

Unconstrained Optimization (One Variable)

12.0

$$z = f(x)$$

FOC: $dz = f'(x) dx = 0$ for an arbitrary nonzero dx .

$$\Rightarrow f'(x) = 0$$

SOC: $d^2(z) \equiv d(dz) = d[f'(x)dx]$
 $= [df'(x)]dx$ (dx is constant)

$$= [f''(x)dx]dx$$

For maximum of z $= f''(x) dx^2 < 0$ for an arbitrary nonzero dx .

$$\Rightarrow f''(x) < 0.$$

Unconstrained Optimization (Two Variables)

12.1.1

At a peak the surface should be flat so that very small move in any direction should leave us at the same altitude

$$df = \underbrace{\frac{\partial f}{\partial x_1}}_{=0} dx_1 + \underbrace{\frac{\partial f}{\partial x_2}}_{=0} dx_2 = 0 \quad \text{for arbitrary } dx_i$$

FOC

SOC: If we move away from the peak the surface should slope downward in every direction

* We also need to take the cross effects into account. Even if $f_{11} < 0$ an increase in x_1 may raise the marginal effects of other variables on f , e.g. $f_{21} > 0$.

$$\frac{d(df)}{dx} = \frac{\partial(df)}{\partial x_1} dx_1 + \frac{\partial(df)}{\partial x_2} dx_2 < 0$$

$$\begin{bmatrix} dx_1 & dx_2 \end{bmatrix} \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} < 0$$

" $x'Ax$

$$\frac{\partial}{\partial x_1} (f_{11} dx_1 + f_{12} dx_2) dx_1 + \frac{\partial}{\partial x_2} (f_{11} dx_1 + f_{12} dx_2) dx_2$$

$$(f_{111} dx_1 + f_{112} dx_2) dx_1 + (f_{211} dx_1 + f_{212} dx_2) dx_2$$

$$(f_{111} dx_1^2 + f_{112} dx_2 dx_1 + f_{211} dx_1 dx_2 + f_{212} dx_2^2)$$

$$(f_{111} dx_1^2 + f_{112} dx_2 dx_1 + f_{211} dx_1 dx_2 + f_{212} dx_2^2)$$

$$= f_{11} dx_1^2 + 2f_{12} dx_1 dx_2 + f_{22} dx_2^2 = \begin{bmatrix} dx_1 & dx_2 \end{bmatrix} \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix}$$

Let $u = dx_1$
 $v = dx_2$
 $a = f_{11}$
 $b = f_{22}$
 $h = f_{12}$

$$q = d^2f = au^2 + 2huv + bv^2$$

$$q \pm \frac{h^2 v^2}{a}$$

$$q = a \left(u^2 + \frac{2huv}{a} + \frac{h^2 v^2}{a^2} \right) + \left(b - \frac{h^2}{a} \right) v^2$$

$$q = a \left(u + \frac{h}{a} v \right)^2 + \frac{ab - h^2}{a} v^2$$

convex

q is positive definite iff $a > 0$ and $ab - h^2 > 0$

is negative definite iff $a < 0$ and $ab - h^2 > 0$

concave

f_{11}

$$\begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix}$$

leading principle minors.

$$\max_{q_1, q_2} U(q_1, q_2)$$

q_1, q_2

$$\text{s.t. } p_1 q_1 + p_2 q_2 = M$$

$$\max_{q_2} U\left(\frac{M - p_2 q_2}{p_1}, q_2\right)$$

$$\frac{dU}{dq_2} = \frac{\partial U}{\partial q_1} \frac{dq_1}{dq_2} + \frac{\partial U}{\partial q_2} = \left(-\frac{p_2}{p_1}\right) \frac{\partial U}{\partial q_1} + \frac{\partial U}{\partial q_2}$$

$$\downarrow$$
$$MRS = -\frac{MU_1}{MU_2} = -\frac{p_1}{p_2} = MRT \quad \rightarrow \text{marginal rate of transformation}$$

Lagrangian Method:

$$\max_{q_1, q_2, \lambda} \mathcal{L} = U(q_1, q_2) + \lambda (M - p_1 q_1 - p_2 q_2)$$

Necessary first order conditions:

$$\frac{\partial \mathcal{L}}{\partial q_1} = \frac{\partial U}{\partial q_1} - \lambda p_1 = MU_1 - \lambda p_1 = 0$$

$$\frac{\partial \mathcal{L}}{\partial q_2} = \frac{\partial U}{\partial q_2} - \lambda p_2 = MU_2 - \lambda p_2 = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = M - p_1 q_1 - p_2 q_2 = 0$$

H is negative definite if $x' H x < 0 \quad \forall x \neq 0$

We should also check that we have a maximum.

for strictly concave U .

$$H(x) = \begin{pmatrix} U''_{11} & U''_{12} \\ U''_{21} & U''_{22} \end{pmatrix}$$

$$U''_{11} < 0 \text{ and } U''_{22} < 0$$

preferences are convex
 $\downarrow$
 U is strictly concave and strictly increasing.

Sufficient condition for local maximizer
Hessian is a negative definite $\Rightarrow U''_{11} < 0$ and $|H(x)| > 0$ ($U''_{22} < 0$)

If U is concave then x is GM
 $\Rightarrow x$ is a stationary point of U

A the optimal solution

$$\frac{MU_1}{p_1} = \frac{MU_2}{p_2} = \lambda \text{ : shadow value}$$

extra pleasure one gets from the last dollar of expenditure on either good.

N-Variable Case

$$z = f(x_1, x_2, \dots, x_n)$$

$$dz = f_1 dx_1 + f_2 dx_2 + \dots + f_n dx_n$$

$$|H| = \begin{vmatrix} f_{11} & f_{12} & \dots & f_{1n} \\ f_{21} & f_{22} & \dots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \dots & f_{nn} \end{vmatrix}$$

Maximum

$$\text{FOCs: } f_1 = f_2 = \dots = f_n = 0$$

$$\text{SOC: } |H_1| < 0; |H_2| > 0;$$

$$|H_3| < 0; \dots;$$

$$(-1)^n |H_n| > 0$$

Minimum

$$f_1 = f_2 = \dots = f_n = 0$$

$$|H_1|, |H_2|, \dots, |H_n| > 0$$

Ex: $z = 2x_1^2 + x_1x_2 + 4x_2^2 + x_1x_3 + x_3^2 + 2$

$$\left. \begin{aligned} f_1 &\rightarrow 4x_1 + x_2 + x_3 = 0 \\ x_1 + 8x_2 &= 0 \\ x_1 + 2x_3 &= 0 \end{aligned} \right\} x_1^* = x_2^* = x_3^* = 0$$

$$|H| = \begin{vmatrix} 4 & 1 & 1 \\ -1 & 8 & 0 \\ -1 & 0 & 2 \end{vmatrix}$$

$$|H_1| = 4 \quad |H_2| = 31 \quad |H_3| = 54 \quad \Rightarrow z^* = 2 \text{ is a minimum.}$$

Ex: $z = -x_1^3 + 3x_1x_3 + 2x_2 - x_2^2 - 3x_3^2$

$$f_1 = -3x_1^2 + 3x_3 \quad f_2 = 2 - 2x_2 \quad f_3 = 3x_1 - 6x_3$$

$$\text{FOCs} \Rightarrow x_2^* = 1 \quad x_1^* = 2x_3^*$$

$$(x_1^*, x_2^*, x_3^*) = \begin{cases} (0, 1, 0), \text{ implying } z^* = 1 \\ (\frac{1}{2}, 1, \frac{1}{4}), \text{ implying } z^* = \frac{17}{16} \end{cases}$$

$$|H| = \begin{vmatrix} -bx_1 & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & -6 \end{vmatrix}$$

for $(\frac{1}{2}, 1, \frac{1}{4})$, $|H_1| = -3$, $|H_2| = 6$ and $|H_3| = -18$

$\Rightarrow z^* = \frac{17}{16}$ is a maximum

$(0, 1, 0) \Rightarrow |H_1| = 0$ does not satisfy SOC's.

Characteristic-Root Test for Sign Definiteness

$$D_{n \times n} X_{n \times 1} = r_{n \times 1} X_{n \times 1}$$

$$(D - rI)X = 0_{n \times 1}$$

for nonsingular solution $|D - rI| = 0$

$q = U'DU$ is positive (negative) definite iff every r of D is positive (negative)

Optimization with equality constraint

$$\max_{q_1, q_2} U(q_1, q_2) \quad \text{s.t.} \quad p_1 q_1 + p_2 q_2 = M$$

$$\max_{q_2} U\left(\frac{M - p_2 q_2}{p_1}, q_2\right)$$

$$\frac{dU}{dq_2} = \frac{\partial U}{\partial q_1} \frac{dq_1}{dq_2} + \frac{\partial U}{\partial q_2} = \left(-\frac{p_2}{p_1}\right) \frac{\partial U}{\partial q_1} + \frac{\partial U}{\partial q_2}$$

$$\Rightarrow MRS = -\frac{MU_1}{MU_2} = -\frac{p_1}{p_2}$$

Lagrangian Method

12.4

(1) $\max U(q_1, q_2)$ s.t. $p_1 q_1 + p_2 q_2 = M$

(2) $\max_{q_1, q_2, \lambda} \mathcal{L} = U(q_1, q_2) + \lambda (M - p_1 q_1 - p_2 q_2)$

necessary FOCs

$$\frac{\partial \mathcal{L}}{\partial q_1} = \frac{\partial U}{\partial q_1} - \lambda p_1 = MU_1 - \lambda p_1 = 0$$

$$\frac{\partial \mathcal{L}}{\partial q_2} = \frac{\partial U}{\partial q_2} - \lambda p_2 = MU_2 - \lambda p_2 = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = M - p_1 q_1 - p_2 q_2 = 0$$

* Stationary values in (2) must be identical with those of (1).

SOCs. $\max_{x,y} f(x,y)$ s.t. $g(x,y) = C$

dx constant
 dy is a function
of x and y

$$dg = g_x dx + g_y dy = 0$$

$$d^2 y \neq (dy)^2$$

$$d^2 z = d(dz) = \frac{\partial (dz)}{\partial x} dx + \frac{\partial (dz)}{\partial y} dy$$

$$= \frac{\partial}{\partial x} (f_x dx + f_y dy) dx + \frac{\partial}{\partial y} (f_x dx + f_y dy) dy$$

$$= \left(f_{xx} dx + \left(f_{xy} dy + f_y \frac{\partial (dy)}{\partial x} \right) \right) dx +$$

$$\left(f_{yx} dx + \left(f_{yy} dy + f_y \frac{\partial (dy)}{\partial y} \right) \right) dy$$

$$\textcircled{1} + \textcircled{6} = f_y d(dy) = f_y d^2 y$$

$$= f_{xx} dx^2 + 2f_{xy} dx dy + f_{yy} dy^2 + f_y \frac{\partial^2 y}{\partial x^2}$$

$$d^2 g = g_{xx} dx^2 + 2g_{xy} dx dy + g_{yy} dy^2 + g_y d^2 y = 0 \quad \uparrow$$

$$d^2 z = \left(f_{xx} - \frac{f_y}{g_y} g_{xx} \right) dx^2 + 2 \left(f_{xy} - \frac{f_y}{g_y} g_{xy} \right) dx dy + \left(f_{yy} - \frac{f_y}{g_y} g_{yy} \right) dy^2$$

$\frac{f_y}{g_y} = \lambda$ from FOCs

$$L_{xx} = f_{xx} - \lambda g_{xx}$$

$$L_{xy} = f_{xy} - \lambda g_{xy} = z_{yx}$$

$$L_{yy} = f_{yy} - \lambda g_{yy}$$

$$\Rightarrow d^2 z = L_{xx} dx^2 + L_{xy} dx dy + L_{yx} dy dx + L_{yy} dy^2$$

sign check only for those dx and dy such that $g_x dx + g_y dy = 0$

The second-order sufficient conditions are

For maximum of z : $d^2 z$ negative definite, s.t. $dg=0$

For minimum of z : $d^2 z$ positive definite, s.t. $dg=0$

Two-variable quadratic form:

$$q = au^2 + 2huv + bv^2$$

$$\text{s.t. } \overset{g_x, dx}{\alpha} u + \overset{g_y, dy}{\beta} v = 0$$

$$\Downarrow \\ v = -\frac{\alpha}{\beta} u$$

$$\Rightarrow q = au^2 - 2h\frac{\alpha}{\beta}u^2 + b\frac{\alpha^2}{\beta^2}u^2 = \underbrace{(a\beta^2 - 2h\alpha\beta + b\alpha^2)}_{-\Omega} \frac{u^2}{\beta^2}$$

$$|\overline{H}| = \begin{vmatrix} 0 & \alpha & \beta \\ \alpha & a & h \\ \beta & h & b \end{vmatrix} = 2h\alpha\beta - a\beta^2 - b\alpha^2 = -\Omega$$

q is $\begin{cases} \text{positive definite} \\ \text{negative definite} \end{cases}$ s.t. $\alpha u + \beta v = 0$ iff $|\overline{H}| \begin{cases} < 0 \\ > 0 \end{cases}$

N-Variable Case

$$z = f(x_1, x_2, \dots, x_n) \quad \text{s.t.} \quad g(x_1, x_2, \dots, x_n) = c$$

$$L = f(l) + \lambda [c - g(l)]$$

$$|\overline{H}| = \begin{vmatrix} 0 & g_1 & g_2 & \dots & g_n \\ g_1 & L_{11} & L_{12} & \dots & L_{1n} \\ g_2 & L_{21} & L_{22} & \dots & L_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_n & L_{n1} & L_{n2} & \dots & L_{nn} \end{vmatrix}$$

bordered leading
Principal minors

$d^2 z \begin{cases} \text{pos} \\ \text{neg} \end{cases}$ s.t. $dg=0$ iff
 $\begin{cases} |\overline{H}_2|, |\overline{H}_3|, \dots, |\overline{H}_n| < 0 \\ |\overline{H}_2| > 0; |\overline{H}_3| < 0; \dots, (-1)^n |\overline{H}_n| < 0 \end{cases}$

$$\text{Max } z = f(x_1, x_2, \dots, x_n) \quad \text{s.t.} \quad g(x_1, x_2, \dots, x_n) = c$$

12.6.

$$L = f(x_1, x_2, \dots, x_n) + \lambda [c - g(x_1, x_2, \dots, x_n)]$$

Maximum

$$L_x = L_1 = L_2 = \dots = L_n = 0$$

FOCs

SOC

$$|\bar{H}_2| > 0; |\bar{H}_3| < 0; |\bar{H}_4| > 0;$$

$$(-1)^n |\bar{H}_n| > 0$$

Minimum

$$L_\lambda = L_1 = L_2 = \dots = L_n = 0$$

$$|\bar{H}_2|, |\bar{H}_3|, \dots, |\bar{H}_n| < 0$$

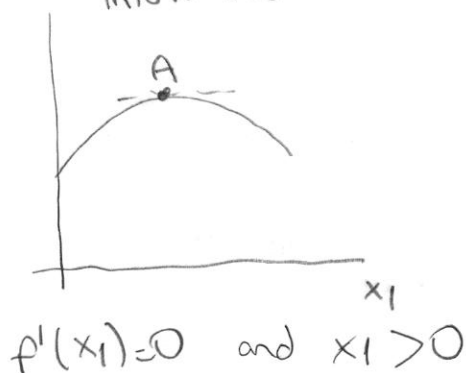
UMP

$$\begin{vmatrix} 0 & p_1 & p_2 \\ p_1 & U_{11} & U_{12} \\ p_2 & U_{12} & U_{22} \end{vmatrix} > 0$$

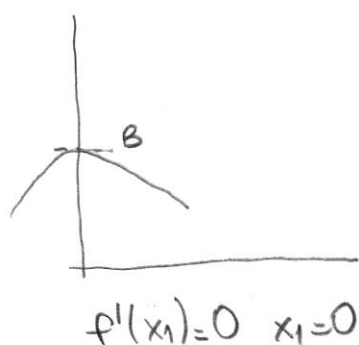
Kuhn-Tucker Conditions

$$\text{max } \pi = f(x_1) \quad \text{s.t.} \quad x_1 \geq 0$$

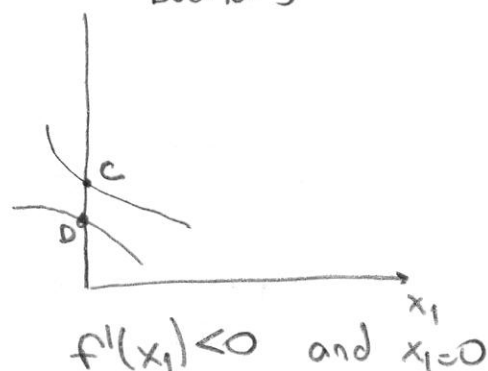
Interior solution



Boundary solution



Boundary solutions



$$f'(x_1) \leq 0 \quad x_1 \geq 0 \quad \text{and} \quad x_1 f'(x_1) = 0$$

complementary slackness condition

Kuhn-Tucker necessary conditions.

Section 4.3 The Effect of Inequality Constraints

- Now consider a problem that includes inequality constraints as well.
- Suppose we have a maximization problem with three choice variables x_1, x_2, x_3 and two constraints:

$$\max \pi = f(x_1, x_2, x_3)$$

$$s.t. \quad g_1(x_1, x_2, x_3) \leq r_1$$

$$g_2(x_1, x_2, x_3) \leq r_2$$

$$and \quad x_1, x_2, x_3 \geq 0$$

- We can convert the inequalities of the constraints by defining *slack* variables s_1, s_2 :

$$s_1 = r_1 - g_1(x_1, x_2, x_3)$$

$$s_2 = r_2 - g_2(x_1, x_2, x_3)$$

- Therefore the original problem can be transformed into the equivalent form:

$$\max \pi = f(x_1, x_2, x_3)$$

$$s.t. \quad g_1(x_1, x_2, x_3) + s_1 = r_1$$

$$g_2(x_1, x_2, x_3) + s_2 = r_2$$

$$and \quad x_1, x_2, x_3, s_1, s_2 \geq 0$$

- If the nonnegativity restrictions are absent we can form the Lagrange function Z :

$$\mathcal{L} = Z = f(x_1, x_2, x_3) + \lambda_1[r_1 - g_1(x_1, x_2, x_3) - s_1] + \lambda_2[r_2 - g_2(x_1, x_2, x_3) - s_2]$$

and write the first-order condition as:

$$\frac{\partial Z}{\partial x_1} = \frac{\partial Z}{\partial x_2} = \frac{\partial Z}{\partial x_3} = \frac{\partial Z}{\partial s_1} = \frac{\partial Z}{\partial s_2} = \frac{\partial Z}{\partial \lambda_1} = \frac{\partial Z}{\partial \lambda_2} = 0$$

- But since x and s variables do have to be nonnegative the F.O.C.s for these variables need to be modified:

$$\frac{\partial Z}{\partial x_j} \leq 0; \quad x_j \geq 0; \quad x_j \frac{\partial Z}{\partial x_j} = 0$$

$$\frac{\partial Z}{\partial s_i} \leq 0; \quad s_i \geq 0; \quad s_i \frac{\partial Z}{\partial s_i} = 0$$

$$\frac{\partial Z}{\partial \lambda_i} = 0$$

where $i = 1, 2$ and $j = 1, 2, 3$.

- Inasmuch as $\partial Z / \partial s_i = -\lambda_i$ we can eliminate the slack variables from the F.O.C.s to obtain:

$$s_i \geq 0; \quad \lambda_i \geq 0; \quad s_i \lambda_i = 0$$

- Since $s_i = r_i - g_i(x_1, x_2, x_3)$ we get:

$$r_i - g_i(x_1, x_2, x_3) \geq 0; \quad \lambda_i \geq 0; \quad \lambda_i[r_i - g_i(x_1, x_2, x_3)] = 0$$

- Therefore we can express the F.O.Cs in an equivalent form without the slack variables:

$$\frac{\partial Z}{\partial x_j} = f_j - (\lambda_1 g_1^j + \lambda_2 g_2^j) \leq 0; \quad x_j \geq 0; \quad x_j \frac{\partial Z}{\partial x_j} = 0 \quad (1)$$

$$r_i - g_i(x_1, x_2, x_3) \geq 0; \quad \lambda_i \geq 0; \quad \lambda_i [r_i - g_i(x_1, x_2, x_3)] = 0$$

where g_i^j denotes $\partial g_i / \partial x_j$.

- For a constraint to bind requires that its Lagrange Multiplier to be strictly positive. For a constraint to be non-binding, its Lagrange multiplier equals zero.
- This is one version of the Kuhn-Tucker conditions for this problem.

Section 4.4 Kuhn-Tucker Conditions

- We can obtain the same set of conditions set out in equation (1) more directly by setting up the Lagrange function without using slack variables

Definition 1 (*Kuhn-Tucker conditions*) *To derive the Kuhn-Tucker conditions for solution of the problem*

$$\begin{aligned} \max \pi &= f(x_1, x_2, x_3) \\ \text{s.t. } r_1 - g_1(x_1, x_2, x_3) &\geq 0 \\ r_2 - g_2(x_1, x_2, x_3) &\geq 0 \\ \text{and } x_1, x_2, x_3 &\geq 0 \end{aligned}$$

where all functions are concave and differentiable, we first form the Lagrange function

$$\begin{aligned} L(x_1, x_2, x_3, \lambda_1, \lambda_2) &= f(x_1, x_2, x_3) + \lambda_1[r_1 - g_1(x_1, x_2, x_3)] \\ &\quad + \lambda_2[r_2 - g_2(x_1, x_2, x_3)] \end{aligned}$$

and then maximize with respect to the variables x_1, x_2 and x_3 subject to the nonnegativity restrictions $x_1, x_2, x_3 \geq 0$

$$\frac{\partial L}{\partial x_j} = f_j - (\lambda_1 g_1^j + \lambda_2 g_2^j) \leq 0; \quad x_j \geq 0; \quad x_j \frac{\partial L}{\partial x_j} = 0$$

and minimize with respect to the variables λ_1 and λ_2 subject to the nonnegativity restrictions $\lambda_1, \lambda_2 \geq 0$

$$\frac{\partial L}{\partial \lambda_i} = r_i - g_i(x_1, x_2, x_3) \geq 0; \quad \lambda_i \geq 0; \quad \lambda_i \frac{\partial L}{\partial \lambda_i} = 0$$

Example 1 $\max f(x_1, x_2) = 4x_1 + 3x_2$ subject to $g(x_1, x_2) = 2x_1 + x_2 \leq 10$ and $x_1, x_2 \geq 0$. We first form the Lagrangian:

$$\mathcal{L} = 4x_1 + 3x_2 + \lambda(10 - 2x_1 - x_2)$$

Hence, the necessary conditions for a point to be a maximum are:

$$\begin{array}{lll} \mathcal{L}_{x_1} = 4 - 2\lambda \leq 0 & x_1 \geq 0 & x_1(4 - 2\lambda) = 0 \\ \mathcal{L}_{x_2} = 3 - \lambda \leq 0 & x_2 \geq 0 & x_2(3 - \lambda) = 0 \\ 2x_1 + x_2 - 10 \leq 0 & \lambda \geq 0 & \lambda(10 - 2x_1 - x_2) = 0 \end{array}$$

We solve this set of inequalities and equations to find points which may be maxima. Let 1 (ii) denote the second condition on the first line etc.

$$1 \text{ (iii): } x_1(4 - 2\lambda) = 0 \Rightarrow x_1 = 0 \text{ or } \lambda = 2$$

Suppose $\lambda = 2$. Then:

$$2 \text{ (i): } 3 - \lambda \leq 0 \Rightarrow 1 \leq 0$$

which is clearly false. Hence, we must have that $x_1 = 0$. Then:

$$2 \text{ (iii): } x_2(3 - \lambda) = 0 \Rightarrow x_2 = 0 \text{ or } \lambda = 3$$

We can see that, if $x_2 = 0$ (together with $x_1 = 0$):

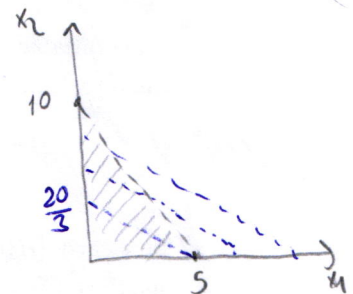
$$3 \text{ (iii): } 10\lambda = 0 \Rightarrow \lambda = 0$$

But this contradicts 1 (i). So, we have $x_1 = 0$ and $\lambda = 3$. Hence:

$$3 \text{ (iii): } 3(10 - x_2) = 0 \Rightarrow x_2 = 10$$

So, we have solved the system and found the only solution, which is $x_1 = 0, x_2 = 10, \lambda = 3$ and hence $f(x_1, x_2) = 30$. As before with equality constraints, λ measures the increase in $f(x_1, x_2)$ for a one unit increase in the constraint (i.e. increasing $c_1 = 10$ by 1).

In general, solving such systems can be very tedious (although not difficult) if there are more than 4 variables and constraints in total. What you need to do is just plug away at every possible combination and eliminate those that do not fit the conditions. When you have eliminated all impossible points, you will be left with a few candidate points which you must then check by substitution into



$f(\mathbf{x})$ to find whether they are the maximum.

For certain types of systems, we can assume the point(s) we find are maxima. This occurs if the function f and the functions g_i satisfy some more conditions, known as the Kuhn-Tucker sufficiency conditions.

Result 2 If the following conditions are satisfied:

1. $f(\mathbf{x})$ is differentiable and concave in the nonnegative orthant
2. each constraint function $g_i(\mathbf{x})$ is differentiable and convex in the nonnegative orthant
3. a point $\mathbf{x}_0$ satisfies the Kuhn-Tucker conditions

then the point $\mathbf{x}_0$ is a maximum. [The nonnegative orthant is the region where each $x_i \geq 0$]

Example 2 Suppose $f(x, y) = 2x + 3y$ and $g(x, y) = x^2 + y^2 \leq 2$. Show that f and g satisfy the Kuhn-Tucker sufficiency conditions and hence find the maxima of $f(x, y)$.

Well, all linear functions are both convex and concave, so f is certainly concave, and is clearly differentiable. As for $g(x, y)$: we can show g is convex by showing the differential d^2g is positive definite. The Hessian matrix associated with g is:

$$H = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

So clearly $|H_1|, |H_2| > 0$, so g is convex. Hence, any candidate points we find will automatically be maxima. As before, we form the Lagrangian:

$$\mathcal{L} = 2x + 3y + \lambda(2 - x^2 - y^2)$$

The Kuhn-Tucker conditions imply that:

$$\begin{array}{lll} (1) & \mathcal{L}_x = 2 - 2\lambda x \leq 0 & x \geq 0 & x(2 - 2\lambda x) = 0 \\ (2) & \mathcal{L}_y = 3 - 2\lambda y \leq 0 & y \geq 0 & y(3 - 2\lambda y) = 0 \\ (3) & x^2 + y^2 \leq 2 & \lambda \geq 0 & \lambda(2 - x^2 - y^2) = 0 \end{array}$$

1 (iii) implies that either $x = 0$ or $\lambda x = 1$. If $x = 0$, then 1 (i) cannot be satisfied, hence $\lambda x = 1$. Similarly, $\lambda y = \frac{2}{3}$ from 2 (iii) (and clearly $x, y, \lambda > 0$). Substituting these values for x and y in 3 (iii), and noting that $\lambda > 0$, we get:

$$2 - x^2 - y^2 = 0 \Rightarrow 2 - \left(\frac{1}{\lambda}\right)^2 - \left(\frac{2}{3\lambda}\right)^2 = 0 \Rightarrow \lambda = \pm \sqrt{\frac{13}{18}}$$

As $\lambda > 0$, we must have $\lambda = \sqrt{\frac{13}{18}}$, hence $y = \sqrt{\frac{8}{13}}$ and $x = \sqrt{\frac{18}{13}}$, and we have a maximum at this point.

Section 4.5 Example

- Consider the following linear-programming problem:

$$\max U = 4S + D$$

$$s.t. \quad S + D \leq 10$$

$$S + 2D \leq 12$$

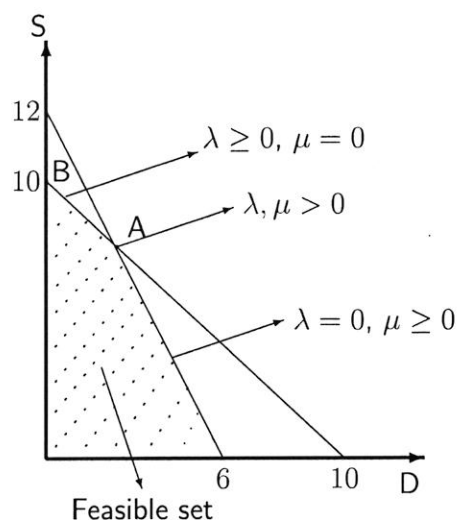
$$S, D \geq 0$$

- Since a linear function is concave and convex, though not strictly concave or strictly convex, a concave-programming problem consisting solely of linear functions that meet the Kuhn-Tucker conditions, will always satisfy the necessary and sufficient condition for a maximum.

- To solve a concave programming problem, first derive the Kuhn-Tucker conditions and then through trial and error find if the guess leads us to a solution or to a contradiction that informs us to try something else.
- Either start by assuming one of the constraints to be non-binding since the related Lagrange multiplier will be zero by complementary slackness, thereby eliminating a variable.
- Or start by trying a zero value for a choice variable since this simplifies the marginal conditions.
- In this example the Lagrange function is:

$$L = 4S + D + \lambda[10 - S - D] + \mu[12 - S - 2D]$$

- The feasible set is depicted below



and the Kuhn-Tucker conditions are:

$$\begin{aligned}
 S^* \frac{\partial L}{\partial S} &= 0; & S^* &\geq 0; & \frac{\partial L}{\partial S} &\leq 0; \\
 D^* \frac{\partial L}{\partial D} &= 0; & D^* &\geq 0; & \frac{\partial L}{\partial D} &\leq 0; \\
 \lambda^* \frac{\partial L}{\partial \lambda} &= 0; & \lambda^* &\geq 0; & \frac{\partial L}{\partial \lambda} &\geq 0; \\
 \mu^* \frac{\partial L}{\partial \mu} &= 0; & \mu^* &\geq 0; & \frac{\partial L}{\partial \mu} &\geq 0;
 \end{aligned}$$

Since:

$$\begin{aligned}
 \frac{\partial L}{\partial S} &= 4 - \lambda - \mu \\
 \frac{\partial L}{\partial D} &= 1 - \lambda - 2\mu \\
 \frac{\partial L}{\partial \lambda} &= 10 - S - D \\
 \frac{\partial L}{\partial \mu} &= 12 - S - 2D
 \end{aligned}$$

we obtain:

$$\begin{aligned}
 S^*[4 - \lambda - \mu] &= 0; & S^* &\geq 0; & 4 - \lambda - \mu &\leq 0; \\
 D^*[1 - \lambda - 2\mu] &= 0; & D^* &\geq 0; & 1 - \lambda - 2\mu &\leq 0; \\
 \lambda^*[10 - S - D] &= 0; & \lambda^* &\geq 0; & 10 - S - D &\geq 0; \\
 \mu^*[12 - S - 2D] &= 0; & \mu^* &\geq 0; & 12 - S - 2D &\geq 0;
 \end{aligned}$$

- Let's guess that $\mu > 0$ and $\lambda > 0$ i.e. both constraints are binding (point A in the above diagram). This implies that:

$$10 = S + D$$

$$12 = S + 2D$$

- Solving for S and D we obtain that $S^* = 8 > 0$ and $D^* = 2 > 0$. This implies that:

$$4 - \lambda - \mu = 0$$

$$1 - \lambda - 2\mu = 0$$

in order to satisfy the complementary slackness conditions.

- Solving for λ and μ we get that $\lambda^* = 7 > 0$ and $\mu^* = -3 < 0$ which contradicts our initial guess that $\mu > 0$!
- Next guess: Let's now guess that $\mu = 0$ and $\lambda > 0$. This implies that

$$10 = S + D \quad (2)$$

$$12 > S + 2D \quad (3)$$

- Try $S > 0$ and $D = 0$ for the choice variables (point B in the above diagram). From equation (2) this implies that $S^* = 10$. it is clear to see that equation (3) is also satisfied when $S = 10$ and $D = 0$. This also implies that

$$4 - \lambda - \mu = 0 \quad (4)$$

$$1 - \lambda - 2\mu < 0 \quad (5)$$

- Since we guessed $\mu = 0$ this implies from equation (4) that $\lambda^* = 4 > 0$ and this implies that equation (5) is satisfied since $-3 < 0$.

Therefore we have satisfied all the Kuhn-Tucker conditions for this maximization problem and the solution $D = 0, S = 10, \lambda = 4, \mu = 0$ is a maximum.

Ex: $z = x_1^2 + x_2^2$ s.t. $x_1 + 4x_2 = 2$

(12', 19)

$$J = x_1^2 + x_2^2 + \lambda(2 - x_1 - 4x_2)$$

$$J_1 = 2x_1 - \lambda = 0$$

$$J_2 = 2x_2 - 4\lambda = 0$$

$$J_\lambda = 2 - x_1 - 4x_2 = 0$$

$$\left. \begin{array}{l} J_1 = 2x_1 - \lambda = 0 \\ J_2 = 2x_2 - 4\lambda = 0 \end{array} \right\} \frac{x_1}{x_2} = \frac{1}{4} \Rightarrow x_1 = \frac{2}{17}$$

$$\Downarrow \\ x_2 = \frac{8}{17}$$

$$|H| = \begin{vmatrix} 0 & 1 & 4 \\ 1 & 2 & 0 \\ 4 & 0 & 2 \end{vmatrix} = -34 < 0$$

d_z^2 is PD

$z^* = \frac{4}{17}$ is minimum.

$\nexists$ interior maximum