_2}$$

The joint pdf of the two new variables is now

$$g(y_1, y_2) = f\left(\frac{y_1}{y_2}, y_2\right) \cdot |\det(M)| = \begin{cases} \left(\frac{y_1}{y_2} + y_2\right) \cdot \frac{1}{y_2} & 0 < y_1 < y_2, 0 < y_2 < 1 \\ 0 & \text{otherwise} \end{cases}$$

To obtain the marginal pdf of Y_1 alone, we must now fix y_1 at some arbitrary value between 0 and 1, and integrate out y_2 . Figure 5.7b shows that we must integrate along the vertical line segment passing through y_1 whose lower limit is y_1 and whose upper limit is 1:

$$g_1(y_1) = \int_{y_1}^1 \left(\frac{y_1}{y_2} + y_2\right) \cdot \frac{1}{y_2} dy_2 = 2(1 - y_1) \quad 0 < y_1 < 1$$

This marginal pdf can now be integrated to obtain any desired probability involving the area. For example, integrating from 0 to .5 gives $P(\text{area} < .5) = .75$. ■

References

- Devore, J.L. and Berk, K.N. (2012) Modern Mathematical Statistics with Applications, 2nd Edition, New York: Springer.
- Wackerly, D.D., Mendenhall, W. and Scheaffer, R.L., (2008) Mathematical Statistics With Applications, 7th Edition, Pacific Grove: Duxbury.