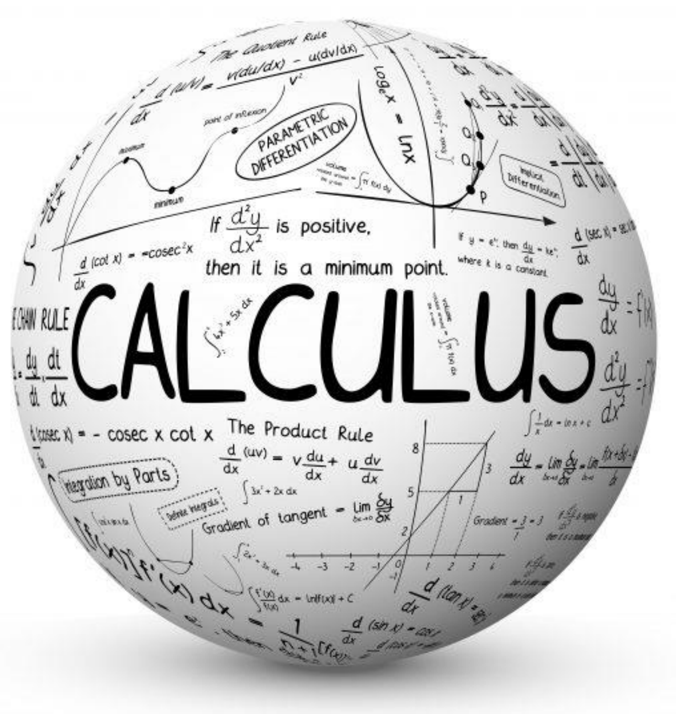
ODTÜ

Calculus with Analytic Geometry

[MATH 119 BOOKLET]

Answer Key



Road to success

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INTRODUCTION

This booklet is for those who are taking Math119 (Calculus with Analytic Geometry).

What does the booklet offer to their students?

1. Sufficient way to **read** the calculus subjects.
2. Organized way to **understand** calculus theorems and some proves as well as some necessary rules.
3. Sample examples to **understand** how to solve the related problems
4. Some suggested problems to **practice**
5. Appendix which contains functions and their graphs Geometric formulas
6. Short answer key of the suggested problems at the end of the booklet as well as full solution in the answer key booklet.

$$Read \dashrightarrow Understand \dashrightarrow Practice \dashrightarrow AA$$

What are the external booklets offered?

1. Notebook Booklet
2. Answer key Booklet (This booklet)
3. Sample midterms and Final exam Booklet
4. Brief Booklet

Note: All Math 119 booklets can be found online via *http://mathbooklet.weebly.com*

To have chance of having mock/sample exams as well as other online extra recourses and support, you may join *http://mathbooklet.weebly.com*

If you have any inquiries, please do not hesitate to contact me via *hassanin.risha @ gmail.com*

I hope such a work helps you to reach AA letter grade.

Best Regards,

Risha

Answer Key:

Section 1: Problems session

Evaluate the following limits. If the limit does not exist, is it ∞, −∞ or neither? (Do not use L'Hôpital's Rule)

1. $\lim_{x\to1}\frac{x^2-2x+1}{x-1}=\lim_{x\to1}\frac{(x-1)\cancel{(x-1)}}{\cancel{x-1}}=\lim_{x\to1}x-1=0$

2. $\lim_{x\to1}\frac{\tan x-x}{x^3}=\lim_{x\to1}\frac{\tan(1)-1}{1^3}=\tan(1)-1$

3. $\lim_{x\to\frac{\pi}{2}^-}\frac{e^x+3x+1}{\sin x-1}=\lim_{x\to\frac{\pi}{2}^-}\frac{e^{\frac{\pi}{2}}+3\left(\frac{\pi}{2}\right)+1}{\sin\left(\frac{\pi}{2}\right)-1}=\frac{e^{\frac{\pi}{2}}+3\left(\frac{\pi}{2}\right)+1}{1-1}=\frac{e^{\frac{\pi}{2}}+3\left(\frac{\pi}{2}\right)+1}{0}=-\infty \qquad \textit{since } e^{\frac{\pi}{2}}+3\left(\frac{\pi}{2}\right)+1 \textit{ exist}$

4. $\lim_{x\to3}\frac{(x^2+x-12)\sqrt{x+3}}{\sqrt{5}(x^2-9)}=\lim_{x\to3}\frac{\cancel{(x-3)}(x+4)\sqrt{x+3}}{\sqrt{5}\cancel{(x-3)}(x+3)}=\frac{1}{\sqrt{5}}\lim_{x\to3}\frac{(x+4)\sqrt{x+3}}{(x+3)}=\frac{1}{\sqrt{5}}\frac{(3+4)\sqrt{3+3}}{(3+3)}=\frac{1}{\sqrt{5}}\frac{7\sqrt{6}}{6}=\frac{7\sqrt{6}}{6\sqrt{5}}\left(\frac{\sqrt{5}}{\sqrt{5}}\right)=\frac{7\sqrt{30}}{30}$

5. $\lim_{x\to0}\frac{1-\cos x}{\sin^2 x}=\lim_{x\to0}\frac{1-\cos x}{\sin^2 x}*\frac{1+\cos x}{1+\cos x}=\lim_{x\to0}\frac{1-\cos^2 x}{(\sin^2 x)\,(1+\cos x)}=\lim_{x\to0}\frac{\sin^2 x}{(\sin^2 x)\,(1+\cos x)}=\lim_{x\to0}\frac{1}{(1+\cos x)}=\frac{1}{1+\cos 0}=\frac{1}{2}$

6. $\lim_{x\to0}x\sin x=0\sin 0=0*0=0$

7. $\lim_{x\to\infty}\frac{e^x}{\sinh x}=\lim_{x\to\infty}\frac{e^x}{\frac{e^x-e^{-x}}{2}}=2\lim_{x\to\infty}\frac{e^x}{e^x-e^{-x}}=2\lim_{x\to\infty}\frac{\frac{e^x}{e^x}}{\frac{e^x}{e^x}-\frac{e^{-x}}{e^x}}=2\lim_{x\to\infty}\frac{1}{1-e^{-2x}}=2\frac{1}{1-0}=2 \qquad \textit{since } e^{-2(\infty)}=0$

8. $\lim_{x\to0}\frac{\arcsin x+3\ln(x+1)}{x+\sin x-\cos x}=\lim_{x\to0}\frac{\arcsin 0+3\ln(0+1)}{0+\sin 0-\cos 0}=\frac{0+3(0)}{0+0-1}=0$

9. $\lim_{x\to2}\frac{x^2-4}{3x|x-2|}=D.N.E\begin{cases}\lim_{x\to2^+}\frac{x^2-4}{3x\,(x-2)}=\lim_{x\to2^+}\frac{(x+2)\cancel{(x-2)}}{3x\,\cancel{(x-2)}}=\lim_{x\to2^+}\frac{(x+2)}{3x}=\frac{4}{6}=\frac{2}{3}\to(1)\\ \lim_{x\to2^-}\frac{x^2-4}{-3x\,(x-2)}=\lim_{x\to2^-}\frac{(x+2)\cancel{(x-2)}}{-3x\,\cancel{(x-2)}}=\lim_{x\to2^-}\frac{(x+2)}{-3x}=-\frac{4}{6}=-\frac{2}{3}\to(2)\end{cases} \qquad (1)\neq(2)$

10. $\lim_{x\to1}\frac{2x-2}{|x^3-x^2|}=D.N.E\begin{cases}\lim_{x\to1^+}\frac{2x-2}{(x^3-x^2)}=\lim_{x\to1^+}\frac{2\cancel{(x-1)}}{x^2\cancel{(x-1)}}=\lim_{x\to1^+}\frac{2}{x^2}=2\to(1)\\ \lim_{x\to1^-}\frac{2x-2}{-(x^3-x^2)}=\lim_{x\to1^-}\frac{2\cancel{(x-1)}}{-x^2\cancel{(x-1)}}=\lim_{x\to1^-}\frac{2}{-x^2}=-2\to(2)\end{cases} \qquad (1)\neq(2)$

11. $\lim_{x\to-2}\frac{|x+2|}{|x|-2}=D.N.E\begin{cases}\lim_{x\to-2^+}\frac{x+2}{-x-2}=\lim_{x\to-2^+}\frac{x+2}{-(x+2)}=-1\to(1)\\ \lim_{x\to-2^-}\frac{-(x+2)}{-x-2}=\lim_{x\to-2^-}\frac{-(x+2)}{-(x+2)}=+1\to(2)\end{cases} \qquad (1)\neq(2)$

12. $\lim_{x\to1}\frac{|x^2-5|-|x+3|}{x-1}=\lim_{x\to1}\frac{-(x^2-5)-(x+3)}{x-1}=\lim_{x\to1}\frac{-(x^2+x-2)}{x-1}=\lim_{x\to1}\frac{-(x+2)\cancel{(x-1)}}{\cancel{x-1}}=\lim_{x\to1}-(x+2)=-3$

13. $\lim_{x\to0}\frac{|2x-3|-|x-3|}{x}=\lim_{x\to0}\frac{-(2x-3)+(x-3)}{x}=\lim_{x\to0}\frac{-2x+x}{x}=\lim_{x\to0}\frac{-x}{x}=-1$

14. $\lim_{x\to0}\frac{|\sin(x)|}{x}=D.N.E\begin{cases}\lim_{x\to0^+}\frac{\sin x}{x}=+1\to(1)\\ \lim_{x\to0^-}\frac{-\sin x}{x}=-1\to(2)\end{cases} \qquad (1)\neq(2)$

15. $\lim_{x\to3}\frac{|x^2-9|}{\sqrt[3]{x-3}}=0\begin{cases}\lim_{x\to3^+}\frac{x^2-9}{\sqrt[3]{x-3}}=\lim_{x\to3^+}\frac{(x-3)(x+3)}{\sqrt[3]{x-3}}=\lim_{x\to3^+}(x-3)^{\frac{2}{3}}(x+3)=0\to(1)\\ \lim_{x\to3^-}\frac{-(x^2-9)}{\sqrt[3]{x-3}}=\lim_{x\to3^+}\frac{-(x-3)(x+3)}{\sqrt[3]{x-3}}=\lim_{x\to3^-}-(x-3)^{\frac{2}{3}}(x+3)=0\to(2)\end{cases} \qquad (1)=(2)$

16. $\lim_{x\to\frac{\pi}{2}} \frac{\arctan(\tan(x))}{\tan(\arctan(x))} = D.N.E$

$$\begin{cases} \lim_{x\to\frac{\pi}{2}^+} \frac{\arctan(\tan(x))}{\tan(\arctan(x))} = \frac{\arctan\left(\tan\left(\frac{\pi}{2}\right)\right)}{\tan\left(\arctan\left(\frac{\pi}{2}\right)\right)} = \frac{\arctan(-\infty)}{\frac{\pi}{2}} = \frac{-\frac{\pi}{2}}{\frac{\pi}{2}} = -1 \to (1) \\ \lim_{x\to\frac{\pi}{2}^-} \frac{\arctan(\tan(x))}{\tan(\arctan(x))} = \frac{\arctan\left(\tan\left(\frac{\pi}{2}\right)\right)}{\tan\left(\arctan\left(\frac{\pi}{2}\right)\right)} = \frac{\arctan(\infty)}{\frac{\pi}{2}} = \frac{\frac{\pi}{2}}{\frac{\pi}{2}} = 1 \to (2) \end{cases} \quad (1) \neq (2)$$

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17. $\lim_{x\to\infty} \frac{2\sin(x^5)}{x^4+1}$ $\quad$ *we have* $-1 \le \sin(x^5) \le 1$ *and so* $\frac{-2}{x^4+1} \le \frac{2\sin(x^5)}{x^4+1} \le \frac{2}{x^4+1}$ *for all* x

Moreover, $\lim_{x\to\infty} \frac{-2}{x^4+1} = \lim_{x\to\infty} \frac{-2}{x^4+1} = 0$ *Thus By squeeze thm, we have* $\lim_{x\to\infty} \frac{2\sin(x^5)}{x^4+1} = 0$

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18. $\lim_{x\to\infty} \frac{cos\,(x\sin x)}{x^2}$ $\quad$ *we have* $-1 \le cos\,(x\sin x) \le 1$ *for all* x

hence $\frac{-1}{x^2} \le \frac{cos\,(x\sin x)}{x^2} \le \frac{1}{x^2}$ *for all* $x \neq 0$ $\quad \because \lim_{x\to\infty} \frac{1}{x^2} = \lim_{x\to\infty} \frac{-1}{x^2} = 0$

so By squeeze thm , then $\lim_{x\to\infty} \frac{cos\,(x\sin x)}{x^2} = 0$

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19. $\lim_{x\to\infty} \frac{sin\,(x^2+1)}{x^2}$ $\quad$ *we have* $-1 \le sin\,(x^2+1) \le 1$ *for all* x

hence $\frac{-1}{x^2} \le \frac{sin\,(x^2+1)}{x^2} \le \frac{1}{x^2}$ *for all* $x \neq 0$ $\quad \because \lim_{x\to\infty} \frac{1}{x^2} = \lim_{x\to\infty} \frac{-1}{x^2} = 0$

so By squeeze thm , then $\lim_{x\to\infty} \frac{sin\,(x^2+1)}{x^2} = 0$

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20. $\lim_{x\to\infty} \frac{sin\,(x^{119})}{\ln(x)}$ $\quad$ *we have* $-1 \le sin\,(x^{119}) \le 1$ *for all* x

hence $\frac{-1}{\ln(x)} \le \frac{sin\,(x^{119})}{\ln(x)} \le \frac{1}{\ln(x)}$ *for all* $x > 0$ $\quad \because \lim_{x\to\infty} \frac{1}{\ln(x)} = \lim_{x\to\infty} \frac{-1}{\ln(x)} = \frac{1}{\infty} = 0$

so By squeeze thm , then $\lim_{x\to\infty} \frac{sin\,(x^{119})}{\ln(x)} = 0$

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21. $\lim_{x\to\infty} \frac{3x + |sin\,(2x)|}{x^2}$ $\quad$ *we have* $0 \le |sin\,(2x)| \le 1$ *for all* x

hence $\frac{3x}{x^2} \le \frac{3x + |sin\,(2x)|}{x^2} \le \frac{1+3x}{x^2}$ *for all* $x > 0$ $\quad \because \lim_{x\to\infty} \frac{3x}{x^2} = \lim_{x\to\infty} \frac{3}{x} = \frac{1}{\infty} = 0$

$\because \lim_{x\to\infty} \frac{1+3x}{x^2} = \lim_{x\to\infty} \frac{1}{x^2} + \lim_{x\to\infty} \frac{3}{x} = \frac{1}{\infty} + \frac{3}{\infty} = 0$ $\quad$ *so By squeeze thm,* $\quad$ *then* $\lim_{x\to\infty} \frac{3x + |sin\,(2x)|}{x^2} = 0$

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22. $\lim_{x\to\infty} \frac{x^2+\sin x}{x^2+\sin^2 x} = \lim_{x\to\infty} \frac{\frac{x^2}{x^2}+\frac{\sin x}{x^2}}{\frac{x^2}{x^2}+\frac{\sin^2 x}{x^2}} = \lim_{x\to\infty} \frac{1+\frac{1}{x}*\frac{\sin x}{x}}{1+\left(\frac{\sin x}{x}\right)^2} = \frac{1+\frac{1}{\infty}*0}{1+(0)^2} = \frac{1+0*0}{1} = 1$ $\quad$ *since* $\lim_{x\to\infty} \frac{\sin x}{x} = 0$

23. $\lim_{x\to\infty}\frac{\cos\left(\sqrt{x^2+1}\right)}{e^x+119}$ $\quad$ *we have* $-1\le\cos\left(\sqrt{x^2+1}\right)\le 1$ *for all x*

$$hence \quad \frac{-1}{e^x+119}\le\frac{\cos\left(\sqrt{x^2+1}\right)}{e^x+119}\le\frac{1}{e^x+119} \quad for\ all\ x \qquad \because \lim_{x\to\infty}\frac{-1}{e^x+119}=\lim_{x\to\infty}\frac{1}{e^x+119}=0$$

so By squeeze thm, then $\lim_{x\to\infty}\frac{\cos\left(\sqrt{x^2+1}\right)}{e^x+119}=0$

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24. $\lim_{x\to\infty}\frac{\ln\left(1+\sin^2 x\right)}{\arctan x+e^x}$ $\quad$ *we have* $0\le\sin^2 x\le 1$ *for all x* $\quad\to\ 1\le 1+\sin^2 x\le 2$ $\quad$ After taking ln to both sides

$$\to\ \ln 1\le\ln\left(1+\sin^2 x\right)\le\ln 2 \quad\to\ 0\le\ln\left(1+\sin^2 x\right)\le\ln 2 \quad hence\quad 0\le\frac{\ln\left(1+\sin^2 x\right)}{\arctan x+e^x}\le\frac{\ln 2}{\arctan x+e^x}$$

$\because \lim_{x\to\infty}\frac{\ln 2}{\arctan x+e^x}=\frac{\ln 2}{\frac{\pi}{2}+\infty}=0$ $\qquad$ *so By squeeze thm, then* $\lim_{x\to\infty}\frac{\ln\left(1+\sin^2 x\right)}{\arctan x+e^x}=0$

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25. $\lim_{x\to\infty}\frac{4x^2-\cos(2x)}{x^2+1}$ $\quad$ *we have* $-1\le\cos(2x)\le 1$ *for all x* $\quad\to\ 4x^2-1\le 4x^2-\cos(2x)\le 4x^2+1$

$$\to\frac{4x^2-1}{x^2+1}\le\frac{4x^2-\cos(2x)}{x^2+1}\le\frac{4x^2+1}{x^2+1}$$

$\because \lim_{x\to\infty}\frac{4x^2-1}{x^2+1}=\lim_{x\to\infty}\frac{4x^2+1}{x^2+1}=4$ $\qquad$ *so By squeeze thm, then* $\frac{4x^2-\cos(2x)}{x^2+1}=4$

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26. $\lim_{x\to0}x^2\,e^{\sin\left(\frac{\pi}{x}\right)}$ $\quad$ *we have* $-1\le\sin\left(\frac{\pi}{x}\right)\le 1$ *for all* $x\ne 0$ $\quad\to\ e^{-1}\le e^{\sin\left(\frac{\pi}{x}\right)}\le e^1$

hence $x^2e^{-1}\le x^2e^{\sin\left(\frac{\pi}{x}\right)}\le x^2e^1$ *for all* $x\ne 0$ $\qquad \because \lim_{x\to 0}x^2e^{-1}=\lim_{x\to 0}x^2e^1=0$

so By squeeze thm, then $\lim_{x\to0}x^2\,e^{\sin\left(\frac{\pi}{x}\right)}=0$

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27. $\lim_{x\to0^+}e^{\sin\left(\frac{1}{x}\right)}\ln(1+x)$ $\quad$ *we have* $-1\le\sin\left(\frac{1}{x}\right)\le 1$ *for all* $x\ne 0$ $\quad\to\ e^{-1}\le e^{\sin\left(\frac{1}{x}\right)}\le e^1$

hence $e^{-1}\ln(1+x)\le e^{\sin\left(\frac{1}{x}\right)}\ln(1+x)\le e^1\ln(1+x)$ *for all* $x>0$

$\because \lim_{x\to 0}e^{-1}\ln(1+x)=\lim_{x\to 0}e^1\ln(1+x)=0$ $\qquad$ *so By squeeze thm,* $\qquad$ *then* $\lim_{x\to0^+}e^{\sin\left(\frac{1}{x}\right)}\ln(1+x)=0$

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28. $$\lim_{x\to\infty}\frac{x+\sqrt[4]{16x^6+x^5-3x+7}}{5x+8-\sqrt{x^3+x-2}}=\lim_{x\to\infty}\frac{\cancel{x^{3/2}}\left(\frac{1}{x^{1/2}}+\sqrt[4]{16+\frac{1}{x}-\frac{3}{x^5}+\frac{7}{x^6}}\right)}{\cancel{x^{3/2}}\left(\frac{5}{x^{1/2}}+\frac{8}{x^{3/2}}-\sqrt{1+\frac{1}{x^2}-\frac{2}{x^3}}\right)}=\frac{0+\sqrt[4]{16+0-0+0}}{0+0-\sqrt{1+0-0}}=\frac{\sqrt[4]{16}}{-1}=-2 \quad where\ \lim_{x\to\infty}\frac{1}{x}=0$$

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29. $$\lim_{x\to\infty}\frac{x^3+\sqrt{x^6+2x^5+1}}{x^3+2x^2+x+1}=\lim_{x\to\infty}\frac{\cancel{x^3}\left(1+\sqrt{1+\frac{2}{x}+\frac{1}{x^6}}\right)}{\cancel{x^3}\left(1+\frac{2}{x}+\frac{1}{x^2}+\frac{1}{x^3}\right)}=\frac{1+\sqrt{1+0+0}}{1+0+0+0}=\frac{1+1}{1}=2 \qquad where\ \lim_{x\to\infty}\frac{1}{x}=0$$

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30. $$\lim_{x\to\infty}\frac{\sqrt{x^4+x+1}}{\sqrt[3]{x^6+1}}=\left(dividing\ by\ x^2=\sqrt{x^4}=\sqrt[3]{x^6}\right)\lim_{x\to\infty}\frac{\sqrt{\frac{x^4}{x^4}+\frac{x}{x^4}+\frac{1}{x^4}}}{\sqrt[3]{\frac{x^6}{x^6}+\frac{1}{x^6}}}=\lim_{x\to\infty}\frac{\sqrt{1+\frac{1}{x^3}+\frac{1}{x^4}}}{\sqrt[3]{1+\frac{1}{x^6}}}=\frac{\sqrt{1+0+0}}{\sqrt[3]{1+0}}=1$$

$,where\ \lim_{x\to\infty}\frac{1}{x}=0$

31. $\lim\limits_{x\to\infty}\frac{\sqrt{x^2+e}}{x^2+7}=\lim\limits_{x\to\infty}\frac{\sqrt{\frac{x^2}{x^4}+\frac{e}{x^4}}}{\frac{x^2}{x^2}+\frac{7}{x^2}}=\lim\limits_{x\to\infty}\frac{\sqrt{\frac{1}{x^2}+\frac{e}{x^4}}}{1+\frac{7}{x^2}}=\frac{\sqrt{0+0}}{1+0}=0$ $\quad$ *where* $\lim\limits_{x\to\infty}\frac{1}{x}=0$

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32. $\lim\limits_{x\to\infty}[\ln(x^2+1)-\ln(x^2-1)]=\lim\limits_{x\to\infty}\ln\left(\frac{x^2+1}{x^2-1}\right)=(\blacksquare)\ln\left(\lim\limits_{x\to\infty}\frac{x^2+1}{x^2-1}\right)=\ln\left(\lim\limits_{x\to\infty}\frac{\frac{x^2}{x^2}+\frac{1}{x^2}}{\frac{x^2}{x^2}-\frac{1}{x^2}}\right)=$

$\ln\left(\frac{1+0}{1-0}\right)=\ln 1=0$ $\qquad(\blacksquare)\ln$ *is continous as* $x\to\infty$, *where* $\lim\limits_{x\to\infty}\frac{1}{x}=0$

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33. $\lim\limits_{x\to\infty}\left(\sqrt{x+\sqrt{x}}-\sqrt{x}\right)=\lim\limits_{x\to\infty}\frac{\left(\sqrt{x+\sqrt{x}}-\sqrt{x}\right)\left(\sqrt{x+\sqrt{x}}+\sqrt{x}\right)}{\left(\sqrt{x+\sqrt{x}}+\sqrt{x}\right)}=\lim\limits_{x\to\infty}\frac{(x+\sqrt{x}-x)}{\left(\sqrt{x+\sqrt{x}}+\sqrt{x}\right)}=\lim\limits_{x\to\infty}\frac{\sqrt{x}}{\sqrt{x}\left(\sqrt{1+\frac{1}{\sqrt{x}}}+1\right)}$

$\lim\limits_{x\to\infty}\frac{1}{\left(\sqrt{1+\frac{1}{\sqrt{x}}}+1\right)}=\frac{1}{1+1}=\frac{1}{2}$ $\quad$ *as* $\frac{1}{\sqrt{x}}=0$ *when* $x\to\infty$

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34. $\lim\limits_{x\to-\infty}\left(x-\sqrt{x^2+4x}\right)=-\infty$ $\quad$ *note that* $x-\sqrt{x^2+4x}\le x$ *for all* $x\le-4$ *and* $\lim\limits_{x\to-\infty}x=-\infty$

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35. $\lim\limits_{x\to-\infty}\left(x+\sqrt{x^2-x-4}\right)=\lim\limits_{x\to-\infty}\frac{(x+\sqrt{x^2-x-4})(x-\sqrt{x^2-x-4})}{(x-\sqrt{x^2-x-4})}=\lim\limits_{x\to-\infty}\frac{x^2-x^2+x+4}{(x-\sqrt{x^2-x-4})}=\lim\limits_{x\to-\infty}\frac{x+4}{(x-\sqrt{x^2-x-4})}$

$\lim\limits_{x\to-\infty}\frac{\frac{x}{x}+\frac{4}{x}}{\left(\frac{x}{x}+\sqrt{\frac{x^2}{x^2}-\frac{x}{x^2}-\frac{4}{x^2}}\right)}=\frac{1}{1+\sqrt{1-0-0}}=\frac{1}{2}$ $\quad$, *where* $\lim\limits_{x\to\infty}\frac{1}{x}=0$

Note: *as* $x\to-\infty$ *we have* $\sqrt{x^2-x-4}=\frac{\sqrt{x^2-x-4}}{|x|}=-\frac{\sqrt{x^2-x-4}}{x}=-\sqrt{\frac{x^2}{x^2}-\frac{x}{x^2}-\frac{4}{x^2}}$

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36. $\lim\limits_{x\to\infty}x-\sqrt{x^2+x+2}=\lim\limits_{x\to\infty}x-\sqrt{x^2+x+2}\left(\frac{x+\sqrt{x^2+x+2}}{x+\sqrt{x^2+x+2}}\right)=\lim\limits_{x\to\infty}\frac{x^2-x^2-x-2}{x+\sqrt{x^2+x+2}}=\lim\limits_{x\to\infty}\frac{-x\,(1+\frac{2}{x})}{x\left(1+\sqrt{1+\frac{1}{x}+\frac{2}{x^2}}\right)}=$

$\lim\limits_{x\to\infty}\frac{-\left(1+\frac{2}{x}\right)}{\left(1+\sqrt{1+\frac{1}{x}+\frac{2}{x^2}}\right)}=-\frac{-(1+0)}{1+\sqrt{1+0+0}}=\frac{-1}{2}$ $\quad$, *where* $\lim\limits_{x\to\infty}\frac{1}{x}=0$

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37. $\lim\limits_{x\to\infty}x-\sqrt{x^2-4x+1}=\lim\limits_{x\to\infty}\frac{(x-\sqrt{x^2-4x+1})(x+\sqrt{x^2-4x+1})}{x+\sqrt{x^2-4x+1}}=\lim\limits_{x\to\infty}\frac{x^2-x^2+4x-1}{x+\sqrt{x^2-4x+1}}=\lim\limits_{x\to\infty}\frac{4x-1}{x+\sqrt{x^2-4x+1}}=$

$=\lim\limits_{x\to\infty}\frac{\frac{4x}{x}-\frac{1}{x}}{\frac{x}{x}+\sqrt{\frac{x^2}{x^2}-\frac{4x}{x^2}+\frac{1}{x^2}}}=\frac{4-0}{1+\sqrt{1-0+0}}=\frac{4}{2}=2$ $\quad$, *where* $\lim\limits_{x\to\infty}\frac{1}{x}=0$

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38. $\lim\limits_{x\to\infty}\sqrt{x^2+4x}-\sqrt{x^2-4x}=\lim\limits_{x\to\infty}\frac{x^2+4x-x^2+4x}{\sqrt{x^2+4x}+\sqrt{x^2-4x}}=\lim\limits_{x\to\infty}\frac{8\,x}{x\left(\sqrt{1+\frac{4}{x}}+\sqrt{1-\frac{4}{x}}\right)}=\frac{8}{2}=4$, *where* $\lim\limits_{x\to\infty}\frac{1}{x}=0$

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39. $\lim\limits_{x\to-\infty}\sqrt{2x^2-x+1}-\sqrt{x^2-x+5}=\lim\limits_{x\to-\infty}\frac{2x^2-x+1-x^2+x-5}{\sqrt{2x^2-x+1}+\sqrt{x^2-x+5}}=\lim\limits_{x\to-\infty}\frac{x^2-4}{\sqrt{2x^2-x+1}+\sqrt{x^2-x+5}}$

$$\lim_{x\to-\infty}\frac{x^2\left(1-\frac{4}{x^2}\right)}{|x|\left(\sqrt{2-\frac{1}{x}+\frac{1}{x^2}}+\sqrt{1-\frac{1}{x}+\frac{5}{x^2}}\right)}=\lim_{x\to-\infty}\frac{(-\infty)^2(1-0)}{(-\infty)(\sqrt{2-0+0}+\sqrt{1-0+0})}=+\infty \quad , where \lim_{x\to\infty}\frac{1}{x}=0$$

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40. $$\lim_{x\to\infty}\sqrt{x^2+x}-\sqrt{x}=\lim_{x\to\infty}\frac{x^2+x-x}{\sqrt{x^2+x}+\sqrt{x}}=\lim_{x\to\infty}\frac{x^2}{|x|\left(\sqrt{1+\frac{1}{x}}+\frac{1}{\sqrt{x}}\right)}=\frac{\infty}{(\sqrt{1+0}+0)}=\infty$$

$|x|=x \ \ as\ x\to\infty$

$, where \lim_{x\to\infty}\frac{1}{x}=0$

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41. $$\lim_{x\to-\infty}\sqrt{x^{10/3}-5x^2}-\sqrt{x^{10/3}+5x^2}=\lim_{x\to-\infty}\frac{\left(\sqrt{x^{10/3}-5x^2}-\sqrt{x^{10/3}+5x^2}\right)\left(\sqrt{x^{10/3}-5x^2}+\sqrt{x^{10/3}+5x^2}\right)}{\sqrt{x^{10/3}-5x^2}+\sqrt{x^{10/3}+5x^2}}$$

$$\lim_{x\to-\infty}\frac{x^{10/3}-5x^2-x^{10/3}-5x^2}{\sqrt{x^{10/3}-5x^2}+\sqrt{x^{10/3}+5x^2}}=\lim_{x\to-\infty}\frac{-10x^2}{\sqrt{x^{10/3}-5x^2}+\sqrt{x^{10/3}+5x^2}}=$$

$$\lim_{x\to-\infty}\frac{-10x^2}{-x^{5/3}\left(\sqrt{\frac{x^{10/3}}{x^{10/3}}-\frac{5x^2}{x^{10/3}}}+\sqrt{\frac{x^{10/3}}{x^{10/3}}+\frac{5x^2}{x^{10/3}}}\right)}=\lim_{x\to-\infty}\frac{-10x^2}{-x^{5/3}(\sqrt{1-0}+\sqrt{1+0})}=5\lim_{x\to-\infty}x^{1/3}=-\infty$$

$Note: \ as\ x\to-\infty\ \ we\ have\ \sqrt{x^{10/3}}=\left|x^{5/3}\right|=-x^{5/3}$

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42. $$\lim_{x\to-\infty}\ln\left(\sqrt{x^2+1}+x\right)=\lim_{x\to-\infty}\ln\left(\frac{(\sqrt{x^2+1}+x)(\sqrt{x^2+1}-x)}{\sqrt{x^2+1}-x}\right)=\lim_{x\to-\infty}\ln\left(\frac{x^2+1-x^2}{\sqrt{x^2+1}-x}\right)=\lim_{x\to-\infty}\ln\left(\frac{1}{\sqrt{x^2+1}-x}\right)$$

$$(\blacksquare)=\ln\left(\lim_{x\to-\infty}\frac{1}{\sqrt{x^2+1}-x}\right)=\ln\left(\lim_{x\to-\infty}\frac{\frac{1}{x}}{-\sqrt{\frac{x^2}{x^2}+\frac{1}{x^2}}-\frac{x}{x}}\right)=\ln\left(\frac{0}{-\sqrt{1+0}-1}\right)=\ln 0=-\infty$$

$(\blacksquare)$*since* $\ln$ *is continous as* $x\to-\infty$ $\quad , where \lim_{x\to\infty}\frac{1}{x}=0$

$Note: \ as\ x\to-\infty\ \ we\ have\ \sqrt{x^2+1}=\frac{\sqrt{x^2+1}}{|x|}=-\frac{\sqrt{x^2+1}}{x}=-\sqrt{\frac{x^2}{x^2}+\frac{1}{x^2}}$

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43. $$\lim_{x\to0}\frac{(1-\cos x)}{x^2}=\lim_{x\to0}\frac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)}=\lim_{x\to0}\frac{1-\cos^2 x}{x^2(1+\cos x)}=\lim_{x\to0}\frac{\sin^2 x}{x^2(1+\cos x)}=\lim_{x\to0}\left(\frac{\sin x}{x}\right)^2\frac{1}{(1+\cos x)}=(1)\left(\frac{1}{2}\right)=\frac{1}{2}$$

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44. $$\lim_{x\to0}\frac{1-\cos(x^2)}{x^2\sin^2(x)}=\lim_{x\to0}\frac{1-\cos(x^2)}{x^2\sin^2(x)}*\frac{1+\cos(x^2)}{1+\cos(x^2)}=\lim_{x\to0}\frac{1-\cos^2(x^2)}{x^2\sin^2(x)(1+\cos(x^2))}=\lim_{x\to0}\frac{\sin^2(x^2)}{x^2\sin^2(x)(1+\cos(x^2))}$$

$$=\lim_{x\to0}\frac{\sin^2(x^2)}{(x^2)^2}*\frac{\cancel{x^2}}{1}*\lim_{x\to0}\frac{1}{\frac{\sin^2(x)}{x^2}*\cancel{x^2}}*\lim_{x\to0}\frac{1}{1+\cos(x^2)}=\frac{1}{2}$$

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45. $$\lim_{x\to0}\frac{\sqrt{1+\tan x}-\sqrt{1+\sin x}}{x^3}*\left(\frac{\sqrt{1+\tan x}+\sqrt{1+\sin x}}{\sqrt{1+\tan x}+\sqrt{1+\sin x}}\right)=\lim_{x\to0}\frac{1+\tan x-1-\sin x}{x^3(\sqrt{1+\tan x}+\sqrt{1+\sin x})}=\lim_{x\to0}\frac{\sin x(\frac{1}{\cos x}-1)}{x^3(\sqrt{1+\tan x}+\sqrt{1+\sin x})}$$

$$=\lim_{x\to0}\frac{\sin x}{x^3}\left(\frac{1-\cos x}{\cos x}\right)*\left(\frac{1+\cos x}{1+\cos x}\right)*\frac{1}{(\sqrt{1+\tan x}+\sqrt{1+\sin x})}=\lim_{x\to0}\left(\frac{\sin x}{x}\right)^3*\left(\frac{1}{\cos x\,(1+\cos x)}\right)*\frac{1}{(\sqrt{1+\tan x}+\sqrt{1+\sin x})}=\frac{1}{4}$$

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46. $$\lim_{x\to\infty}\frac{2-x+\sin(x)}{x+\cos(x)}=\lim_{x\to\infty}\frac{\frac{2}{x}-\frac{x}{x}+\frac{\sin(x)}{x}}{\frac{x}{x}+\frac{\cos(x)}{x}}=\frac{0-1+0}{1+0}=-1$$

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47. $$\lim_{x\to\infty}\frac{2x^2+\sin^2 x}{x^2}=\lim_{x\to\infty}\frac{2x^2}{x^2}+\frac{\sin^2 x}{x^2}=(\blacksquare)\lim_{x\to\infty}2+\lim_{x\to\infty}\left(\frac{\sin x}{x}\right)^2=2+\left(\lim_{x\to\infty}\frac{\sin x}{x}\right)^2=2+0^2=2$$

(◘) we can separete the limit since each of the following limits exists

48. $\lim\limits_{x\to 0}\frac{\sin 5x-\sin 3x}{\sin x}=\lim\limits_{x\to 0}\frac{5\frac{\sin 5x}{5x}-3\frac{\sin 3x}{3x}}{\frac{\sin x}{x}}=\frac{5(1)-3(1)}{(1)}=5-3=2$

49. $\lim\limits_{x\to 0}\frac{x^3\sin\left(\frac{1}{x}\right)}{\sin(x)}=\lim\limits_{x\to 0}\left[\frac{x}{\sin(x)}*x^2\sin\left(\frac{1}{x}\right)\right]=1*0=0$ $\quad$ *where* $\lim\limits_{x\to 0}x^2\sin\left(\frac{1}{x}\right)=0$ (◘)

(◘) $-1\le\sin\frac{1}{x}\le 1$ *so* $-x^2\le x^2\sin\left(\frac{1}{x}\right)\le x^2$ $\lim\limits_{x\to 0}x^2=\lim\limits_{x\to 0}-x^2=0$ *By squeeze thm* $\lim\limits_{x\to 0}x^2\sin\left(\frac{1}{x}\right)=0$

50. $\lim\limits_{x\to 0^+}\frac{(\tan\sqrt{x})(\sqrt{2x+1})}{x}=\lim\limits_{x\to 0^+}\frac{(\tan\sqrt{x})(\sqrt{2x+1})}{x}=\lim\limits_{x\to 0^+}\frac{\sin\sqrt{x}}{\cos\sqrt{x}}*\frac{\sqrt{2x+1}}{x}=\lim\limits_{x\to 0^+}\frac{\sin\sqrt{x}}{\sqrt{x}}*\frac{1}{\cos\sqrt{x}}*\frac{\sqrt{2x+1}}{\sqrt{x}}=$

$$\lim_{x\to 0^+}\frac{\sin\sqrt{x}}{\sqrt{x}}*\lim_{x\to 0^+}\frac{1}{\cos\sqrt{x}}*\lim_{x\to 0^+}\sqrt{2+\frac{1}{x}}=1*1*\infty=\infty$$

51. $\lim\limits_{x\to\frac{1}{2}}\frac{\arccos(x)-\frac{\pi}{3}}{x-\frac{1}{2}}=\lim\limits_{x\to\frac{1}{2}}\frac{f(x)-f\left(\frac{1}{2}\right)}{x-\frac{1}{2}}=f'\left(\frac{1}{2}\right)=\frac{-1}{\sqrt{1-\left(\frac{1}{2}\right)^2}}=\frac{-1}{\sqrt{\frac{3}{4}}}=\frac{-2}{\sqrt{3}}$ $\quad$ *where* $f(x)=\arccos(x)\,,f'(x)=\frac{-1}{\sqrt{1-x^2}}$

52. $\lim\limits_{x\to 1}\frac{\arcsin(x^3-1)}{x-1}=\lim\limits_{x\to 1}\frac{f(x)-f(1)}{x-1}=f'(1)=3$ $\quad$ *where* $f(x)=\arcsin(x^3-1)\,,f'(x)=\frac{3x^2}{\sqrt{1-(x^3-1)^2}}$

53. $\lim\limits_{x\to 0}\frac{\cos x-e^x}{x}=\lim\limits_{x\to 0}\frac{f(x)-f(0)}{x-0}=f'(0)=-1$ $\quad$ *where* $f(x)=\cos x-e^x\,,f'(x)=-\sin x-e^x$

54. $\lim\limits_{x\to 0}\frac{3^{(2+x)^2}-81}{x}=\lim\limits_{x\to 0}\frac{f(x)-f(0)}{x-0}=f'(0)=81*4*\ln 3=324\ln 3$

where $f(x)=3^{(2+x)^2},f'(x)=\left[3^{(2+x)^2}\right][2(2+x)](\ln 3)$

55. $\lim\limits_{x\to 0}\frac{2^x-1}{\sin 4x}=\frac{1}{4}*\lim\limits_{x\to 0}\frac{4x}{\sin 4x}*\lim\limits_{x\to 0}\frac{2^x-1}{x}=\frac{1}{4}*1*\lim\limits_{x\to 0}\frac{f(x)-f(0)}{x-0}=\frac{1}{4}*f'(0)=\frac{1}{4}*2^0\ln 2=\frac{\ln 2}{4}$

where $f(x)=2^x,f'(x)=2^x\ln 2$

56. $\lim\limits_{x\to 0}\frac{\cos[(x+2)^2]-\cos 4}{x}=\lim\limits_{x\to 0}\frac{f(x)-f(0)}{x-0}=f'(0)=-2(0+2)\sin[(0+2)^2]=-4\sin 4$

where $f(x)=\cos[(x+2)^2]\,,f'(x)=-2(x+2)\sin[(x+2)^2]$

57. $\lim\limits_{x\to 1}\frac{e^{\sqrt{x}}-e}{x^3-1}=\lim\limits_{x\to 1}\frac{e^{\sqrt{x}}-e}{(x-1)(x^2+x+1)}=\lim\limits_{x\to 1}\frac{1}{x^2+x+1}*\lim\limits_{x\to 1}\frac{e^{\sqrt{x}}-e}{(x-1)}=\frac{1}{3}*\lim\limits_{x\to 0}\frac{f(x)-f(1)}{x-1}=\frac{1}{3}f'(1)=\frac{1}{3}e^{\sqrt{1}}\left(\frac{1}{2\sqrt{1}}\right)=\frac{e}{6}$

where $f(x)=e^{\sqrt{x}},\ f'(x)=e^{\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\right)$

Question1: $f(x) = \begin{cases} \frac{\sin(2x^2)+x^3}{x}, & x \neq 0 \\ 0 & , x = 0 \end{cases}$

a) Show that the function f is continuous at $x = 0$

$$\lim_{x\to 0^+} \frac{\sin(2x^2)+x^3}{x} = \lim_{x\to 0^-} \frac{\sin(2x^2)+x^3}{x} = \lim_{x\to 0^-} \left((2x)\frac{\sin(2x^2)}{2x^2} + x^2 \right) = 0 * 1 + 0 = 0 \quad and \quad f(0) = 0$$

$since \lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0) = 0$ The function f is continuous at $x = 0$

b) Find $f'(0)$ if it exists.

$$f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{f(x)-0}{x-0} = \lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} \frac{\frac{\sin(2x^2)+x^3}{x}}{x} = \lim_{x\to 0} \frac{\sin(2x^2)+x^3}{x^2}$$

$$\lim_{x\to 0} \left(2\frac{\sin(2x^2)}{2x^2} + x \right) = 2 + 0 = 2 \qquad since\ each\ limit\ exists \qquad f'(0) = 2$$

Question2: let $f: R \to R$ be a function defined by

$$f(x) = \begin{cases} mx - b^2, & x < 119 \\ x^2 & if \quad x \geq 119 \end{cases}$$

Using the definition of derivative, find all values of $\boldsymbol{m}$ and $\boldsymbol{b}$ that make f differentiable at 119.

Since f is going to be differentiable at 119, we must have

$$\lim_{t\to 0^+} \frac{f(119+t)-f(119)}{t} = \lim_{t\to 0^-} \frac{f(119+t)-f(119)}{t}$$

$$\lim_{t\to 0^+} \frac{f(119+t)-f(119)}{t} = \lim_{t\to 0^+} \frac{f(119+t)-119^2}{t} = \lim_{t\to 0^+} \frac{(119+t)^2-119^2}{t} = \lim_{t\to 0^+} \frac{2(119)t+t^2}{t} = 2(119) \to (1)$$

$$\lim_{t\to 0^-} \frac{f(119+t)-f(119)}{t} = \lim_{t\to 0^-} \frac{m(119+t)-b^2-119^2}{t} = \lim_{t\to 0^-} \left[\frac{119m-b^2-119^2}{t} + m \right] = 2(119)$$

But this limit exists if and only if $119m - b^2 - 119^2 = 0$ and in this case $\boxed{m = 2(119)}$

Hence $119 * 2(119) - b^2 - 119^2 = 0 \quad \to \boxed{b = \pm 119}$

Question3: let $f: R \to R$ be a function defined by

$$f(x) = \begin{cases} x + x^2 & if \quad x < 1 \\ x^3 + 1 & if \quad x \geq 1 \end{cases}$$

A. Find the derivative $f'(x)$ for $x \neq 1$.

$$f'(x) = \begin{cases} 1 + 2x & if \quad x < 1 \\ 3x^2 & if \quad x > 1 \end{cases}$$

B. Find $f'(1)$ if it exists.

$$\lim_{x\to1^+}\frac{f(x)-f(1)}{x-1}=\lim_{x\to1^+}\frac{x^3+1-2}{x-1}=\lim_{x\to1^+}\frac{(x-1)(x^2+x+1)}{x-1}=3\ \to(1)$$

$$\lim_{x\to1^-}\frac{f(x)-f(1)}{x-1}=\lim_{x\to1^-}\frac{x+x^2-2}{x-1}=\lim_{x\to1^-}\frac{(x-1)(x+2)}{x-1}=3\ \to(2)$$

$$From\ (1)\ and\ (2)\quad we\ have\ f'(1)=\lim_{x\to1}\frac{f(x)-f(1)}{x-1}=3$$

C. Is the **derivative** function $y=f'(x)$ continuous at $x=1$? Explain

From part (a), we have $\lim_{x\to1} f'(x)=f'(1)=3\ \to(1)$ and $(2)\to\begin{cases}\lim_{x\to1^+} f'(x)=\lim_{x\to1^+} 3x^2=3\\ \lim_{x\to1^-} f'(x)=\lim_{x\to1^-} 1+2x=3\end{cases}$

$From\ (1)\ and\ (2)\ \ we\ have\ \ f'(x)$ is continuous at $x=1$

Question4: let $f: R\to R$ be a function defined by

Consider $f(x)=\begin{cases}\frac{1-\cos x}{x} & if\quad -\frac{\pi}{2}\le x<0\\ ax+b & if\quad 0\le x\le 1\end{cases}$

Determine the values of a,b such that $f'(x)$ exists for all x with $-\frac{\pi}{2}<x<1$

$\because\ f'(x)$ exists $\therefore$ we have step (1) and (2)

$$\to(1)\quad\therefore f(x)\ is\ cont.\ at\ x=0\qquad f(0)=b=\lim_{x\to0^-}\frac{1-\cos x}{x}=0\qquad Hence\ b=0$$

$$\to(2)\ \ \therefore f(x)\ is\ diff.\ at\ x=0\ \ f'(0)=\lim_{x\to0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to0}\frac{f(x)}{x}\ \ \therefore\lim_{x\to0^+}\frac{f(x)}{x}=\lim_{x\to0^-}\frac{f(x)}{x}$$

$$\lim_{x\to0^+}\frac{ax}{x}=a=\lim_{x\to0^-}\frac{1-\cos x}{x^2}*\frac{1+\cos x}{1+\cos x}=\lim_{x\to0^-}\frac{1-\cos^2 x}{x^2(1+\cos x)}=\lim_{x\to0^-}\left(\frac{\sin x}{x}\right)^2*\frac{1}{(1+\cos x)}=\frac{1}{2}=a$$

Question5: $find\ f'(0)\ if\ f(x)=\begin{cases}\frac{x^3\sin\left(\frac{1}{x}\right)}{\sin x} & if\ \ x\neq0\\ 0 & if\ \ x=0\end{cases}$

Show your work.

$$f'(0)=\lim_{x\to0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to0}\frac{f(x)}{x}=\lim_{x\to0}\frac{x^2\sin\left(\frac{1}{x}\right)}{\sin x}=\left[\lim_{x\to0}\frac{x}{\sin x}\right]*\left[\lim_{x\to0}x\sin\left(\frac{1}{x}\right)\right]=1*0=0$$

$$\lim_{x\to0}-x\le\lim_{x\to0}x\sin\left(\frac{1}{x}\right)\le\lim_{x\to0}x\qquad since\ \lim_{x\to0}x=\lim_{x\to0}-x=0\qquad By\ squeeze\ thm,\ \ \lim_{x\to0}x\sin\left(\frac{1}{x}\right)=0$$

Question6: Consider $f(x)$ is a continuous function for all x. Find $\boldsymbol{a}$, where

$$f(x)=\begin{cases}2x+(x-1)\sin\frac{1}{x-1} & if\ \ x<1\\ ax+2 & if\ \ x\ge1\end{cases}$$

$$\lim_{x\to1^+}ax+2=a+2\ \ \to(1)\quad\lim_{x\to1^-}2x+(x-1)\sin\frac{1}{x-1}=2\ \ (*)\ \to(2)$$

$$(*) \lim_{x\to 1^-} 2x-(x-1) \le \lim_{x\to 1^-} 2x+(x-1)\sin\frac{1}{x-1} \le \lim_{x\to 1^-} 2x+(x-1)$$

$$since \lim_{x\to 1^-} 2x+(x-1) = \lim_{x\to 1^-} 2x-(x-1) = 2 \quad By\ squeeze\ thm, \ \lim_{x\to 1^-} 2x+(x-1)\sin\frac{1}{x-1} = 2$$

$\because f(x)$ is a continuous at $x=1$ $\quad\therefore$ we have $(1) = (2)\ steps \quad a+2=2 \quad \therefore \mathrm{a}=0$

Question7: Find the number $a \neq 0$ such that the function

$$f(x) = \begin{cases} \dfrac{\sin(ax)}{x} + \cos x & for \ \ x<0 \\ x^2+3e^{2x} & for \ \ x \ge 0 \end{cases}$$

Is continuous **everywhere**

$$f(0) = \lim_{x\to 0^+} x^2+3e^{2x} = 3 \ \to (1) \quad \lim_{x\to 0^-} \frac{\sin(ax)}{x} + \cos x = \lim_{x\to 0^-} \frac{a\sin(ax)}{ax} + \cos x = a+1 \to (2)$$

$\because f(x)$ is a continuous at $x=0$ $\quad\therefore$ we have $(1) = (2)\ steps \quad a+1=3 \quad \therefore \mathrm{a}=2$

Question8:

$$\text{Let } f(x) = \begin{cases} x^2+\arcsin x & for \ \ x \ge 0 \\ x+e^{x^2} & for \ \ x<0 \end{cases}$$

Find $f'(0)$ if it exists, or explain why it does NOT exist

$$\left(f(0) = \lim_{x\to 0^-} x+e^{x^2} = 1\right) \neq \left(\lim_{x\to 0^+} x^2+\arcsin x = 0\right) \ f\ is\ NOT\ cont.\ and\ hence\ f\ is\ Not\ diff.\ \ at\ x=0$$

Question9:

$$\text{Let } f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & for \ \ x<0 \\ x^2 & for \ \ x \ge 0 \end{cases}$$

a) Show that the function $f(x)$ is continuous at $x=0$

$$since\ f(0) = \lim_{x\to 0^+} x^2 = \lim_{x\to 0^+} x\sin\left(\frac{1}{x}\right) = 0 \qquad hence\ f\ is\ cont.\ \ at\ x=0$$

b) Use the definition of derivative to determine whether $f`(0)$ exists.

$$f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h} \quad since\ f(0)=0$$

$$\lim_{h\to 0^+} \frac{h^2}{h} = \lim_{h\to 0^+} h = 0 \ \to (1) \quad \lim_{h\to 0^-} \frac{h\sin\left(\frac{1}{h}\right)}{h} = \lim_{h\to 0^-} \sin\left(\frac{1}{h}\right)\ D.N.E\ (\infty) \to (2) \quad So\ (1)\ and\ (2)\ f'(0)\ D.N.E$$

Question10:

$$let\ f(x) = \begin{cases} e^x(x^2+a) & if\ x<0 \\ 2 & if\ x=0 \\ bx^2+2 & if\ x>0 \end{cases}$$

$If\ possible, determine\ a, b\ so\ that\ f`(0)\ exists.\ If\ not\ possible\ explain\ why?$

$\Rightarrow when\ f`(0)\ exists\ \therefore f(x)\ is\ continuous\ and\ differentiable\ at\ x=0$

$when\ f(x)\ is\ continuous\ at\ x=0\ \therefore \lim_{x\to0^+} f(x) = \lim_{x\to0^-} f(x) = f(0)$

$\lim_{x\to0^+} bx^2+2 = 2 \quad and \quad \lim_{x\to0^-} e^x(x^2+a) = a \quad and\ \ f(0)=0 \qquad \therefore a = 2$

$$\Rightarrow when\ f(x)\ is\ differentiable\ at\ x=0\ \therefore f`(0) = \lim_{x\to0^+}\frac{f(x)-f(0)}{x} = \lim_{x\to0^-}\frac{f(x)-f(0)}{x}$$

$$\lim_{x\to0^+}\frac{f(x)-f(0)}{x} = \lim_{x\to0^+}\frac{e^x(x^2+2)-2}{x} = f`(0), \quad f(x) = e^x(x^2+2)\ \ f`(x) = e^x(x^2+2)+2x\,e^x$$

$$\therefore \lim_{x\to0^+}\frac{f(x)-f(0)}{x} = f`(0) = 2$$

$$\lim_{x\to0^-}\frac{f(x)-f(0)}{x} = \lim_{x\to0^-}\frac{bx^2+2-2}{x} = \lim_{x\to0^-} bx = 0$$

$$\lim_{x\to0^+}\frac{f(x)-f(0)}{x} \neq \lim_{x\to0^-}\frac{f(x)-f(0)}{x} \quad ,that\ is,\ \ 2\neq0 \quad \boldsymbol{Thus, f(x)\ is\ NOT\ \ differentiable\ at\ x=0}$$

$$\boldsymbol{Therefore,\ \ there\ is\ \ NO\ such\ a\ \&\ b\ points}$$

Section 3: Problems session

Part I: Evaluate the following derivatives. There is **no** need to simplify your answer.

a) $\frac{d}{dx}[\arcsin(e^x+x^e)] = \frac{e^x+ex^{e-1}}{\sqrt{1-(e^x+x^e)^2}}$

b) $\frac{d}{dx}[(\sin x)^{\cos x}] = \frac{d}{dx}\left[e^{\cos x\ln(\sin x)}\right] = (\sin x)^{\cos x}\left[-(\sin x)\ln(\sin x) + \frac{\cos^2 x}{\sin x}\right]$

c) $\frac{d}{dx}[\tan^2(\sec(x^2))] = (2\tan(\sec(x^2))) * (\sec^2(\sec(x^2))) * (\sec(x^2)\tan(x^2)) * 2x$

d) $\frac{d}{dx}\left[(1+x)^{\sin x}\right] = \frac{d}{dx}\left[e^{\sin x\ln(1+x)}\right] = (1+x)^{\sin x}\left[\cos x\ \ln(1+x) + \frac{\sin x}{1+x}\right]$

e) $\frac{d}{dx}[x^3\arcsin(x^2-1)] = 3x^2\arcsin(x^2-1) + x^3\frac{2x}{\sqrt{1-(x^2-1)^2}}$

f) $\frac{d}{dx} f(x)$ *at point* $(x, f(x)) = (1,1)$ *where* f *is a differentiable function satisfying the equation*

$e^{xf(x)} + \ln f(x) = e \quad$ *for all* $x \quad \rightarrow \quad$ *taking* $\frac{d}{dx}$ *to both side* $\quad e^{xf(x)}(f(x) + xf'(x)) + \frac{f'(x)}{f(x)} = 0$

since $(x, f(x)) = (1,1) \quad e(1 + f'(1)) + f'(1) = 0 \qquad$ *Hence* $f'(1) = \frac{-e}{1+e}$

Part II: For the given function $f(x)$, find the derivative function $f'(x)$. DO NOT SIMPLIFY:

a) $f(x) = \cos^2 \sqrt{x} + \frac{1}{\ln x} \qquad f'(x) = 2(\cos\sqrt{x})(-\sin\sqrt{x}) * \frac{1}{2\sqrt{x}} + \frac{-1/x}{\ln^2 x}$

b) $f(x) = \frac{\arcsin\left(\frac{1}{1+x^2}\right)}{1+\cos x}$

$$f'(x) = \frac{\frac{1}{\sqrt{1 - 1/(1+x^2)^2}}\left(\frac{-2x}{(1+x^2)^2}\right)(1+\cos x) \; - \; \arcsin\left(\frac{1}{1+x^2}\right) * (-\sin x)}{(1+\cos x)^2}$$

c) $f'(x) = \left(1 + \frac{3}{x} + \frac{5}{x^2}\right)^x = \frac{d}{dx}\left[e^{x\ln\left(1+\frac{3}{x}+\frac{5}{x^2}\right)}\right] = \left(1+\frac{3}{x}+\frac{5}{x^2}\right)^x \left[\ln\left(1+\frac{3}{x}+\frac{5}{x^2}\right) + (x)\left(\frac{1}{1+\frac{3}{x}+\frac{5}{x^2}}\right)\left(\frac{-3}{x^2}+\frac{-10}{x^3}\right)\right]$

d) $f'(x) = \frac{d}{dx}\left[\frac{\ln(1+x^2)}{\sqrt{e^{2x}+5^x}}\right] = \frac{\left(\frac{2x}{1+x^2}\right)\left(\sqrt{e^{2x}+5^x}\right) \; - \; \ln(1+x^2)\left[\frac{1}{2\sqrt{e^{2x}+5^x}}\left(2e^{2x}+5^x\ln 5\right)\right]}{e^{2x}+5^x}$

e) $f'(x) = \frac{d}{dx}\left[e^{\cos(x^3)\ln(\sin x)}\right] = (\sin x)^{\cos(x^3)}\left[-\sin(x^3)(3x^2)\ln(\sin x) + \cos(x^3)\left(\frac{1}{\sin x}\right)(\cos x)\right]$

Part III: Find the derivative $\frac{dy}{dx}$ for the following functions.

a) $y = \sqrt{\frac{\sin(x^2)}{\tan x}} \qquad \frac{dy}{dx} = y' = \frac{1}{2\sqrt{\frac{\sin(x^2)}{\tan x}}}\left[\frac{2x\,cos(x^2) - \sin(x^2)\sec^2 x}{\tan^2 x}\right]$

b) $y = (x+x^2)^{\ln x} = e^{\ln x \ln(x+x^2)} \qquad \frac{dy}{dx} = y' = (x+x^2)^{\ln x}\left[\frac{\ln(x+x^2)}{x} + \ln x\frac{(1+2x)}{1+x}\right]$

Part IV: This question has <u>unrelated</u> parts about derivatives.

a) Evaluate $\frac{d}{dx}\left[\sin^2(2x) + \frac{e^x}{\ln x}\right] = 2\,sin(2x)\,cos(2x) * 2 + \frac{e^x \ln x - \frac{e^x}{x}}{(\ln x)^2} = 2\,sin(4x) + \frac{e^x(x\ln x - 1)}{x\,(\ln x)^2}$

b) Evaluate $\frac{d}{dx}\left[\ln\left(\frac{\sin x}{\sqrt[3]{2x+x^2}}\right)\right] = \frac{d}{dx}\left[\ln(\sin x) - \frac{1}{3}\ln(2x+x^2)\right] = \frac{\cos x}{\sin x} - \frac{1}{3}\frac{2+2x}{2x+x^2}$

c) Find $f'(x) = \frac{dy}{dx} \quad$ *if* $\quad y = f(x) = \frac{x^\pi + \cos x}{1+x^3} \qquad f'(x) = \frac{(\pi x^{\pi-1} - \sin x)(1+x^3) - (x^\pi + \cos x)(3x^2)}{(1+x^3)^2}$

d) Find $f'(x) = \frac{dy}{dx} \quad$ *if* $\quad y = f(x) = \sec(sec\,x) \quad f'(x) = \sec(sec\,x) * \tan(sec\,x) * sec\,x * tan\,x$

e) *if* $\;y = f(x) = 3^x + \log_x 3 = 3^x + \frac{\ln 3}{\ln x} \quad$ *then* $\;f'(x) = 3^x \ln 3 + \ln 3\,\frac{-1}{x\,(\ln x)^2} = \ln 3\left[3^x - \frac{1}{x\ln^2 x}\right]$

f) Suppose that $h(x) = f(x^2 + g(x))$, *where* $f(3) = 6,\; f'(3) = -4, f(-2) = 0,\; f'(-2) = 1,\; g(3) = 1,$

$g'(3) = -1, \qquad g(-2) = -1$ *and* $g'(-2) = 2.$ *Compute* $h'(-2)$.

$h(x) = f\left(x^2 + g(x)\right) \leftrightarrow h'(x) = f'\left(x^2 + g(x)\right) * \left(2x + g'(x)\right)$

$h'(-2) = f'\left(4 + g(-2)\right) * \left(-4 + g'(-2)\right) = f'(3) * (-2) = (-4)(-2) = 8 \qquad h'(-2) = 8$

g) Find $f'(1) = \left.\frac{dy}{dx}\right|_{x=1}$ $\quad if \quad$ $y = f(x) = \frac{x^{9x}(x-2)^3}{x^4 e^x}$

$\ln y = 9x \ln x + 3 \ln\ (x-2) - 4 \ln x \qquad \frac{y'}{y} = 9(\ln x + 1) + \frac{3}{x-2} - \frac{4}{x} - 1$

$y' = \frac{x^{9x}(x-2)^3}{x^4 e^x}\left[9(\ln x + 1) + \frac{3}{x-2} - \frac{4}{x} - 1\right] \qquad f'(1) = \left.\frac{dy}{dx}\right|_{x=1} = y'|_{x=1} = \frac{-1}{e}$

Part V: Question 1

Let $f(x) = \arctan(x) + e^x$

a) Show that f is invertible, f^{-1} is a function.

$f'(x) = \frac{1}{1+x^2} + e^x > 0\,; hence\ f(x)\ is\ strictly\ increasing\ for\ all\ x.\ Thus, f(x)\ is\ \ one\ to\ one\ function$

$Thus, f(x)$ is invertible.

b) Find the **derivative of the inverse** of f at 1, that is, find $(f^{-1})'(1)$.

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = (*) = \frac{1}{f'(a)} = \frac{1}{f'(0)} = \frac{1}{\frac{1}{1+0^2} + e^0} = \frac{1}{1+1} = \frac{1}{2}$$

$(*)\ f^{-1}(1) = a\ \rightarrow f(a) = 1\ \rightarrow \arctan(a) + e^a = 1\ \ thus\ a = 0$

Question 2:

Let $f(x) = 2x + \cos(x)$.

a) Show that f is invertible.

$f'(x) = 2 - \sin x\ > 0\,; hence\ f(x)\ is\ strictly\ increasing\ for\ all\ x.\ Thus, f(x)\ is\ \ one\ to\ one\ function$

$Thus, f(x)$ is invertible.

b) Find $(f^{-1})'(\pi)$ and the derivative of the inverse of f at π, that is, find $(f^{-1})'(\pi)$.

$$(f^{-1})'(\pi) = \frac{1}{f'(f^{-1}(\pi))} = (*) = \frac{1}{f'(a)} = \frac{1}{f'(\pi/2)} = \frac{1}{2 - \sin(\pi/2)} = \frac{1}{2-1} = 1$$

$(*)\ f^{-1}(\pi) = a\ \rightarrow f(a) = 1\ \rightarrow 2a + \cos(a) = \pi\ \ thus\ a = \pi/2$

Question 3:

Show that $f(x) = 1 + \arctan x + e^x$ has an inverse function. Find the derivative of the inverse of $f(x)$ at $x = 2$, that is, find $(f^{-1})'(2)$.

$f'(x) = \frac{1}{1+x^2} + e^x > 0$; $hence\ f(x)\ is\ strictly\ increasing\ for\ all\ x.\ Thus, f(x)\ is\ \ one\ to\ one\ function$

$Thus, f(x)$ is invertible.

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = (*) = \frac{1}{f'(a)} = \frac{1}{f'(0)} = \frac{1}{\frac{1}{1+0^2} + e^0} = \frac{1}{1+1} = \frac{1}{2}$$

$(*)\ f^{-1}(2) = a\ \rightarrow f(a) = 1\ \rightarrow\ \ 1 + \arctan a + e^a = 2\ \ thus\ a = 0$

Section 4: Problems session

Question 1

Find an equation of the tangent line to the curve $119\left(\frac{x}{y}\right) + 1891\left(\frac{y}{x}\right)^3 = 2010\ \ at\ the\ point\ (-1,-1).$

By taking the derivative of both sides: $119\left(\frac{y-xy'}{y^2}\right) + 1891 * 3\left(\frac{y}{x}\right)^2\left(\frac{y'x-y}{x^2}\right) = 0\ \ since\ x = y = -1$

$119(-1+y') + 1891 * 3(-y'+1) = 0 \Rightarrow\ (1-y')(1891 * 3 - 119) = 0\ \ \Rightarrow (1-y') = 0 \Rightarrow\ y' = 1\ (slope)$

Tangent line is $y = m(x-a) + b \qquad where\ a = b = -1 \qquad y = 1(x+1) - 1 \qquad y = x\ (tangent\ eqn)$

Question 2

Let f and g be differentiable functions such that $f(9) = -2,\ g(3) = 8,\ f'(9) = 7, g'(3) = 2.$

(a) Show that $y' = -11$ at the point (3,1) if $yg(x) + f(x^2) = y^4 + 5$.

$\frac{d}{dx}\left(yg(x) + f(x^2)\right) = \frac{d}{dx}(y^4+5) \quad \Rightarrow y'g(x) + yg'(x) + 2x\,f'(x^2)\ = 4\,y^3y' \qquad x = 3\,, y = 1$

$y'g(3) + g'(3) + 6\,f'(9)\ = 4\,y' \quad \Rightarrow\ \ 8y' + 2 + 42\ = 4\,y' \qquad \Rightarrow\ \ 4\,y' = -44 \qquad \Rightarrow \qquad y' = -11$

(b) Find an equation of the line tangent to the curve $yg(x) + f(x^2) = y^4 + 5$ at the point (3,1).

$y' = -11\ (slope) \quad point\ (3{,}1)$ line tangent eqn is $y = 11(x-3) + 1$

Question 3

Consider the curve given by the equation $xy^2 + \sin\left((x-1)y\right) = 4$

a) Find all points $(1, y)$ on this curve.

$x = 1\ \ y^2 + \sin(0) = 4 \qquad y^2 = 4 \qquad y = \pm 2 \qquad\qquad points\ are\ \ (1,+2)\ and\ (1,-2)$

b) Find the equations of the tangent lines to the curves at the points you found in part (a).

$$\frac{d}{dx}\left[xy^2 + \sin\left((x-1)y\right)\right] = \frac{d}{dx}[4] \qquad y^2 + 2xyy' + \cos\left((x-1)y\right)[y + (x-1)y'] = 0$$

$$(1,+2):\ \ y'|_{(1,2)} = \frac{-3}{2}\ \ then\ \ y = \frac{-3}{2}(x-1) + 2\ \ and\ \ (1,-2):\ \ y'|_{(1,-2)} = \frac{1}{2}\ \ then\ y = \frac{1}{2}(x-1) - 2$$

c) Are any of the tangent lines you found in part (b) parallel? Explain!

$Since\ \left(y'|_{(1,2)} = \frac{-3}{2}\right) \neq \left(y'|_{(1,-2)} = \frac{1}{2}\right) \quad so,\ \ these\ line\ are\ not\ parallel$

Question 4

Suppose the f is a function for which $f\left(\frac{\pi}{2}\right) = 2$ *and* $f'\left(\frac{\pi}{2}\right) = 1$.   *Let* $g(x) = f(x)^{\cos x}$

Find an equation of the line tangent to the graph of $g(x)$ *at* $x = \frac{\pi}{2}$

$$g'(x) = \left(e^{\cos x \ln f(x)}\right)' = f(x)^{\cos x}\left[-\sin x \ln f(x) + \cos x \frac{f'(x)}{f(x)}\right]$$

$$g'\left(\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}\right)^{\cos\left(\frac{\pi}{2}\right)}\left[-\sin\left(\frac{\pi}{2}\right)\ln f\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right)\frac{f'\left(\frac{\pi}{2}\right)}{f\left(\frac{\pi}{2}\right)}\right] = -\ln 2 \ \left(slope\ of\ tangent\ line\ at\ x = \frac{\pi}{2}\right)$$

$$at\ x = \frac{\pi}{2} \quad g\left(\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}\right)^{\cos\left(\frac{\pi}{2}\right)} = 1 \qquad Thus,\ \ y = -\ln 2\left(x - \frac{\pi}{2}\right) + 1 \ \ tangent\ line\ eqn.$$

Question 5

Verify that the point $P_0(\pi, 0)$ is on the curve C defined implicitly by the equation $\sin(x+y) = xy$. Then find the two lines which are **normal** and **tangent** to C at P_0.

$x = \pi, y = 0$ *so*, $\sin(\pi + 0) = \pi(0)$ *Then*, $\sin\pi = 0$ *so* $P_0(\pi, 0)$ is on the curve C

$\cos(x+y)(1+y') = y + xy'$ *at* $P_0 \ \to m = y'|_{P_0} = \dfrac{-1}{\pi+1}$ **Tangent** line to C at P_0 $y = \dfrac{-1}{\pi+1}(x-\pi)$

$m\ (slope\ of\ normal) = \pi + 1$ **Normal** line to C at P_0 $y = (\pi+1)(x-\pi)$

Question 6

Find the equations of the tangent lines to the curve $y = x^3 + x$ which pass through the point (2,2).

$2 \neq 2^3 + 2$ *Thus*, $(2,2) \notin curve\ y = x^3 + x$ *let* $x = a$ *then the point is* $(a, f(a))$ *where* $x \neq 2$

The **slope** of the **tangent line** with the point $(a, f(a))$ must be $y' = 3x^2 + 1 \ \ \therefore y'|_{x=a} = 3a^2 + 1 \ \to (1)$

Because the line passes through the point (2,2), the slope of the tangent line is **also**:

$\frac{f(a)-2}{a-2} = \frac{a^3+a-2}{a-2} \to (2) \quad \because (1) = (2) \quad \therefore \frac{a^3+a-2}{a-2} = 3a^2 + 1 \ \to \ a^3 + a - 2 = 3a^3 + a - 6a^2 - 2$

$\to \ 2a^3 - 6a^2 = 0 \quad \to 2a^2(a-3) = 0 \quad a_1 = 0 \to f(0) = 0$ *and* $a_2 = 3 \to f(3) = 30$

$\therefore y'|_{x=0,\ y=0} = 1$ *Tangent line eqn is* $y = x$

$\therefore y'|_{x=3,\ y=30} = 28$ *Tangent line eqn is* $y = 28(x-3) + 30$

Question 7

Find the equations of any lines that pass through the point $(-1,0)$ and are tangent to the curve $y = \frac{x-1}{x+1}$.

$0 \neq \dfrac{-1-1}{-1+1}$ *Thus*, $(-1,0) \notin curve\ y = \dfrac{x-1}{x+1}$ *let* $x = a$ *then the point is* $(a, f(a))$ *where* $x \neq -1$

The **slope** of the **tangent line** with the point $(a, f(a))$ must be $y' = \frac{(x-1)(1)-(x-1)(1)}{(x+1)^2} \ \ \therefore y'|_{x=a} = \frac{2}{(a+1)^2} \to (1)$

Because the line passes through the point $(-1,0)$, the slope of the tangent line is **also**:

$\frac{f(a)-0}{a+1} = \frac{\frac{a-1}{a+1}}{a+1} = \frac{a-1}{(a+1)^2} \rightarrow (2) \quad \because (1) = (2) \quad \therefore \frac{a-1}{(a+1)^2} = \frac{2}{(a+1)^2} \rightarrow a-1=2 \quad \rightarrow a=3$

$\therefore y'|_{a=3} = \frac{1}{8} \qquad Tangent\ line\ eqn\ is\ y = \frac{1}{8}(x+1)$

<u>Question8</u>

Let $f(x) = |x^3|$. Find $f'(0)$ by using the definition of derivative and find an equation of the tangent line to the graph of $f(x)$ at the point where $x = 0$.

$$f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{|x^3|-0}{x-0} = \lim_{x\to 0}\frac{|x^3|-0}{x-0} = \lim_{x\to 0}\frac{x^2|x|}{x} = \lim_{x\to 0} x|x| = 0$$

Section 5: Problems session

$Q1: Show\ that\ \lim_{x\to 119}(2x+1) = 239\ \ by\ using\ the\ formal\ definition\ of\ the\ limit.$

$For\ every\ \epsilon > 0, choose\ \delta = \frac{\epsilon}{2}\ such\ that\ if\ 0 < |x-119| < \delta = \frac{\epsilon}{2}, \quad then\ |2x+1-239| = |2x-238| = 2|x-119| < 2\frac{\epsilon}{2} = \epsilon$

$Q2: Use\ the\ definition\ of\ the\ limit\ to\ show\ that\ \lim_{x\to 1}(2x+119) = 121.$

$For\ every\ \epsilon > 0,\ choose\ \delta = \frac{\epsilon}{2}\ \ such\ that\ \ if\ 0 < |x-1| < \delta = \frac{\epsilon}{2}, \quad then\ |2x+119-121| = |2x-2| = 2|x-1| < 2\frac{\epsilon}{2} = \epsilon$

$Q3: Use\ the\ definition\ of\ the\ limit\ to\ prove\ that\ \lim_{x\to 2}(3x-1) = 5.$

$For\ every\ \epsilon > 0,\ choose\ \delta = \frac{\epsilon}{3}\ \ such\ that\ \ if\ 0 < |x-2| < \delta = \frac{\epsilon}{3},\ then\ |3x-1-5| = |3x-6| = 3|x-2| < 3\frac{\epsilon}{3} = \epsilon$

$Q4: Use\ the\ definition\ of\ the\ limit\ to\ prove\ that\ \lim_{x\to 0} x^3 = 0.$

$For\ every\ \epsilon > 0,\ choose\ \delta = \sqrt[3]{\epsilon}\ \ such\ that\ \ if\ 0 < |x-0| < \delta = \sqrt[3]{\epsilon}, \quad then\ |x^3-0| = |x^3| < (\sqrt[3]{\epsilon})^3 = \epsilon$

$Q5: Suppose\ that\ 4 \le f'(x) \le 6\ \ for\ all\ x \in \mathbb{R}\ and\ f(0) = 119.\ Show\ that\ 121 \le f\left(\frac{1}{2}\right) \le 122.$

Since f is differentiable for all $x \in \mathbb{R}$ we have, by MVT, $\frac{f\left(\frac{1}{2}\right)-f(0)}{\frac{1}{2}-0} = f'(c)\ \ for\ some\ c \in \left(0, \frac{1}{2}\right)$

$, \frac{f\left(\frac{1}{2}\right)-119}{\frac{1}{2}-0} = [4 \le f'(c) \le 6\ ] \qquad 4 \le \frac{f\left(\frac{1}{2}\right)-119}{\frac{1}{2}-0} \le 6 \qquad 121 \le f\left(\frac{1}{2}\right) \le 122$

$Q6: If\ f(1) = -2\ and\ 3 \le f'(x) \le 5\ \ for\ all\ x,\ \ Show\ that\ 1 \le f(2) \le 3.$

Since f is differentiable for all $x \in \mathbb{R}$ we have, by MVT, $\frac{f(2)-f(1)}{2-1} = f'(c)\ \ for\ some\ c \in (1,2\ )$

$, \frac{f(2)+2}{2-1} = [3 \le f'(c) \le 5\ ] \qquad 3 \le \frac{f(2)+2}{2-1} \le 5 \qquad 1 \le f(2) \le 3$

$Q7: Prove\ that\ \ln\ (1+x^2) \le 2x\ \ for\ every\ real\ number\ x \in (0,1).$

Let $f(x) = \ln\ (1+x^2) - 2x\ \ which\ is\ cont.on\ [0,1]\ and\ diff.on\ (0,1) \quad and\ f(0) = \ln 1 = 0$

For any $x \in (0,1\ ), since\ [0,x] \subseteq [0,1]\ , f\ is\ cont.on\ [0,x]\ and\ diff.on\ (0,x). Therefore,$

By MVT, we have $\frac{f(x)-f(0)}{x-0} = f'(c)\ \ for\ some\ c \in (0, x\ )\ \ note: f'(x) = \frac{2x}{1+x^2} - 2\ and\ \ f'(c) = \frac{2c}{1+c^2} - 2$

$$\frac{\ln\ (1+x^2)-2x-0}{x-0}=\frac{2c}{1+c^2}-2=\frac{\ln\ (1+x^2)}{x}-2$$

$\Rightarrow \frac{\ln\ (1+x^2)}{x}=\frac{2c}{1+c^2}=2\left(\frac{c}{1+c^2}\le 1\right)\le 2 \quad c\in(0,1)$ Thus, $\frac{\ln\ (1+x^2)}{x}\le 2 \quad \ln\ (1+x^2)\le 2x \quad x\in(0,1)$

$Q8$: *Prove that* $\ln\ x<x-1$ *for all* $x>1$.

Let $f(x)=\ln\ x$ *which is cont. on* $[1,x]$ *and diff. on* $(1,x)$ *and* $f(1)=0$

By MVT, we have $\frac{f(x)-f(1)}{x-1}=f'(c)$ *for some* $c\in(1,x)$ *note*: $f'(x)=\frac{1}{x}$ *and* $f'(c)=\frac{1}{c}$

$\frac{\ln\ x-0}{x-1}=\frac{1}{c}$ *since* $\left(\frac{1}{c}\right)<1$ *then* $\frac{\ln\ x-0}{x-1}<1$ Thus, $\ln\ x<x-1$ *for all* $x>1$

$Q9$: *If* $f:\mathbb{R}\to\mathbb{R}$ *satisfies* $f'(x)=e^{-x^2}$ *for all* x *and* $f(0)=0$ *then prove that* $|f(x)|\le|x|$ *for all* x.

$f'(x)=e^{-x^2}$ *for all* x *which means* $f(x)$ *is cont. on* $[0,x]$ *and diff. on* $(0,x)$ *for* $x>0$ *and* $f(0)=0$ *and for* $x<0$ $f(x)$ *is also cont. on* $[x,0)$ *and diff. on* $(x,0)$

By MVT, we have $\frac{f(x)-f(0)}{x-0}=f'(c)$ *for some* $c\in(0,x)\ \&\ (x,0)$ *note*: $f'(x)=e^{-x^2}$ *and* $f'(c)=e^{-c^2}$

$$\frac{f(x)-0}{x-0}=e^{-c^2}=\frac{f(x)}{x} \quad \text{since} \begin{cases} c^2>0 \\ c\ne 0 \\ e^0=1 \end{cases} \text{then } 0<e^{-c^2}<1 \quad \text{thus} \quad 0<\frac{f(x)}{x}<1 \quad \text{so } 0<f(x)<x$$

$|f(x)|<|x|$ *for all* x *when* $x=0$ *so* $|f(0)|=|0|=0$ *thus* $|f(x)|\le|x|$ *for all* x

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$Q10$: *Show that the equation* $x^3+x^2+2x+5=0$ *has a root, but no more than one.*

Let $f(x)=x^3+x^2+2x+5$ *then* $f(x)$ *is cont. and diff. for all* x

$\begin{cases} f(0)=5>0 \\ f(-2)=-3<0 \end{cases}$ *thus by IVT* $f(x)$ *is cont. on* $[-2,0]$ *and there is a point* $c_1\in(-2,0)$

such that $f(c_1)=0$ *which means* $x=c_1$ *is a root of* $f(x)$

$f'(x)=3x^2+2x+2\ge 0$ *for all* x *so* $f(x)$ *is a strictly increasing function and hence one to to one*

Therefore, the equation $f(x)$ has at most one solution. $f(x)$ *has only one root which is* c_1

$Q11$: *Prove that the equation* $\ln(x)=\frac{1}{x}$ *for* $x>0$ *has a unique solution. Explain.*

Let $f(x)=\ln(x)-\frac{1}{x}$ *then* $f(x)$ *is cont. and diff. for all* $x>0$

$$\begin{cases} f(e)=1-\frac{1}{e}>0 \\ f\left(\frac{1}{e}\right)=-1-e<0 \end{cases} \text{thus by IVT } f(x) \text{ is cont. on } \left[e,\frac{1}{e}\right] \text{ and there is a point } c_1\in\left(e,\frac{1}{e}\right)$$

such that $f(c_1)=0$ *which means* $x=c_1$ *is a a solution of* $f(x)$

$f'(x)=\frac{1}{x}+\frac{1}{x^2}\ge 0$ *for all* $x>0$ *so* $f(x)$ *is a strictly increasing function and hence one to to one*

Therefore, the equation $f(x)$ has at most one solution. $f(x)$ *has a unique solution which is* c_1

$Q12: Show\ that\ the\ equation\quad \arctan x = 2 - x - x^3\ \ has\ exactly\ ONE\ soluation.$

$Let\ f(x) = \arctan x - 2 + x + x^3\ \ then\ f(x)\ is\ cont.\ and\ diff.\ for\ all\ x$

$$\begin{cases} f(1) = \frac{\pi}{4} > 0 \\ f(0) = -2 < 0 \end{cases}\ \ thus\ by\ IVT\ \ f(x)\ is\ cont. on\ [0,1]\ and\ there\ is\ a\ point\ c_1 \in (0,1)$$

$such\ that\ f(c_1) = 0\quad which\ means\ x = c_1\ \ is\ a\ a\ solution\ of\ f(x)$

$f'(x) = \frac{1}{x^2+1} + 1 + 3x^2 \geq 0\ for\ all\ x\ \ so\ f(x)\ is\ a\ strictly\ increasing\ function\ and\ hence\ one\ to\ to\ one$

Therefore, the equation $f(x)$ has at most one solution. $f(x)\ has\ a\ unique\ solution\ which\ is\ c_1$

$Q13: let\ f\ be\ a\ continuous\ function\ on\ [0,2]\ and\ assume\ that\ f'\ and\ f''\ exists\ on\ (0,2).$

$a)\ If\ f(0) = 0\ and\ f(1) = 1\ \ then\ \ show\ that\ there\ exists\ a\ number\ a \in (0,1)\ such\ that\ f'(a) = 1.$

Since $f(x)\ is\ cont. on\ [0,1]\ and\ diff. on\ (0,1)\ \ and\ f(0) = 0\ \&\ f(1) = 1\ \ \ Thus, by\ MVT$

we have $\frac{f(1)-f(0)}{1-0} = f'(a) = \frac{1-0}{1-0} = 1\quad for\ some\ a \in (0,1)$

$b)\ In\ addition\ conditions\ in\ part\ (a)\ if\ f(2) = 2\ show\ that\ there\ exists\ a\ number\ c \in (0,2)\ such\ that\ f''(c) = 0$

Since $f(x)\ is\ cont. on\ [1,2]\ and\ diff. on\ (1,2)\ \ and\ f(2) = 2\ \ \ Thus, by\ MVT$

we have $\frac{f(2)-f(1)}{2-1} = f'(b) = \frac{2-1}{2-1} = 1\quad for\ some\ b \in (1,2)$

$Note\ that\ a \in (0,1)\ and\ b \in (1,2)\ so\ a \neq b\quad and\ also\ f'(a) = f'(b) = 1$

Since $f'(x)\ is\ cont. on\ [a,b]\ and\ diff.\ on\ (a,b)\ \ \ Thus, by\ Rolle`s\ Thm\ we\ have\ f''(c) = 0\ \ c \in (a,b) \subseteq (0,2)$

$Q14: Consider\ the\ function\ f(x) = e^x + 2x + x^4.$

$(a)\ Find\ the\ line\ of\ \boldsymbol{tangent}\ to\ the\ graph\ of\ \ y = f(x)\ at\ the\ point\ P(0,1).$

$f'(x) = e^x + 2 + 4x^3\ \ so\ the\ slope\ of\ the\ tangent\ line\ at\ (0,1)\ is\ f'(0) = 3\ \ so\ tangent\ line\ is\ \ y = 3(x-0) + 1$

$(b)\ Show\ that\ there\ exists\ a\ point\ Q(x,y)\ on\ \ the\ graph\ of\ \ y = f(x)\ at\ which\ the\ \boldsymbol{tangent}\ line\ is$

$\boldsymbol{parallel}\ to\ the\ line\ y = 2x + 1.\ \ (Do\ NOT\ find\ the\ point\ Q.)$

At such a point, slopes must be equal. Therefore, $f'(x) = 2\ \ and\ f'(x) = e^x + 2 + 4x^3\ cont.\ everywhere$

$$\begin{cases} f'(0) = 3 > 2 \\ f'(-1) = \frac{1}{e} + 2 - 4 < 2 \end{cases}\quad thus,\ \ By\ IVT\ \ f'(x)\ cont. [-1,0]\ \ then\ there\ exists\ -1 < x < 0\ such\ that\ \ f'(x) = 2$$

$(c)\ In\ \boldsymbol{Part}\ (\boldsymbol{b}), show\ that\ Q\ is\ unique, i.e. there\ is\ ONLY\ ONE\ such\ point.$

$f''(x) = e^x + 12x^2 \geq 0\ \ for\ all\ x\ so\ f'(x)\ is\ a\ strictly\ increasing\ function\ and\ hence\ one\ to\ to\ one$

Consequently, there can be only one point at which we can have $f'(x) = 2$

$Q15: show\ that\qquad e\ln x\ \leq x\quad for\ all\ x > 0.$

Let $f(x) = e\ln x - x\quad x \in (0,\infty)\qquad f'(x) = \frac{e}{x} - 1$

$f'(x) < 0 \ for\ all\ x \in (0, e) \ \Rightarrow f\ is\ decreasing\ \ and\ \ f'(x) > 0\ for\ all\ x \in (e, \infty) \ \Rightarrow f\ is\ increasing$

$x \in (0, e) \qquad f(x) > f(e) \quad \Rightarrow \quad x - e \ln x > 0 \quad \Rightarrow e \ln x < x$

$x \in (e, \infty) \qquad f(x) > f(e) \quad \Rightarrow \quad x - e \ln x > 0 \quad \Rightarrow e \ln x < x$

$x = e \qquad f(e) = f(e) \quad Thus \qquad e \ln x \ \leq x \ \ for\ all\ x > 0$

Section 6: Problems session

Q1: The height of a rectangular box decreases at 1 $cm\ per\ hour$, its width decreases at 0.5 $cm\ per\ hour$ and its length decreases at 2 $cm\ per\ hour$. When the shape of the box become a cube, its volume decreases at the rate of 14 $cm^3\ per\ hour$. Find the volume of the box when it **becomes a cube**.

Volume of a box is given by $V(t) = x(t) * \ y(t) * z(t)$

$$\frac{dV}{dt} = \frac{dx}{dt} y(t) * z(t) + x(t) * \frac{dy}{dt} * z(t) + x(t) * \ y(t) * \frac{dz}{dt} \ \rightarrow (1) \ \ when\ x(t) = \ y(t) = z(t) = a$$

$$and\ \frac{dV}{dt} = -14\ cm^3/h \ \ and\ \frac{dx}{dt} = -1\ cm/h \quad \frac{dy}{dt} = -0.5\ cm/h \quad \frac{dz}{dt} = -2\ cm/h$$

$Step\ (1): \ -14 = -1 * a * a - 0.5 * a * a - 2 * a * a = -3.5\, a^2 \ \ so\ a^2 = 4 \ \rightarrow \ \ a = 2\ cm$

When it is a cube, we have the volume $V = 2 * 2 * 2 = 8\ cm^3$

Q2: A point is moving to the right along the first-quadrant portion of the curve $x^2y^3 = 72$. When the point has coordinates (3,2), its horizontal velocity is 2 $unites/second$.

What is the rate of change in the distance of the particle to the origin?

Let $x = x(t)$ and $y = y(t)$ be coordinate functions parametrized by time t. So that $x^2y^3 = 72$ and

$\left.\frac{dx}{dt}\right|_{(3,2)} = 2\ unit/sec$ let $D(t) = \sqrt{x^2 + y^2}$ be the distance of the particle to the origin. $(D \geq 0)$

To find $\left.\frac{dD}{dt}\right|_{(3,2)}$ we need to $\frac{d}{dt}(x^2y^3) = \frac{d}{dt}(72) \ \rightarrow \ 2xy^3\frac{dx}{dt} + 3xy^2\frac{dy}{dt} = 0 \ \ then\ \left.\frac{dy}{dt}\right|_{(3,2)} = -\frac{8}{9}\ unit/sec$

$$D^2 = x^2 + y^2 \ \ so \ \ \frac{d}{dt}(D^2) = \frac{d}{dt}(x^2 + y^2)\ \ thus\ 2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \ \ then \ \left.\frac{dD}{dt}\right|_{(3,2)} = \frac{38}{9\sqrt{13}}\ unit/sec$$

Q3: A point (x, y) is moving on the curve $y = e^{x^2}$ so that its $x - coordinate$ is decreasing at rate of 2 $units/sec$. Find the rate of change of the slope of the tangent line to the curve $y = e^{x^2}$ at the moment when $x = 0$.

Let m be the slope. Then, $m = \frac{dy}{dx} = 2xe^x \quad so\ \frac{dm}{dt} = 2\left(e^{x^2} + 2x^2e^{x^2}\right)\frac{dx}{dt} = 2e^{x^2}(1 + 2x^2)(-2)$

$when\ x = 0 \ \ \frac{dm}{dt} = -4\ unit/\sec^2$ (the slope is decreasing at a rate of 4 $unit/sec^2$)

Q4: If the short side of a rectangle is increasing at a rate of 3 $cm\ per\ second$ and the long side is decreasing at a rate of 2 $cm\ per\ second$. Determine how fast the area of the rectangle is changing when the short side is 4 cm and the long side is 6 cm.

The area of rectangle is $A(t) = s(t) * l(t)\ \ for\ time\ t \quad Note\colon \frac{ds}{dt} = +3\ cm/\sec \quad and \ \ \frac{dl}{dt} = -2\ cm/\sec$

$\frac{dA}{dt} = \frac{ds}{dt}l(t) + \frac{dl}{dt}s(t) \ \ when\ s = 4\ and\ l = 6 \ \ thus\ \frac{dA}{dt} = 3 * 6 + 4 * -2 = 10\ cm^2/sec$ (the area is increasing)

Q5: A particle moves along the curve $y = 4x - x^3$ and it`s $x - coordinate$ is decreasing at a constant rate of $\frac{1}{3}$ $unites\ per\ second$.

a) At what rate does the $y - coordinate$ of the particle change when $x = 2$?

When time is equal to t, particle is at the point $(x(t), y(t))$ where $y(t) = 4x(t) - x^3(t)$ $\because \frac{dx}{dt} = \frac{-1}{3}\ unit/sec$

When $x = 2$ then $y = 4 * 2 - 2^3 = 0$ and $y'(t) = 4x'(t) - 3x^2(t)\, x'(t)$ $Thus, y'(t) = +\frac{8}{3}\ unit\ /sec$

b) At what rate does the slope of the tangent line change when $x = 2$?

Slope $= y' = \frac{dy}{dt} = 4 - 3x^2$ $\frac{dy'}{dt} = -6\, x(t)x'(t)$ $\left.\frac{dy'}{dt}\right|_{x(t)=2} = -6 * 2 * \frac{-1}{3} = +4\ unit/sec$

Q6: Consider the right triangle ABC at the angle C. It`s side-lengths $a, b, c$ are changing in such a way that it`s area is always $16\ cm^2$. If the side-length a is increasing at a rate of $1/2\ cm/sec$ when a is $8\ cm$, at the rate $(radian/sec)$ is the measure of θ the angle $\hat{B}$ changing at that moment?

$$Area = A(t) = \frac{a(t) * b(t)}{2} = 16\ cm^2\ at\ time\ t \qquad so\ we\ need\ to\ find\ \left.\frac{d\theta(t)}{dt}\right|_{a=8} = ?$$

Since $A(t) = 16\ cm^2$ $thus\ b(t) = \frac{32}{a(t)}$ and $\tan\theta(t) = \frac{b(t)}{a(t)} = \frac{32}{a^2(t)}$ $When\ we\ diff.\ wrt\ time\ t,\ we\ have$

$$\sec^2\theta(t)\,\theta'(t) = \frac{-64}{a^3(t)}a'(t) \qquad we\ have\ \ a(t) = 8,\ \ a'(t) = \frac{1}{2},\ \ b(t) = \frac{32}{8} = 4\ \ and\ \ \sec\theta(t) = \frac{\sqrt{80}}{8}$$

$$\frac{80}{64}\,\theta'(t) = \frac{-64}{8^3} * \frac{1}{2} \qquad so\theta'(t)|_{a=8} = \left.\frac{d\theta(t)}{dt}\right|_{a=8} = \frac{-1}{20}\ \ radian\ /sec$$

Q7: A particle is moving around the ellipse $4x^2 + 16y^2 = 64$. At any time, t, its $x -$ and $y - coordinates$ are given by $x(t) = 4cos\ t$ and $y(t) = 2sin\ t$. At what rate is the particle`s distance to the point (2,0) changing at any time t? $(\rightarrow part\ I)$ At what rate is the distance changing when $t = \frac{\pi}{2}$? $(\rightarrow part\ II)$

$s(t)$: distance from particle to point (2,0) at any time t

$$s^2 = (x-2)^2 + (y-0)^2, by\ diff.wrt\ t\ we\ have\ \ 2s\frac{ds}{dt} = 2(x-2)\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{ds}{dt} = \frac{(x-2)}{s}\frac{dx}{dt} + \frac{y}{s}\frac{dy}{dt}\ since\ \begin{cases} x(t) = 4\cos t \quad then\ \frac{dx}{dt} = -4\sin t \\ y(t) = 2\sin t \quad then\ \frac{dy}{dt} = 2\cos t \end{cases}$$

$$\frac{ds}{dt} = \frac{(4\cos t - 2)}{s}(-4\sin t) + \frac{2\sin t}{s}(2\cos t) = \frac{1}{s}(8\sin t - 6\sin 2t) \qquad \rightarrow part\ I$$

At $t = \frac{\pi}{2}$ $\left.s^2\right|_{t=\frac{\pi}{2}} = \left.(x-2)^2 + (y)^2\right|_{t=\frac{\pi}{2}} = \left.(4\cos t - 2)^2 + (2\sin t)^2\right|_{t=\frac{\pi}{2}} = 8$ $thus\ s = 2\sqrt{2}$

$$\left.\frac{ds}{dt}\right|_{t=\frac{\pi}{2}} = \frac{1}{s}(8\sin t - 6\sin 2t) = \frac{1}{2\sqrt{2}}\left(8\sin\frac{\pi}{2} - 6\sin\pi\right) = 2\sqrt{2}\ \ unit/sec$$

Evaluate each of the the following limits if it exists :

1) $\lim\limits_{x\to 0}\frac{e^x-1-x}{x^2}\left(\frac{0}{0}\right) =^{(L`H)} \lim\limits_{x\to 0}\frac{e^x-1}{2x}\left(\frac{0}{0}\right) =^{(L`H)} \lim\limits_{x\to 0}\frac{e^x}{2} = \frac{e^0}{2} = \frac{1}{2}$

2) $\lim\limits_{x\to\infty}\frac{(\ln x)^2}{\sqrt{x}}\left(\frac{\infty}{\infty}\right) =^{(L`H)} \lim\limits_{x\to\infty}\frac{2(\ln x)\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = 4\lim\limits_{x\to\infty}\frac{(\ln x)}{\sqrt{x}}\left(\frac{\infty}{\infty}\right) =^{(L`H)} 4\lim\limits_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = 8\lim\limits_{x\to\infty}\frac{1}{\sqrt{x}} = 0$

3) $\lim\limits_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right) = \lim\limits_{x\to 1}\frac{x-1-\ln x}{(x-1)\ln x}\left(\frac{0}{0}\right) =^{(L`H)} \lim\limits_{x\to 1}\frac{1-\frac{1}{x}}{\ln x+(x-1)\frac{1}{x}} = \lim\limits_{x\to 1}\frac{x-1}{x\ln x+x-1}\left(\frac{0}{0}\right) =^{(L`H)} \lim\limits_{x\to 1}\frac{1}{\ln x+1+1} = \frac{1}{2}$

4) $\lim\limits_{x\to\infty}(e^x-x^2) = \lim\limits_{x\to\infty}e^x\left(1-\frac{x^2}{e^x}\right) = \infty * 1 = \infty \;\; since \lim\limits_{x\to\infty}\frac{x^2}{e^x}\left(\frac{\infty}{\infty}\right) =^{(L`H)} \lim\limits_{x\to\infty}\frac{2x}{e^x}\left(\frac{\infty}{\infty}\right) =^{(L`H)} \lim\limits_{x\to\infty}\frac{2}{e^x} = 0$

5) $\lim\limits_{x\to 0^+}(\sin x)^{\frac{2}{Inx}} \overset{*}{=} e^{\lim\limits_{x\to 0^+}\frac{2}{Inx}\ln\sin x} = e^2 \;\; where \lim\limits_{x\to 0^+}\frac{2\ln\sin x}{Inx}\left(\frac{\infty}{\infty}\right) =^{(L`H)} \lim\limits_{x\to 0^+}\frac{\frac{2\cos x}{\sin x}}{\frac{1}{x}} = \lim\limits_{x\to 0^+}2\frac{x}{\sin x}\cos x = 2$

* where e^x is continues

6) $\lim\limits_{x\to\infty}\left(\cos\left(\frac{2}{x}\right)\right)^x \overset{*}{=} e^{\lim\limits_{x\to\infty}x\ln\cos\left(\frac{2}{x}\right)} = e^0 = 1 \;\; where \lim\limits_{x\to\infty}\frac{\ln\cos\left(\frac{2}{x}\right)}{\frac{1}{x}}\left(\frac{0}{0}\right) =^{(L`H)} \lim\limits_{x\to\infty}\frac{\frac{-\sin\left(\frac{2}{x}\right)}{\cos\left(\frac{2}{x}\right)}\left(\frac{-2}{x^2}\right)}{\frac{-1}{x^2}} = \lim\limits_{x\to\infty}\frac{-2\sin\left(\frac{2}{x}\right)}{\cos\left(\frac{2}{x}\right)} = 0$

* where e^x is continues

7) $\lim\limits_{x\to\left(\frac{\pi}{2}\right)^-}(\tan x)^{\cos x} \overset{*}{=} e^{\lim\limits_{x\to\left(\frac{\pi}{2}\right)^-}\cos x\ln\tan x} = e^0 = 1 \;\; where \lim\limits_{x\to\left(\frac{\pi}{2}\right)^-}\frac{\ln\tan x}{\sec x}\left(\frac{\infty}{\infty}\right) =^{(L`H)} \lim\limits_{x\to\left(\frac{\pi}{2}\right)^-}\frac{\frac{\sec^2 x}{\tan x}}{\sec x\,\tan x}$

$\lim\limits_{x\to\left(\frac{\pi}{2}\right)^-}\frac{\sec x}{\tan^2 x} = \lim\limits_{x\to\left(\frac{\pi}{2}\right)^-}\frac{1}{\cos x}*\frac{\cos^2 x}{\sin^2 x} = \lim\limits_{x\to\left(\frac{\pi}{2}\right)^-}\frac{\cos x}{\sin^2 x} = 0$ * where e^x is continues

8) $\lim\limits_{x\to 0^+}(1+4x+x^2)^{\frac{2}{x}} \overset{*}{=} e^{\lim\limits_{x\to 0^+}\frac{2}{x}\ln(1+4x+x^2)} = e^8 \;\; where \lim\limits_{x\to 0^+}\frac{2\ln(1+4x+x^2)}{x}\left(\frac{0}{0}\right) =^{(L`H)} \lim\limits_{x\to 0^+}\frac{2(4+2x)}{1+4x+x^2} = 8$

* where e^x is continues

9) $\lim\limits_{x\to 0^+}(1+\sin x)^{\cot x} \overset{*}{=} e^{\lim\limits_{x\to 0^+}\cot x\ln(1+\sin x)} = e^1 = e \;\; where \lim\limits_{x\to 0^+}\frac{\ln(1+\sin x)}{\tan x}\left(\frac{0}{0}\right) =^{(L`H)} \lim\limits_{x\to 0^+}\frac{\frac{\cos x}{1+\sin x}}{\sec^2 x}$

$= \lim\limits_{x\to 0^+}\frac{\cos^3 x}{1+\sin x} = 1$ * where e^x is continue

10) $\lim_{x\to 0} \frac{1-\cos\left(\frac{\sin^2 x}{x}\right)}{\left(\frac{\sin^2 x}{x}\right)^2}$ $let\ u = \frac{\sin^2 x}{x};\ \lim_{x\to 0} u = \lim_{x\to 0} \sin x \frac{\sin x}{x} = 0 * 1 = 0\ \ Thus, \lim_{u\to 0} \frac{1-\cos(u)}{(u)^2} \left(\frac{0}{0}\right) =^{(L`H)} \lim_{u\to 0} \frac{\sin(u)}{2\,u} = \frac{1}{2}$

11) $\lim_{x\to\infty} \frac{5^x-2^x}{5^{x+1}+2^x} = \lim_{x\to\infty} \frac{1-\frac{2^x}{5^x}}{5+\frac{2^x}{5^x}} = \lim_{x\to\infty} \frac{1-\left(\frac{2}{5}\right)^x}{5+\left(\frac{2}{5}\right)^x} = \frac{1}{5}$ $\quad where\ \lim_{x\to\infty} \left(\frac{2}{5}\right)^x = 0\ \ and\ \ \frac{2}{5} < 1$

12) $\lim_{x\to\infty} x\,(2\tan^{-1} x - \pi) = \lim_{x\to\infty} \frac{2\tan^{-1} x - \pi}{\frac{1}{x}} \left(\frac{0}{0}\right) =^{(L`H)} \lim_{x\to\infty} \frac{\frac{2}{1+x^2}}{\frac{-1}{x^2}} = \lim_{x\to\infty} \frac{-2\,x^2}{1+x^2} = \lim_{x\to\infty} \frac{-2}{\frac{1}{x^2}+1} = -2$

13) $\lim_{x\to\infty} \frac{119^x - e^x}{119^{x+1} + e^x} = \lim_{x\to\infty} \frac{1 - \frac{e^x}{119^x}}{119 + \frac{e^x}{119^x}} = \lim_{x\to\infty} \frac{1 - \left(\frac{e}{119}\right)^x}{119 + \left(\frac{e}{119}\right)^x} = \frac{1}{119}$ $\quad where\ \lim_{x\to\infty} \left(\frac{e}{119}\right)^x = 0\ \ and\ \ \frac{e}{119} < 1$

14) $\lim_{x\to 1^+} \left(\tan\left(\frac{\pi x}{4}\right)\right)^{\left(\frac{1}{1-x}\right)} \overset{*}{=} e^{\lim_{x\to 1^+}\left(\frac{1}{1-x}\right)\ln\left(\tan\left(\frac{\pi x}{4}\right)\right)} = e^{-\frac{\pi}{2}}$

$where\ \lim_{x\to 1^+} \left(\frac{1}{1-x}\right)\ln\left(\tan\left(\frac{\pi x}{4}\right)\right) = \lim_{x\to 1^+} \frac{\ln\left(\tan\left(\frac{\pi x}{4}\right)\right)}{1-x} \left(\frac{0}{0}\right) =^{(L`H)} \lim_{x\to 1^+} \frac{\frac{\frac{\pi}{4}\sec^2\left(\frac{\pi x}{4}\right)}{\tan\left(\frac{\pi x}{4}\right)}}{-1} = -\frac{\pi}{2}$

* where e^x is continues

15) $\lim_{x\to 0^+} \frac{x - \sin(x)}{(1-\cos(x))^{3/2}} \left(\frac{0}{0}\right) =^{(L`H)} \lim_{x\to 0^+} \frac{1-\cos(x)}{\frac{3}{2}(1-\cos(x))^{1/2}(\sin x)} = \frac{2}{3} \lim_{x\to 0^+} \frac{(1-\cos(x))^{1/2}}{(\sin x)}$

$= \frac{2}{3} \lim_{x\to 0^+} \frac{(1-\cos(x))^{1/2}}{(\sin x)} * \frac{(1+\cos(x))^{\frac{1}{2}}}{(1+\cos(x))^{\frac{1}{2}}} = \frac{2}{3} \lim_{x\to 0^+} \frac{(1-\cos^2(x))^{\frac{1}{2}}}{(\sin x)(1+\cos(x))^{\frac{1}{2}}} = \frac{2}{3} \lim_{x\to 0^+} \frac{|\sin x|}{(\sin x)(1+\cos(x))^{\frac{1}{2}}}$

$= \frac{2}{3} \lim_{x\to 0^+} \frac{1}{(1+\cos(x))^{\frac{1}{2}}} = \frac{2}{3}\left(\frac{1}{\sqrt{2}}\right) = \frac{\sqrt{2}}{3}$

Q2: *let* $f(x) = \dfrac{x^2-2}{(x-1)^2}$ *. You are given that* $f'(x) = \dfrac{4-2x}{(x-1)^3}$ *and* $f''(x) = \dfrac{4x-10}{(x-1)^4}$

- ***Find the domain of*** $\boldsymbol{f(x)}$

The domain of $f(x)$ *is* $\mathbb{R} - \{1\}$ *hint*: $(x-1)^2 \neq 0 \ \therefore x-1 \neq 0 \ \therefore x \neq 1$

- ***Find both the x and y intercepts of*** $\boldsymbol{f(x)}$

x intercept: $\dfrac{x^2-2}{(x-1)^2} = 0 \ \rightarrow \ x^2-2 = 0 \ \rightarrow \ x^2 = 2 \ \rightarrow \ x = \pm\sqrt{2}$ *x intercept point is* $(\pm\sqrt{2}, 0)$

y intercepts: $f(0) = \dfrac{0^2-2}{(0-1)^2} = -2 \ \rightarrow \ f(0) = -2$ *y intercept point is* $(0, -2)$

- ***Determine all asymptotes of*** $\boldsymbol{f(x)}$.

Vertical asymp. $\lim_{x\to 1^+} f(x) = -\infty$ & $\lim_{x\to 1^-} f(x) = -\infty$ $\therefore x = 1$ *is Vertical asymp.*

Horizontal asymp. $\lim_{x\to \infty} f(x) = 1$ & $\lim_{x\to -\infty} f(x) = 1$ $\therefore y = 1$ *is Horizontal asymp.*

There is no Oblique asymp.

- ***Find all local maximum and local minimum of*** $\boldsymbol{f(x)}$; ***and determine intervals of increase and decrease*** .

$f'(x) = \frac{4-2x}{(x-1)^3}$ *when* $f'(x) = 0 \ \therefore \frac{4-2x}{(x-1)^3} = 0 \ \rightarrow \ 4-2x = 0 \ \rightarrow$ *at* $x = 2$

$f(x)$ *has a critical point. There is no singular point.*

$f(x)$ *is increasing on* $(1,2]$

$f(x)$ *is decreasing on* $(-\infty, 1)$ & $[2, \infty)$

$f(x)$ *has local* max *point at* $x = 2$ & $f(x)$ *has no local* min *point* *The critical point is* $(2, f(2))$

x	1	2	
$f'(x)$	$-$	$+$	$-$
$f(x)$	↘	↗	↘

- ***Determine all inflection point(s) and the intervals where*** $\boldsymbol{f(x)}$ ***concave up and concave down***.

$f''(x) = \dfrac{4x-10}{(x-1)^4}$ *when* $f''(x) = 0 \ \therefore \dfrac{4x-10}{(x-1)^4} = 0 \ \rightarrow \ 4x-10 = 0$

at $x = \dfrac{5}{2}$ $f(x)$ *has an inflection point*

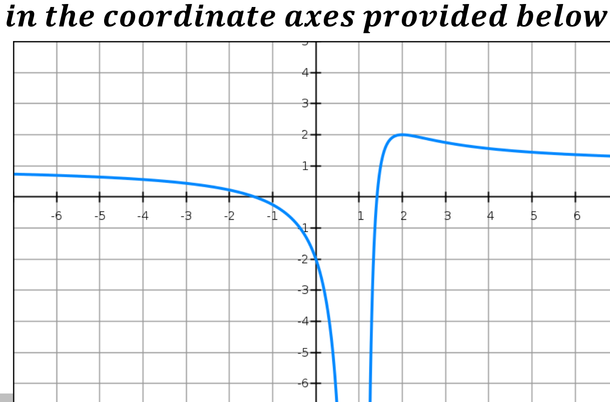
$f(x)$ *is concaving up on* $(-1,1) \cup \left(\frac{5}{2}, \infty\right)$

$f(x)$ *is concaving down on* $\left(1, \frac{5}{2}\right) \cup (-\infty, 1)$ *The inflection point is* $\left(\frac{5}{2}, f\left(\frac{5}{2}\right)\right)$

x	1	$\frac{5}{2}$	
$f''(x)$	$-$	$-$	$+$
$f(x)$	$\cap$	$\cap$	$\cup$

- ***Using the parts above, sketch the graph*** $\boldsymbol{y = f(x)}$ ***in the coordinate axes provided below***.

x	1	2	$\frac{5}{2}$	
$f'(x)$	$-$	$+$	$-$	$-$
$f''(x)$	$\cap$	$\cap$	$\cap$	$\cup$
$f(x)$				



Q3: $let\ f(x) = (x-3)^2\, e^x$. $You\ are\ given\ that\ f'(x) = e^x(x-3)(x-1)\ and\ f''(x) = e^x(x^2-2x-1)$

- ***Find the domain of*** $\boldsymbol{f(x)}$

The domain of $f(x)$ *is* $\mathbb{R}$

- ***Find both the x and y intercepts of*** $\boldsymbol{f(x)}$

$x\ intercept$: $(x-3)^2\, e^x = 0 \rightarrow (x-3)^2 = 0 \rightarrow x-3=0 \rightarrow x = 3$ $\quad x\ intercept\ point\ is (3,0)$

$y\ intercepts$: $f(0) = (0-3)^2\, e^0 = 9 \rightarrow f(0) = 9$ $\quad y\ intercept\ point\ is\ (0,9)$

- ***Determine all asymptotes of*** $\boldsymbol{f(x)}$.

There is no Vertical asymp. since the domain of $f(x)$ *is* $\mathbb{R}$

Horizontal asymp. $\lim_{x\to\infty} f(x) = \infty$ & $\lim_{x\to-\infty} f(x) = 0$ $\quad \therefore y = 0$ *is ONLY Horizontal asymp. on the left*

There is no Oblique asymp.

- ***Find all local maximum and local minimum of*** $\boldsymbol{f(x)}$; ***and determine intervals of increase and decrease*** .

$f'(x) = e^x(x-3)(x-1)$ *when* $f'(x) = 0 \ \therefore\ e^x(x-3)(x-1) = 0 \rightarrow (x-3)(x-1) = 0 \rightarrow$ *at* $x = 1\ \&\ 3$
$f(x)$ *has a critical point. There is no singular point.*

x	1	3	
$f'(x)$	$+$	$-$	$+$
$f(x)$	↗	↘	↗

$f(x)$ *is increasing on* $(-\infty, 1] \cup [3,\infty)$

$f(x)$ *is decreasing on* $[1,3]$

$f(x)$ *has no local* max *point at* $x = 3$ & $f(x)$ *has local* min *point at* $x = 1$
The critical point are $\big(1, f(1)\big)$ *and* $\big(3, f(3)\big)$

- ***Determine all inflection point(s) and the intervals where*** $\boldsymbol{f(x)}$ ***concave up and concave down***.

$f''(x) = e^x(x^2-2x-1)$*when* $f''(x) = 0 \ \therefore\ e^x(x^2-2x-1) = 0 \rightarrow (x^2-2x-1) = 0 \qquad x = \dfrac{2\pm\sqrt{8}}{2}$

at $x = 1\pm\sqrt{2}$ $f(x)$ *has an inflection point*

x	$1-\sqrt{2}$	$1+\sqrt{2}$	
$f''(x)$	$+$	$-$	$+$
$f(x)$	$\cup$	$\cap$	$\cup$

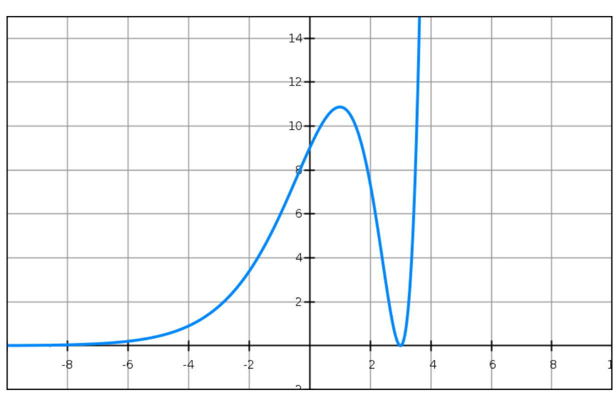
$f(x)$ *is concaving up on* $(-\infty, 1-\sqrt{2}) \cup \big(1+\sqrt{2},\infty\big)$

$f(x)$ *is concaving down on* $(1-\sqrt{2},\ 1+\sqrt{2})$

The inflection point is $(-1, f(-1))$

- ***Using the parts above, sketch the graph*** $\boldsymbol{y = f(x)}$ ***in the coordinate axes provided below***.

x	$1-\sqrt{2}$	1	$1+\sqrt{2}$	3	
$f'(x)$	$+$	$+$	$-$	$-$	$+$
$f''(x)$	$\cup$	$\cap$	$\cap$	$\cup$	$\cup$
$f(x)$					



$\boldsymbol{Q4}$: $let\ f(x) = \dfrac{x^3-1}{x^3+1}$. $You\ are\ given\ that\ f'(x) = \dfrac{6x^2}{(x^3+1)^2}$ $and\ f''(x) = \dfrac{12x(1-2x^3)}{(x^3+1)^3}$

- ***Find the domain of $f(x)$***

The domain of $f(x)$ *is* $\mathbb{R} - \{-1\}$ $\qquad$ $hint: x^3 + 1 \neq 0 \ \therefore x^3 \neq -1 \ \therefore x \neq -1$

- ***Find both the x and y intercepts of $f(x)$***

$x\ intercept: \dfrac{x^3-1}{x^3+1} = 0 \rightarrow x^3 - 1 = 0 \rightarrow x^3 = 1 \rightarrow x = 1$ $\qquad$ $x\ intercept\ point\ is (1,0)$

$y\ intercepts: f(0) = \dfrac{0^3-1}{0^3+1} = -1 \rightarrow f(0) = -1$ $\qquad$ $y\ intercept\ point\ is\ (0,-1)$

- ***Determine all asymptotes of $f(x)$.***

$Vertical\ asymp.$ $\quad \lim_{x\to -1^+} f(x) = -\infty \quad \& \quad \lim_{x\to -1^-} f(x) = \infty \quad \therefore x = -1\ is\ Vertical\ asymp.$

$Horizontal\ asymp.$ $\quad \lim_{x\to\infty} f(x) = 1 \quad \& \quad \lim_{x\to-\infty} f(x) = 1 \quad \therefore y = 1\ is\ Horizontal\ asymp.$

There is no Oblique asymp.

- ***Find all local maximum and local minimum of $f(x)$; and determine intervals of increase and decrease .***

$f'(x) = \dfrac{6x^2}{(x^3+1)^2}$ $when\ f'(x) = 0 \ \therefore \dfrac{6x^2}{(x^3+1)^2} = 0 \rightarrow 6x^2 = 0 \rightarrow at\ x = 0$

$f(x)$ *has a critical point. There is no singular point.*

$f(x)$ *is always increasing on* $(-\infty,-1) \cup (-1,\infty)$

$f(x)$ *has no both local* max *point and local* min $\qquad$ *The critical point is* $(-1, f(-1))$

x	-1	0	
$f'(x)$	$+$	$+$	$+$
$f(x)$	↗	↗	↗

- ***Determine all inflection point(s) and the intervals where $f(x)$ concave up and concave down.***

$f''(x) = \dfrac{12x\left(1-2x^3\right)}{(x^3+1)^3}$ $when\ f''(x) = 0 \ \therefore \dfrac{12x\left(1-2x^3\right)}{(x^3+1)^3} = 0 \rightarrow 12x\left(1-2x^3\right) = 0$

$at\ x = 0\ \&\ \frac{1}{\sqrt[3]{2}}$ $f(x)$ *has an inflection point*

$f(x)$ *is concaving up on* $(-\infty,-1) \cup \left(0,\dfrac{1}{\sqrt[3]{2}}\right)$

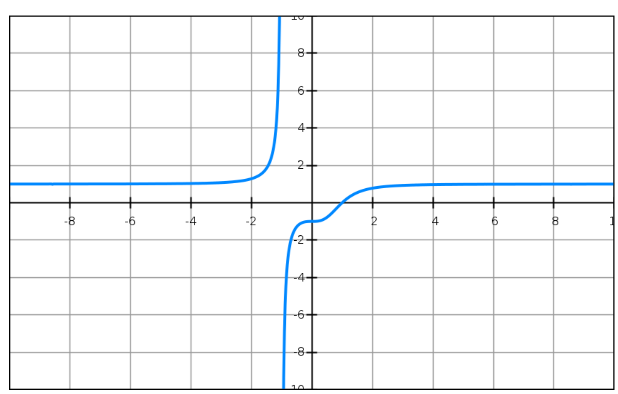
$f(x)$ *is concaving down on* $(-1,0) \cup \left(\dfrac{1}{\sqrt[3]{2}},\infty\right)$

The inflection point are $(0,f(0))$ *and* $\left(\dfrac{1}{\sqrt[3]{2}}, f\left(\dfrac{1}{\sqrt[3]{2}}\right)\right)$

x	-1	0	$\frac{1}{\sqrt[3]{2}}$	
$f''(x)$	$+$	$-$	$+$	$-$
$f(x)$	$\cup$	$\cap$	$\cup$	$\cap$

- ***Using the parts above, sketch the graph***

x	-1	0	$\frac{1}{\sqrt[3]{2}}$	
$f'(x)$	$+$	$+$	$+$	$+$
$f''(x)$	$\cup$	$\cap$	$\cup$	$\cap$
$f(x)$				



$\boldsymbol{Q5}$: let $f(x)=\dfrac{x^3-3x^2+1}{x^3}$. You are given that $f'(x)=\dfrac{3(x^2-1)}{x^4}$ and $f''(x)=\dfrac{3(4-2x^2)}{x^5}$

- **Find the domain of $f(x)$**

The domain of $f(x)$ *is* $\mathbb{R}-\{0\}$ *hint*: $x^3\neq 0 \quad \therefore x\neq 0$

- **Find the y intercepts of $f(x)$**

y intercepts: there is no y intercept point since the domain of $f(x)$ *is* $\mathbb{R}-\{0\}$

- **Determine all asymptotes of $f(x)$.**

Vertical asymp. $\lim\limits_{x\to 0^+} f(x)=\infty$ & $\lim\limits_{x\to 0^-} f(x)=-\infty$ $\therefore x=0$ *is Vertical asymp.*

Horizontal asymp. $\lim\limits_{x\to\infty} f(x)=1$ & $\lim\limits_{x\to-\infty} f(x)=1$ $\therefore y=1$ *is Horizontal asymp.*

There is no Oblique asymp.

- **Find all local maximum and local minimum of $f(x)$; and determine intervals of increase and decrease.**

$f'(x)=\frac{3(x^2-1)}{x^4}$ *when* $f'(x)=0 \quad \therefore \frac{3(x^2-1)}{x^4}=0 \;\rightarrow\; 3(x^2-1)=0 \;\rightarrow\;$ *at* $x=\pm1$

$f(x)$ *has a critical point. There is no singular point.*

$f(x)$ *is increasing on* $(-\infty,-1]\cup[1,\infty)$

$f(x)$ *is decreasing on* $[-1,0)\cup(0,1]$

$f(x)$ *has local* max *point at* $x=-1$ & $f(x)$ *has local* min *point at* $x=1$

The critical point are $(\pm1, f(\pm1))$

x	-1	0	1	
$f'(x)$	$+$	$-$	$-$	$+$
$f(x)$	↗	↘	↘	↗

- **Determine all inflection point(s) and the intervals where $f(x)$ concave up and concave down.**

$$f''(x)=\frac{3\left(4-2x^2\right)}{x^5} \text{ when } f''(x)=0 \quad \therefore \frac{3\left(4-2x^2\right)}{x^5}=0 \;\rightarrow\; 3\left(4-2x^2\right)=0$$

at $x=\pm\sqrt{2}$ $f(x)$ *has an inflection point*

$f(x)$ *is concaving up on* $(-\infty,-\sqrt{2})\cup(0,\sqrt{2})$

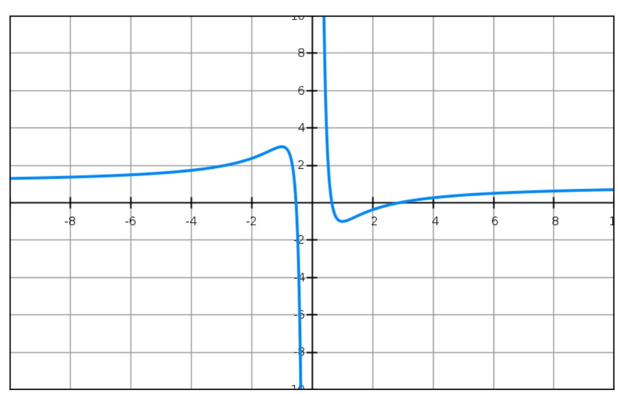
$f(x)$ *is concaving down on* $(-\sqrt{2},0)\cup(\sqrt{2},\infty)$

The inflection point is $(\pm\sqrt{2}, f(\pm\sqrt{2}))$

x	$-\sqrt{2}$	0	$\sqrt{2}$	
$f''(x)$	$+$	$-$	$+$	$-$
$f(x)$	∪	∩	∪	∩

- **Using the parts above, sketch the graph $y=f(x)$ in the coordinate axes provided below.**

x	$-\sqrt{2}$	-1	0		1	$\sqrt{2}$
$f'(x)$	$+$	$+$	$-$	$-$	$+$	$+$
$f''(x)$	∪	∩	∩	∪	∪	∩
$f(x)$						



Q6: *let* $f(x) = \dfrac{x^3}{x^2-1}$ *.You are given that* $f'(x) = \dfrac{x^2(x^2-3)}{(x^2-1)^2}$ *and* $f''(x) = \dfrac{2x(x^2+3)}{(x^2-1)^3}$

- ***Find the domain of*** $f(x)$

The domain of $f(x)$ *is* $\mathbb{R} - \{\pm 1\}$ $\qquad$ *hint*: $x^2 - 1 \neq 0 \ \therefore x^2 \neq 1 \ \therefore x \neq \pm 1$

- ***Find both the x and y intercepts of*** $f(x)$

x intercept: $\dfrac{x^3}{x^2-1} = 0 \ \rightarrow \ x^3 = 0 \ \rightarrow \ x = 0$ $\qquad$ *x intercept point is* $(0,0)$

y intercepts: $f(0) = \dfrac{0^3}{0^2-1} = 1 \ \rightarrow \ f(0) = 0$ $\qquad$ *y intercept point is* $(0,0)$

- ***Determine all asymptotes of*** $f(x)$.

Vertical asymp. $\qquad \lim\limits_{x \to \pm 1^+} f(x) = \infty \quad \& \quad \lim\limits_{x \to \pm 1^-} f(x) = \infty \qquad \therefore x = \pm 1$ *are Vertical asymp.*

There is no horizontal asymp.

There is Oblique asymp. at $y = x$ $\qquad$ *since* $\lim\limits_{x \to \pm\infty} f(x) - x = \lim\limits_{x \to \pm\infty} \frac{x}{x^2-1} = 0$

- ***Find all local maximum and local minimum of*** $f(x)$; ***and determine intervals of increase and decrease .***

$f'(x) = \frac{x^2(x^2-3)}{(x^2-1)^2}$ *when* $f'(x) = 0 \ \therefore \ \frac{x^2(x^2-3)}{(x^2-1)^2} = 0 \ \rightarrow \ x^2(x^2-3) = 0 \ \rightarrow$ *at* $x = 0 \ \& \ \pm\sqrt{3}$

$f(x)$ has a critical point. There is no singular point.

x	$-\sqrt{3}$ $\quad -1$ $\quad 0$ $\quad 1$ $\quad \sqrt{3}$					
$f'(x)$	$+$	$-$	$-$	$-$	$-$	$+$
$f(x)$	↗	↘	↘	↘	↘	↗

$f(x)$ is increasing on $]-\infty, -\sqrt{3}] \cup [\sqrt{3}, \infty[$

$f(x)$ is decreasing on $[-\sqrt{3}, -1[\ \cup \ ]-1,1] \ \cup \ ]1, \sqrt{3}]$

$f(x)$ has local max *point at* $x = \sqrt{3}$ & *$f(x)$ has local* min *point at* $x = \sqrt{3}$

The critical point are $(0,0)$ *and* $\left(\pm\sqrt{3}, f\left(\pm\sqrt{3}\right)\right)$

- ***Determine all inflection point(s) and the intervals where*** $f(x)$ ***concave up and concave down.***

$f''(x) = \dfrac{2x(x^2+3)}{(x^2-1)^3}$ *when* $f''(x) = 0 \ \therefore \ \dfrac{2x(x^2+3)}{(x^2-1)^3} = 0 \ \rightarrow \ 2x(x^2+3) = 0$

at $x = 0$ *$f(x)$ has an inflection point*

$f(x)$ is concaving up on $(-1,0) \cup (1, \infty)$

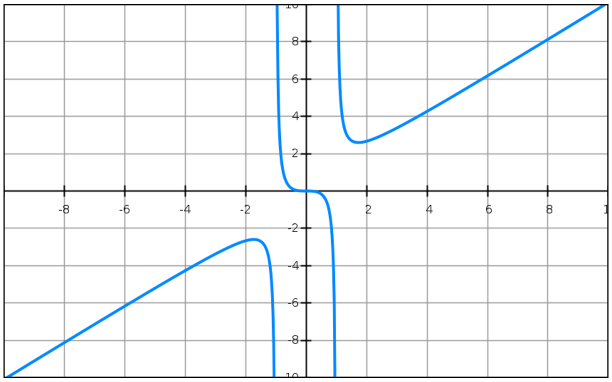
$f(x)$ is concaving down on $(-\infty, -1) \cup (0,1)$

The inflection point is $(0, 0)$

x	-1 $\quad 0$ $\quad 1$			
$f''(x)$	$-$	$+$	$-$	$+$
$f(x)$	$\cap$	$\cup$	$\cap$	$\cup$

- ***Using the parts above, sketch the graph***

x	$-\sqrt{3}$ $\quad -1$ $\quad 0$ $\quad 1$ $\quad \sqrt{3}$					
$f'(x)$	$+$	$-$	$-$	$-$	$-$	$+$
$f''(x)$	$\cap$	$\cap$	$\cup$	$\cap$	$\cup$	$\cup$
$f(x)$						



Q7: $let\ f(x) = \dfrac{e^x}{x-1}$. $You\ are\ given\ that\ f'(x) = \dfrac{e^x(x-2)}{(x-1)^2}$ $and\ f''(x) = \dfrac{e^x(x^2-4x+5)}{(x-1)^3}$

- ***Find the domain of $f(x)$***

$The\ domain\ of\ f(x)\ is\ \mathbb{R} - \{1\}$ $\qquad hint: x - 1 \neq 0 \ \therefore x \neq 1$

- ***Find both the x and y intercepts of $f(x)$***

$x\ intercept: \dfrac{e^x}{x-1} = 0 \ \rightarrow \ e^x = 0 \ \rightarrow \ no\ solution \quad Then, there\ is\ no\ x\ intercept\ point$

$y\ intercepts: f(0) = \dfrac{e^0}{0-1} = -1 \ \rightarrow \ f(0) = -1 \qquad y\ intercept\ point\ is\ (0,-1)$

- ***Determine all asymptotes of $f(x)$.***

$Vertical\ asymp. \quad \lim_{x\to 1^+} f(x) = \infty \quad \& \quad \lim_{x\to 1^-} f(x) = -\infty \quad \therefore x = 1\ is\ \ Vertical\ asymp.$

$Horizontal\ asymp. \quad \lim_{x\to \infty} f(x) = \infty \quad \& \quad \lim_{x\to -\infty} f(x) = 0 \quad \therefore y = 0\ \ is\ \ Only\ Horizontal\ asymp.\ at\ the\ left$

$There\ is\ no\ Oblique\ asymp.$

- ***Find all local maximum and local minimum of $f(x)$; and determine intervals of increase and decrease .***

$f'(x) = \frac{e^x(x-2)}{(x-1)^2} \ \ when\ f'(x) = 0 \ \ \therefore \ \frac{e^x(x-2)}{(x-1)^2} = 0 \ \rightarrow \ e^x(x-2) = 0 \ \rightarrow \ at\ x = 2$

$f(x)\ has\ a\ critical\ point.\ There\ is\ no\ singular\ point.$

$f(x)\ is\ increasing\ on\ [2,\infty)$

$f(x)\ is\ decreasing\ on\ (-\infty,1) \ \cup \ (1,2]$

$f(x)\ has\ no\ both\ local\ \max point\ and\ f(x)\ has\ local\ \min point \qquad The\ critical\ point\ is\ \big(2, f(2)\big)$

x	1	2	
$f'(x)$	$-$	$-$	$+$
$f(x)$	↘	↘	↗

- ***Determine all inflection point(s) and the intervals where $f(x)$ concave up and concave down.***

$f''(x) = \dfrac{e^x(x^2-4x+5)}{(x-1)^3} \ when\ f''(x) = 0 \ \ \therefore \dfrac{e^x(x^2-4x+5)}{(x-1)^3} = 0 \ \rightarrow \ e^x(x^2-4x+5) = 0$

$f(x)\ has\ no\ inflection\ point\ \ since\ \Delta\ (x^2 - 4x + 5) = -4 < 0$

$f(x)\ is\ concaving\ up\ on\ \ (-\infty,1)$

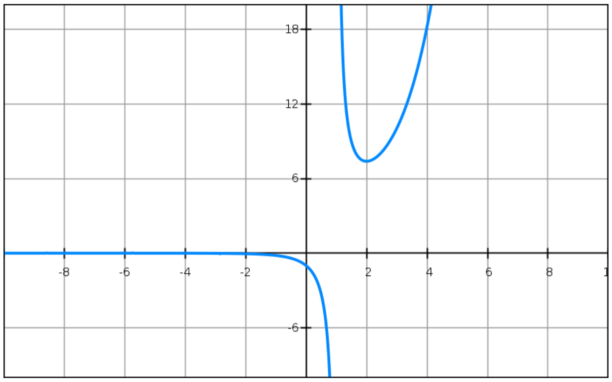
$f(x)\ is\ concaving\ down\ on\ \ (1,\infty)$

$The\ inflection\ point\ is\ (-1, f(-1))$

x	1	
$f''(x)$	$-$	$+$
$f(x)$	$\cap$	$\cup$

- ***Using the parts above, sketch the graph $y = f(x)$ in the coordinate axes provided below.***

x	1	2	
$f'(x)$	$-$	$-$	$+$
$f''(x)$	$\cap$	$\cup$	$\cup$
$f(x)$			



$\boldsymbol{Q8}$: $let\ f(x)\ be\ the\ function\ defined\ for\ all\ real\ numbers. suppose\ that\ f(x)\ satisfies\ the\ following\ conditions$:

$f(x)\ is\ not\ continous\ at\ x = \pm 1$,

$f'(x)\ is\ not\ defined\ at\ x = -2$,

$f'(2) = 0, f''(0) = f''(3) = 0$,

x		-2		-1		0		1		2		3	
$f(x)$		-1		-2		0		4		1		2	
$f'(x)$	$+$		$-$		$-$		$-$		$-$	0	$+$		$+$
$f''(x)$	$+$		$+$		$+$	0	$-$		$+$		$+$	0	$-$

$\lim_{x\to-\infty} f(x) = -\infty$, $\lim_{x\to\infty} f(x) = 3$, $\lim_{x\to-1^+} f(x) = \infty$, $\lim_{x\to-1^-} f(x) = -2$, $\lim_{x\to1^+} f(x) = 4$, $\lim_{x\to1^-} f(x) = -\infty$

Answer the following questions using the given information:

(a) Find all vertical and horizontal asymptotes of $f(x)$.

$since\ \lim_{x\to\infty} f(x) = 3\ \therefore y = 3\ is\ a\ horizontal\ asymptote\ (H.A.)\ ONLY\ from\ the\ right\ side$

$since\ \lim_{x\to-1^-} f(x) = -2\ and\ \lim_{x\to1^+} f(x) = 4\ \therefore x = \pm 1\ are\ the\ vertical\ asymptotes\ (V.A.)$

(b) Give all intervals where $f(x)$ is increasing.

$From\ the\ table$: $(-\infty, -2)\ (2, \infty)$

(c) Is $f(x)$ is decreasing on $(-2,2)$? Explain.

$No!\ \ 0\ and\ 1\ are\ in\ (-2,2)\ and\ f(0) = 0 < 4 = f(1)$ $where\ f(x)\ is\ not\ continous\ at\ x = \pm 1$

(d) Give all intervals where $f(x)$ is concave down.

$From\ the\ table$: $(0,1)\ (3, \infty)$

(e) Find all local maximum and local minimum points of $f(x)$.

$From\ the\ table$: $f(-2) = -1\ is\ local\ maximum.\ f(2) = 1\ is\ local\ minimum.$

$Also, since\ f(x)\ is\ decreasing\ on\ (-2,-1)\ and\ \lim_{x\to-1^+} f(x) = \infty\ , f(-1) = -2\ \ is\ local\ minimum$

$Similarly, since\ f(x)\ is\ decreasing\ on\ (1,2)\ and\ \lim_{x\to1^-} f(x) = -\infty\ , f(1) = 4\ \ is\ local\ maximum$

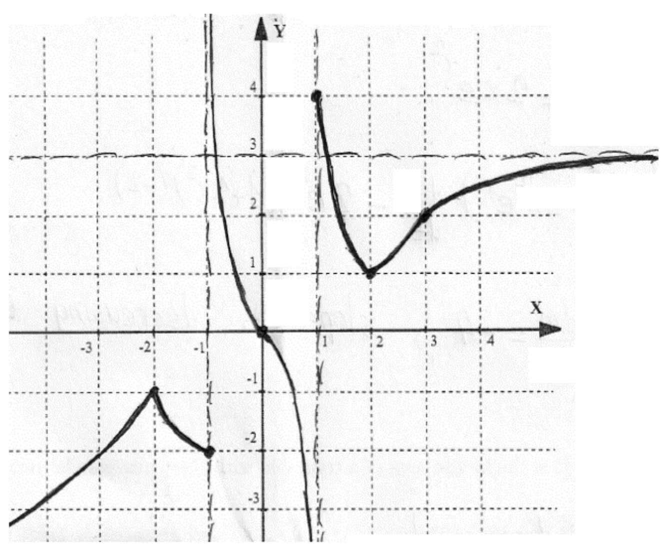
(f) Find all inflection points of $f(x)$.

$f(0) = 0\ , f(3) = 2\ \ are\ inflection\ poins\ from\ the\ table$

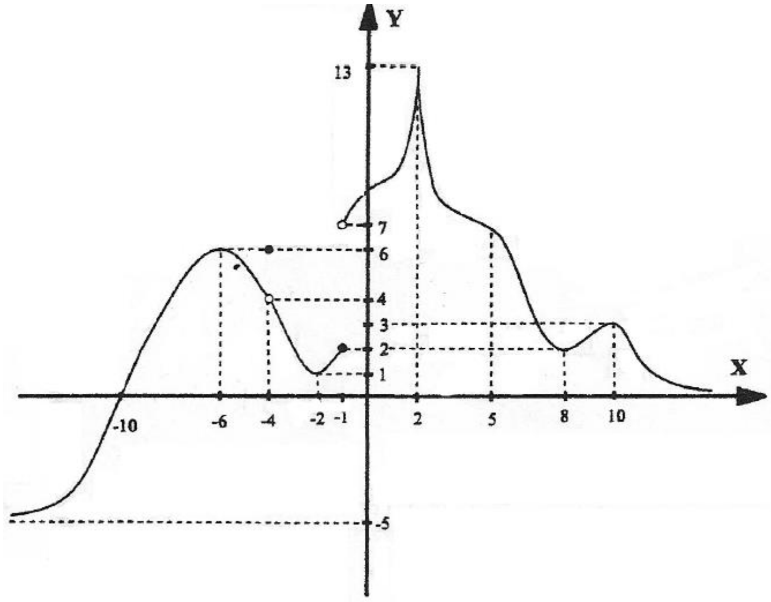
$f(x)\ is\ not\ continous\ at\ x = 1$

$\therefore there\ is\ no\ inflection\ point\ at\ x = 1$

(g) Sketch a graph of $f(x)$.



$\boldsymbol{Q9}$: *For the function* $f(x)$ *whose graph is given, find the following*:



$(a)\ \lim_{x\to 2} f(x) = 13$

$(b)\ \lim_{x\to -4} f(x) = 4$

$(c)\ \lim_{x\to -\infty} f(x) = -5$

$(d)\ \lim_{x\to -1} f(x) =$ *Does Not Exist* $(D.N.E)$

(e) *Points where* $f'(x) = 0$

$x = -6\ , -2\ , 8\ , 10$

(f) *Intervals where* $f'(x) > 0$

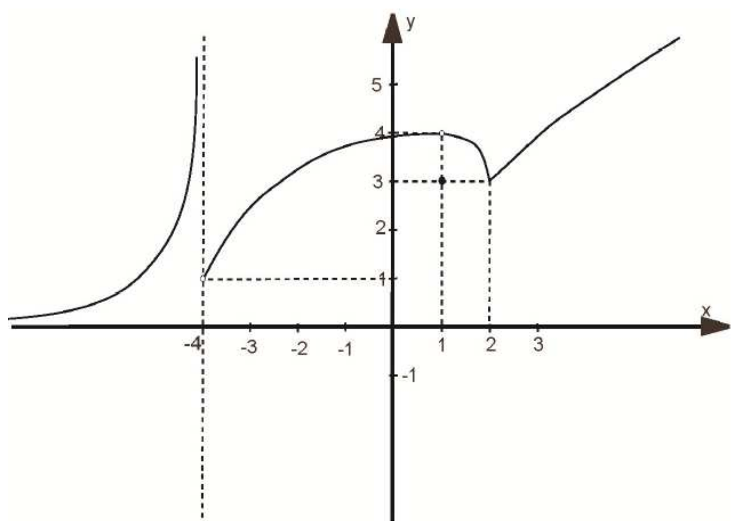
$(-\infty,-6), (-2,-1), (-1,2), (8,10)$

(g) *Intervals where* f *is continous*. $(-\infty,-4), (-4,-1), (-1,\infty)$

(h) *Intervals where* f *is differentiable*. $(-\infty,-4), (-4,-1), (-1,2), (2,\infty)$

$(i)\ \lim_{x\to 2^-} f'(x) = \infty$

$\boldsymbol{Q10}$: *The graph of the function* $f(x)$ *is given below, Answer the following questions*:



(a) *Find the limit* $\lim_{x\to -4^+} f(x)$ *if it exists.*

$\lim_{x\to -4^+} f(x) = 1$

(b) *Find the limit* $\lim_{x\to 1} f(x)$ *if it exists.*

$\lim_{x\to 1} f(x) = 4$

(c) *Find all intervals which* $f(x)$ *is continous*.

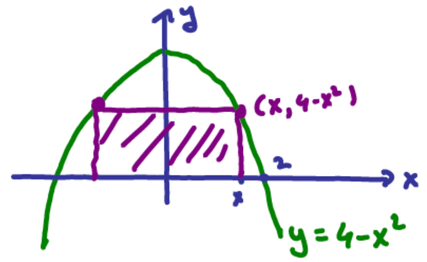
$(-\infty,-4),\ \ (-4,1),\ \ (1,\infty)$

(d) *Find all intervals which* $f(x)$ *is differentiable*.

$(-\infty,-4),\ \ (-4,1),\ \ (1,2),\ \ (2,\infty)$

Part 1:

Q1: Find the dimensions of the rectangle of largest area that has its base on the $x-axis$ and its other two vertices are above the $x-axis$ on the parabola $y=4-x^2$.



Area= $2x\,(4-x^2)\quad \therefore A(x)=8x-2x^3\ \ x\in[0,2]$

End points: $A(0)=A(2)=0$

Critical Point(s) in $(0,2)$: $A'(x)=8-6x^2\ \ when\ A'(x)=0\ \ \therefore\ x=\frac{2}{\sqrt{3}}$

$$A\left(\frac{2}{\sqrt{3}}\right)=8\left(\frac{2}{\sqrt{3}}\right)-2\left(\frac{2}{\sqrt{3}}\right)^3=\frac{32}{3\sqrt{3}}\ \ Thus, the\max area\ of\ rectangle\ is\ A\left(\frac{2}{\sqrt{3}}\right)>[A(0)=A(2)]$$

Hence, dimensions of largest area of rectangle are $\frac{4}{\sqrt{3}}$ and $\frac{8}{3}$

Q2: A cylinder with an open top has total surface area $27\pi\ cm^2$. Find the radius of such a cylinder which has maximum volume. Explain why your answer gives the maximum volume.

$$27\pi=2\pi\,r\,h+\pi\,r^2\quad\Rightarrow h=\frac{27\pi-\pi\,r^2}{2\pi\,r}=\frac{27-\,r^2}{2\,r}\qquad Volume\colon V=\pi\,r^2\,h$$

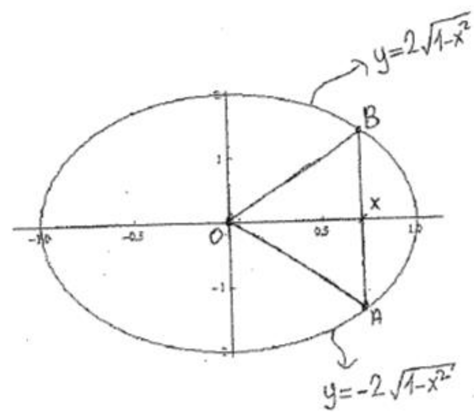
$$V(r)=\pi\,r^2\left(\frac{27-\,r^2}{2\,r}\right)=\frac{\pi}{2}r\,(27-\,r^2)=\frac{\pi}{2}(27\,r-\,r^3)\ \ where\ 0\le r\le\sqrt{27}$$

End points: $V(0)=V(\sqrt{27})=0$

Critical Point(s) in $(0,\sqrt{27})$: $V'(r)=\frac{\pi}{2}(27\,-\,3r^2)\ \ when\ V'(r)=0\ \ \therefore\ r=3\ \ is\ only\ critical\ point$

$[V(3)=27\pi]>\ [V(0)=V(\sqrt{27})=0]\qquad so\ r=3\ cm\ gives\ max.\ volume.$

Q3: Find the area of the largest triangle with the following properties: one vertex is $(0,0)$, the other vertices are on the ellipse $4x^2+y^2=4$ and one side is parallel to the $y-axis$.



$4x^2+y^2=4\quad\therefore y=\pm2\sqrt{1-x^2}\ \ Consider\ the\ triangle\ AOB$

The height is x unit. The base (AB) is $4\sqrt{1-x^2}$

$A(x)=2x\sqrt{1-x^2}\quad x\in(0,1)$

$$A'(x)=2\sqrt{1-x^2}+2\frac{-2x}{\sqrt{1-x^2}}=\frac{2(1-2x^2)}{\sqrt{1-x^2}}\ when\ A'(x)=0\ \ \therefore\ x=\frac{1}{\sqrt{2}}$$

$$A\left(\frac{1}{\sqrt{2}}\right)=1\qquad Note\ that\ \left[A\left(\frac{1}{\sqrt{2}}\right)=1\right]>\left[\lim_{x\to0^+}A(x)=\lim_{x\to1^-}A(x)=0\right]$$

Thus, $A\left(\frac{1}{\sqrt{2}}\right)=1$ is the largest area for such a triangle.

Q4: there are 50 orange trees in a garden. Each tree produces 800 oranges. For each additional tree planted in the garden, the output per tree **drops** by 10 oranges. How many trees should be added to the existing garden in order to maximize the total output of trees?

$x =$ number of orange trees. Output per tree $= 800 - 10x$

$T(x) = (50 + x)(800 - 10x) = 40000 + 300x - 10x^2 \qquad 0 \le x \le 80$

End points: $T(0) = 40000 \;\;,\;\; T(80) = 0$

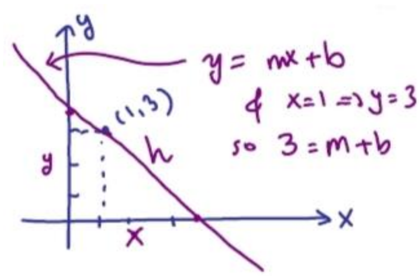
Critical Point(s) in $(0, 80)$: $T'(x) = 300 - 20x \quad when\; T'(x) = 0 \;\therefore x = 15 \qquad T(15) = 42250$

$[T(15) = 42250] > [T(0) = 40000 \;\;,\;\; T(80) = 0]$

Hence, 15 more trees should be planted to maximize the total.

Q5: Among all triangles formed by the positive $x - axis$, the positive $y - axis$ and a line passing through the point (1,3), find the dimensions of the triangle with the shortest hypotenuse.

We are asked to find x, y & $h = \sqrt{x^2 + y^2}$ with $h -$ minimum



Line eqn: $y = mx + b \;\; where\; 3 = m + b \;\; then\; y = mx + (3 - m) \rightarrow (1)$

$When\; x = 0 \;\rightarrow [\,y = 3 - m\,] \;\;\&\;\; when\; y = 0 \rightarrow \left[x = \frac{m-3}{m} = 1 - \frac{3}{m}\right] \; apply \rightarrow (1)$

Let $f(m) = h^2 = x^2 + y^2 \;\; thus\; f(m) = \left(1 - \frac{3}{m}\right)^2 + (3 - m)^2 \quad m \in (-\infty, 0)$

$f'(m) = 2\left(1 - \frac{3}{m}\right)\left(\frac{3}{m^2}\right) - 2(3 - m) \;\; when\; f'(m) = 0 \;\therefore\; 6(m - 3) - 2(3m^3 - m^4) = 0$

$\therefore\; 6(m - 3) + 2m^3(m - 3) = 0 \;\;\therefore\; 2(m - 3)(3 + 2m^3) = 0 \quad \therefore m = 3 \notin (-\infty, 0) \;\; and \;\; m = -3^{\frac{1}{3}} \in (-\infty, 0)$

$Thus,\;\; m = -3^{\frac{1}{3}} \; is\; only\; critical\; point. \;\; Note\; that \left[\lim_{m \to -\infty} f(m) = \lim_{m \to 0^-} f(m) = +\infty\right] > \left[m = -3^{\frac{1}{3}}\right]$

$Hence, m = -3^{\frac{1}{3}}\, gives\; the\; minimum\; value\; of\; hypotenuse \;\therefore x = 1 - \frac{3}{-3^{\frac{1}{3}}} = 1 + 3^{\frac{2}{3}} \;\;\&\;\; y = 3 + 3^{\frac{1}{3}}$

$$h = \sqrt{\left(1 + 3^{\frac{2}{3}}\right)^2 + \left(3 + 3^{\frac{1}{3}}\right)^2} = \left(1 + 3^{\frac{3}{2}}\right)^{3/2}$$

<u>Part 2</u>: *(commonly asked in final exam)*

Q1: Find the absolute mininmum and absolute maximum points and the values of $f(x) = (x^3 - 9x^2 + 15x)^{2/3}$ on $[-1,6]$.

Since $f(x)$ is continous on the close interval $[-1,6]$, by extreme value theorem there exist both absolute minimum and absloute maximum points of $f(x)$ on $[-1,6]$.

$f'(x) = \frac{2}{3}(x^3 - 9x^2 + 15x)^{-1/3}\,(3x^2 - 18x + 15) = \frac{2}{3}\frac{3x^2 - 18x + 15}{\sqrt[3]{x^3 - 9x^2 + 15x}}$

Singular points: $f'(x)$ is not defined i.e $x^3 - 9x^2 + 15x = x(x^2 - 9x + 15) = 0 \;\;\therefore x = 0 \; or\; x = \frac{9 \pm \sqrt{21}}{2}$

But $x=\frac{9+\sqrt{21}}{2}\notin(-1,6)$ $so\ x=0\ and\ x=\frac{9-\sqrt{21}}{2}$ $are\ only\ the\ singular\ points\ in\ (-1,6)$

$$f(0)=f\left(\frac{9-\sqrt{21}}{2}\right)=0\ \rightarrow(2)$$

Critical points: $f'(x)=0\ \therefore 3x^2-18x+15=0\ \therefore (x-6)(x-1)=0\ so, x=1\ and\ x=5\ are\ critical\ pts.$

$f(1)=7^{2/3}\ \ and\ \ f(5)=25^{2/3}\ \ \rightarrow(2)$

End points: $f(-1)=25^{2/3}\ \ and\ \ f(6)=18^{2/3}\ \ \rightarrow(3)$

From $(1),(2)\ and\ (3), we\ have$: $f(-1)=f(5)=25^{2/3}\ is\ abs.\ max.value\ \ \&\ x=-1\ and\ x=5\ are\ abs\ max.\ pts$

$$f(0)=f\left(\frac{9-\sqrt{21}}{2}\right)=0\ is\ abs.\ min.value\ \ \&\ x=0\ and\ x=\frac{9-\sqrt{21}}{2}\ are\ abs\ min.\ pts$$

Q2: Find the absolute mininmum and absolute maximum (if they exist) of $f(x)=e^{-\left(\frac{1+x^2}{x}\right)}$ on $(0,\infty)$.

$$\lim_{x\to\infty}e^{-\left(\frac{1+x^2}{x}\right)}=e^{-\lim_{x\to\infty}\left(\frac{1+x^2}{x}\right)}=e^{-\infty}=0 \qquad \lim_{x\to0^+}e^{-\left(\frac{1+x^2}{x}\right)}=e^{-\lim_{x\to0^+}\left(\frac{1+x^2}{x}\right)}=e^{-\infty}=0$$

$$f'(x)=e^{-\left(\frac{1+x^2}{x}\right)}\left(\frac{1}{x^2}-1\right)$$

Singular points: $f'(x)$ is not defined. Since $f'(x)\ exists\ for\ all\ x>0: no\ singular\ points$

Critical points: $f'(x)=0\ \therefore\left(\frac{1}{x^2}-1\right)=0\ \therefore\frac{1}{x^2}=1\ \therefore x=\pm1\ \ so\ x=1\ is\ the\ only\ critical\ point\ on\ (0,\infty)$

$[f(1)=e^{-2}]>\left[\lim_{x\to\infty}f(x)=\lim_{x\to0^+}f(x)=0\right]\ \ so\ at\ x=1\ the\ f(x)\ has\ abs\ max.point$

OR $since\ f(x)\ is\ increasing\ on\ (0,1]\ and\ decreasing\ on\ [1,\infty)\ \ so\ at\ x=1\ the\ f(x)\ has\ abs\ max.point$

$However, there\ is\ no\ abs\ min.point\ on\ (0,\infty)$

Q3: Consider the function $f(x)=\begin{cases}|x^2-1| & x\le2\\ 2+\cos(x-2) & x>2\end{cases}$

(a) Without finding the absolute extreme values, show that f has absolute maximum and minimum values on the interval $[0,6]$.

$x_0\in[0,2)\ \ \lim_{x\to x_0}f(x)=\lim_{x\to x_0}|x^2-1|=|{x_0}^2-1|=f(x_0)\ \ \rightarrow(1)$

$x_1\in(2,6]\ \ \lim_{x\to x_1}f(x)=\lim_{x\to x_1}2+\cos(x-2)=2+\cos(x_1-2)=f(x_1)\ \rightarrow(2)$

$x=2:\ \lim_{x\to2^-}f(x)=\lim_{x\to2^-}|x^2-1|=3\ \ and\ \lim_{x\to2^+}f(x)=\lim_{x\to2^+}2+\cos(x-2)=3\ \ \therefore\ \lim_{x\to2}f(x)=3=f(2)\ \rightarrow(3)$

$From\ (1),(2)\ \&\ (3),\ \ f(x)\ is\ cont.on\ \ [0,6].\ So, by\ extreme\ value\ thm, there\ exist\ both\ max.\&\ min.on\ [0,6]$

(b) Find the absolute maximum and minimum values of f on the interval $[0,6]$.

End points: $f(0)=1\quad f(6)=2+\cos4\ \ \rightarrow(1)$

Critical points: $f'(x)=\begin{cases}-2x & 0<x<1\\ 2x & 1<x<2\\ -\sin(x-2) & x>2\end{cases}\quad\Rightarrow f'(x)=0\begin{cases}x=0 & \notin & 0<x<1\\ x=0 & \notin & 1<x<2\\ x-2=\pi\ \therefore x=2+\pi & \in & x>2\end{cases}$

$so\ the\ only\ Critical\ point\ is\ \ f(2+\pi) = 2+\cos\pi = 1 \ \ \rightarrow (2)$

Singular points: $\lim_{x\to 1^-} f^{'}(x) \neq \lim_{x\to 1^-} f^{'}(x)\ \ and \lim_{x\to 2^-} f^{'}(x) \neq \lim_{x\to 2^-} f^{'}(x)\ \ so\ at\ x=1\ and\ x=2\ \ are\ singular\ points$

$so\ f(1) = 0\ and\ f(2) = 3\ \ are\ singular\ points\ \rightarrow (3)$

$From\ (1),(2)\ \&\ (3), we\ have:$

$f(1) = 0\ \ is\ the\ abs\ min.value\ of\ f(x)\ on\ [0,6]\ \ at\ x=1\ is\ abs\ min.\ point$

$f(2) = 3\ \ is\ the\ abs\ max.value\ of\ f(x)\ on\ [0,6]\ \ at\ x=2\ is\ abs\ max.\ point$

Section 10: Problems session

1) The equation $\cos\left(\frac{\pi y}{x}\right) = \frac{x^2}{y} - 4\ \ defines\ a\ funcation\ y = f(x). Find\ the\ lineriztion\ of\ f\ at\ \ (3,3)$

$and\ use\ it\ to\ approximate\ f(3.1)$

$\boldsymbol{Ans}: using\ the\ lineriztion\ rule: L(x) = f(a) + f\,`(a)\ (x-a)\qquad hint:\quad a=3\ \ and\ x=3.1$

$f(a) = f(3) = 3$

$\frac{d}{dx}[\ \cos\left(\frac{\pi y}{x}\right)\ ] = \frac{d}{dx}[\frac{x^2}{y} - 4]\ \ \therefore \sin\left(\frac{\pi y}{x}\right).\left(\frac{\pi x y` - \pi y}{x^2}\right) = \frac{2yx - x^2y`}{y^2}\ y`_{(3,3)} = 2$

$\therefore the\ lineriztion\ of\ f\ at\ \ a=3\ \overset{i.e}{\rightarrow}\ L(x) = f(3) + f\,`(3)\ (x-3) = 3 + 2*(x-3) = 2x-3$

$using\ the\ approximation\ rule, we\ have\ f(3.1) \approx L(3.1) =\ 2(3.1) - 3 = 3.2$

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2) The equation $\sin(xy) + \pi = x + y\ \ defines\ a\ funcation\ y = f(x).\ Find\ the\ lineriztion\ of\ f\ at$

$(\pi, 0)\ \ Also, find\ an\ approximation\ for\ f(3)$

$\boldsymbol{Solution}: using\ the\ lineriztion\ rule: L(x) = f(a) + f\,`(a)\ (x-a)\qquad hint:\quad a=\pi\ \ and\ x=3$

$from\ (\boldsymbol{\pi}, \boldsymbol{0})\ since\ \ y=0\ \overset{so}{\rightarrow} f(a) =\ \boldsymbol{f}(\boldsymbol{\pi}) = 0$

$\frac{d}{dx}[\ \sin(xy) + \pi\ ] = \frac{d}{dx}[\ x+y\ ]\ \ \therefore \cos(xy).(y + xy`) = 1 + y`\ \ using\ (\boldsymbol{\pi}, \boldsymbol{0}) \overset{so}{\rightarrow} \boldsymbol{f}\,`(\boldsymbol{\pi}) = y`_{at\ x=\pi} = \frac{1}{\pi+1}$

$\therefore the\ \boldsymbol{lineriztion}\ of\ f\ at\ \ a=\pi\ \overset{i.e}{\rightarrow}\ L(x) = f(\pi) + f\,`(\pi)\ (x-\pi) = 0 + \frac{1}{\pi+1} * (x-\pi) = \frac{x-\pi}{\pi+1}$

$using\ the\ approximation\ rule, we\ have\ f(3) \approx L(3) = \frac{3-\pi}{\pi+1}$

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3) $Using\ linear\ approximation, estimate\ \sqrt[5]{33}$

$\boldsymbol{Solution}: using\ the\ lineriztion\ rule: L(x) = f(a) + f\,`(a)\ (x-a)\qquad hint:\quad a=32\ \ and\ x=33$

Let $f(x) = \sqrt[5]{x}\qquad f(32) = \sqrt[5]{32} = 2\quad f\,`(x) = \frac{1}{5}x^{-\frac{4}{5}}\ \ then\ \ f\,`(32) = \frac{1}{5(16)} = \frac{1}{80}$

$\therefore the\ lineriztion\ of\ f\ at\ \ a=32\ \overset{i.e}{\rightarrow}\ L(x) = f(32) + f\,`(32)\ (x-32) = 2 + \frac{1}{80}(x-32)$

$using\ the\ approximation\ rule, we\ have\ f(33) \approx L(33) = 2 + \frac{1}{80}(33-32) = 2 + \frac{1}{80} = \frac{161}{80}$

4) $let\ f(x) = \int_1^x \frac{dt}{1+t^6}. Use\ the\ linearization\ of\ f\ at\ a = 1\ to\ approximate\ f\left(\frac{3}{2}\right)$

$\boldsymbol{Solution}: using\ the\ lineriztion\ rule: L(x) = f(a) + f\,`(a)\,(x-a) \quad hint: \quad a = 1\ and\ x = \frac{3}{2}$

$f(1) = \int_1^1 \frac{dt}{1+t^6} = 0\ \ and\ by\ F.T.C\ we\ have\ f\,`(x) = \frac{1}{1+x^6}\ \ so\ f\,`(1) = \frac{1}{2}$

$\therefore the\ lineriztion\ of\ f\ at\ \ a = 1 \overset{i.e}{\rightarrow} L(x) = f(1) + f\,`(1)\,(x-1) = 0 + \frac{1}{2}(x-1) = \frac{1}{2}(x-1)$

$using\ the\ approximation\ rule, we\ have\ f\left(\frac{3}{2}\right) \approx L\left(\frac{3}{2}\right) = \frac{1}{2}\left(\frac{3}{2} - 1\right) = \frac{1}{4}$

5) $let\ f(x) = \int_1^{x^2} e^{t^2} dt\ \ be\ given. Using\ the\ linear\ (tangent\ line)\ approximation\ of\ F,$

$and\ estimate\ \int_1^{1.21} e^{t^2} dt$

Since $x^2 = 1.21\ \ then\ x = 1.1\ so\ \ \ a(\approx x = 1.1) = 1$

$f(1) = \int_1^{1^2} e^{t^2} dt = 0\ \ \ and\ by\ F.T.C\ we\ have\ f\,`(x) = \ 2x\ e^{x^4}\ \ \ so\ f\,`(1) = 2e$

$\therefore the\ lineriztion\ of\ f\ at\ \ a = 1 \overset{i.e}{\rightarrow} L(x) = f(1) + f\,`(1)\,(x-1) = 0 + 2e(x-1) = 2e(x-1)$

$using\ the\ approximation\ rule, we\ have\ f(1.1) \approx L(1.1) = \ 2e(1.1-1) = 0.2e = \frac{e}{5}$

Section 11: Problems session

$Q3) \lim_{x\to\infty} \frac{1}{n}\left(\sqrt{1-\frac{1}{n}} + \sqrt{1-\frac{2}{n}} + \cdots + \sqrt{1-\frac{n}{n}}\right) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \sqrt{1-\frac{1}{n}i}$

$\boldsymbol{ans}: \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \sqrt{1-\frac{1}{n}i} = \lim_{n\to\infty} \Delta x \sum_{i=1}^{n} f(x_i^*) \overset{\blacksquare}{\Rightarrow} \int_0^1 \sqrt{1-x}\,dx = (*) \int_0^1 \sqrt{u}\,du = \frac{2}{3}u^{2/3}\Big|_0^1 = \frac{2}{3}$

$$\begin{cases} let\ u = 1-x \qquad du = -dx \\ x = 1\ \rightarrow u = 1\ and\ \ x = 1\ \rightarrow u = 0 \end{cases}$$

$$\blacksquare\begin{Bmatrix} \Delta x \ = \ \frac{1}{n} \ = \ \frac{b-a}{n}\ , \quad x_i^* = \frac{1}{n}i = a + i\,\Delta x = a + i\,\frac{1}{n}\,, \quad P_n = \{\,x_0^* = 0, \ldots\ldots.., x_n^* = 1\} \\ x_0^* = \ 0 = a\,, \quad x_n^* = 1\ = b \\ let\ f(x) = \sqrt{1-x}\ \ which\ is\ cont.\ on\ \ [0,1] \end{Bmatrix}$$

$Q4) \lim_{x\to\infty} \frac{1}{n}\left[\left(1+\frac{1}{n}\right)^5 + \left(1+\frac{2}{n}\right)^5 + \cdots + \left(1+\frac{n}{n}\right)^5\right] = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \left(1+\frac{i}{n}\right)^5$

$\boldsymbol{ans}: \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \left(1+\frac{i}{n}\right)^5 = \lim_{n\to\infty} \Delta x \sum_{i=1}^{n} f(x_i^*) \overset{\blacksquare}{\Rightarrow} \int_1^2 x^5\,dx = \frac{1}{6}x^6\Big|_1^2 = \frac{21}{2}$

$$\blacksquare\left\{\begin{array}{c}\Delta x = \frac{1}{n} = \frac{b-a}{n}\ , \quad x_i^* = 1+\frac{i}{n} = a + i\,\Delta x = a + i\,\frac{1}{n}\ , \quad P_n = \{\, x_0^* = 1, \ldots\ldots.., x_n^* = 2\} \\ x_0^* = 1 = a\ , \quad x_n^* = 2 = b \\ let\ f(x) = x^5 \quad which\ is\ cont.\ on\ [1,2]\end{array}\right\}$$

$$Q5)\ \lim_{x\to\infty}\frac{1}{n}\left[\left(\frac{1}{n}\right)^{2/3} + \left(\frac{2}{n}\right)^{2/3} + \cdots + \left(\frac{n-1}{n}\right)^{\frac{2}{3}} + 1\right] = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^{2/3}$$

$$\boldsymbol{ans}: \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^{2/3} = \lim_{n\to\infty}\Delta x\sum_{i=1}^{n} f(x_i^*) \overset{\blacksquare}{\Rightarrow} \int_0^1 x^{2/3}\,dx = \frac{3}{5}u^{5/3}\Big|_0^1 = \frac{3}{5}$$

$$\blacksquare\left\{\begin{array}{c}\Delta x = \frac{1}{n} = \frac{b-a}{n}\ , \quad x_i^* = \frac{1}{n}i = a + i\,\Delta x = a + i\,\frac{1}{n}\ , \quad P_n = \{\, x_0^* = 0, \ldots\ldots.., x_n^* = 1\} \\ x_0^* = 0 = a\ , \quad x_n^* = 1 = b \\ let\ f(x) = x^{2/3} \quad which\ is\ cont.\ on\ [0,1]\end{array}\right\}$$

$$Q6)\ \lim_{n\to\infty}\sum_{k=1}^{n}\frac{\pi}{n}\sin\left(\frac{k\pi}{n}\right)$$

$$\boldsymbol{ans}: \lim_{n\to\infty}\sum_{k=1}^{n}\frac{\pi}{n}\sin\left(\frac{k\pi}{n}\right) = \lim_{n\to\infty}\Delta x\sum_{k=1}^{n} f(x_k^*) \overset{\blacksquare}{\Rightarrow} \int_0^{\pi} \sin x\,dx = -\cos x|_0^{\pi} = 2$$

$$\blacksquare\left\{\begin{array}{c}\Delta x = \frac{\pi}{n} = \frac{b-a}{n}\ , \quad x_k^* = \frac{k\pi}{n} = a + k\,\Delta x = a + k\,\frac{\pi}{n}\ , \quad P_n = \{\, x_0^* = 0, = \ x_1^* = \frac{\pi}{n} \ldots\ldots.., x_n^* = \pi\} \\ x_0^* = 0 = a\ , \quad x_n^* = \pi = b \\ let\ f(x) = \sin x \quad which\ is\ cont.\ on\ [0,\pi]\end{array}\right\}$$

$$Q7)\ \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n}\sqrt{1-\frac{i^2}{n^2}} = \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n}\sqrt{1-\frac{i^2}{n^2}} + \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sqrt{1-\frac{i^2}{n^2}} = 0 + \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sqrt{1-\frac{i^2}{n^2}}$$

$$\boldsymbol{ans}: \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sqrt{1-\left(\frac{i}{n}\right)^2} = \lim_{n\to\infty}\Delta x\sum_{i=1}^{n} f(x_i^*) \overset{\blacksquare}{\Rightarrow} \int_0^1 \sqrt{1-x^2}\,dx = \frac{1^2\pi}{4} = \frac{\pi}{4}$$

$$\blacksquare\left\{\begin{array}{c}\Delta x = \frac{1}{n} = \frac{b-a}{n}\ , \quad x_i^* = \frac{1}{n}i = a + i\,\Delta x = a + i\,\frac{1}{n}\ , \quad P_n = \{\, x_0^* = 0, = \ x_1^* = \frac{\pi}{n} \ldots\ldots.., x_n^* = 1\} \\ x_0^* = 0 = a\ , \quad x_n^* = 1 = b \\ let\ f(x) = \sqrt{1-x} \quad which\ is\ cont.\ on\ [0,1]\end{array}\right\}$$

$$Q6)\ \lim_{n\to\infty}\sum_{i=1}^{n}\frac{4(4j-2)}{n^2}\sqrt{\left(\frac{4j-2}{n}\right)^2 + 1} = \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n}\frac{(4j-2)}{n}\sqrt{\left(\frac{4j-2}{n}\right)^2 + 1}$$

$$\boldsymbol{ans}: \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n}\frac{(4j-2)}{n}\sqrt{\left(\frac{4j-2}{n}\right)^2 + 1} = \lim_{n\to\infty}\Delta x\sum_{i=1}^{n} f(x_i^*) \overset{\blacksquare}{\Rightarrow} \int_0^4 x\sqrt{x^2+1}\,dx \left\{\begin{array}{c} let\ x^2+1 = u \\ du = 2x\,dx \\ as\ x\to 0 \quad \therefore u \to 1 \\ as\ x\to 4 \quad \therefore u\to 17\end{array}\right\}$$

$$= \frac{1}{2}\int_1^{17}\sqrt{u}\,du = \frac{1}{2}\left[\frac{2}{3}u^{\frac{3}{2}}\Big|_1^{17}\right] = \frac{1}{3}\left[u^{\frac{3}{2}}\Big|_1^{17}\right] = \frac{1}{3}\left[(17)^{\frac{3}{2}} - 1\right]$$

$$\blacksquare \left\{ \begin{array}{c} \Delta x = \frac{4}{n} = \frac{b-a}{n}, \quad x_j^* = \frac{4j-2}{n} = a + j\,\Delta x = a + j\,\frac{4}{n}, \quad P_n = \left\{ x_0^* = -\frac{2}{n},\ x_1^* = \frac{2}{n}, \dots\dots\dots, x_n^* = \frac{4n-2}{n} = 4 - \frac{2}{n} \right\} \\ x_0^* = \lim_{n\to\infty} -\frac{2}{n} = 0 = a, \qquad x_n^* = \lim_{n\to\infty} 4 - \frac{2}{n} = 4 = b \\ let\ f(x) = x\sqrt{x^2+1}\ \ which\ is\ cont.\ on\ [0,4] \end{array} \right\}$$

$Q(B)$: *For the funcation* $f(x) = (\sin x)^{x+1}$

write a Riemann Sum obtained by dividing the interval $[1,3]$ *into n equal length subintervals.*

ans: *we have* $\int_1^3 (\sin x)^{x+1}\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x \Rightarrow$ (*using Riemann Sum*)

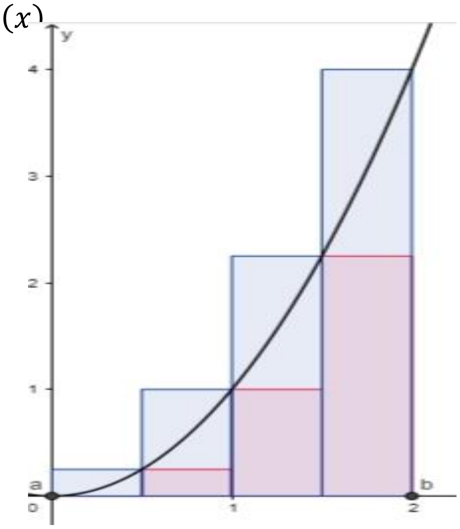
we have $\Rightarrow \left\{ \begin{array}{c} \Delta x = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}, \quad P_n = \{x_0^* = 3, \dots\dots, x_n^* = 0\},\ then\ x_i^* = 1 + \frac{2}{n} i\ where\ x_i^* = a + i\,\Delta x \\ where\ f(x) = (\sin x)^{x+1}\ \ which\ is\ cont.\ on\ [1,3] \end{array} \right\}$

$$\int_1^3 (\sin x)^{x+1}\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(1 + \frac{2}{n} i\right) \frac{2}{n} \Rightarrow (using\ Riemann\ Sum) \Rightarrow \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left(\sin\left(1 + \frac{2}{n} i\right)\right)^{\left(1+\frac{2}{n}i\right)+1}$$

Section 12: Problems session

$Q1$: *calculate the upper and lower riemann sum for the funcation* $f(x) = x^2$, *corresponding to*

the partition P_n *of* $[0,2]$ *into 4 subintervals of equal lenght.*



$$\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$$

$$L(f,p) = \Delta x\,[f(l_1) + f(l_2) + f(l_3) + f(l_4)]$$

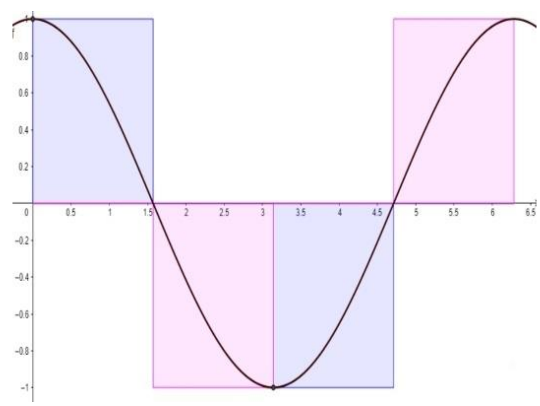
$$= \frac{1}{2}\left[f(0) + f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right)\right] = \frac{1}{2}\left[0 + \frac{1}{2}^2 + 1^2 + \frac{3}{2}^2\right] = \frac{7}{4}$$

$$U(f,p) = \Delta x\,[f(u_1) + f(u_2) + f(u_3) + f(u_4)]$$

$$= \frac{1}{2}\left[f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right) + f(2)\right] = \frac{1}{2}\left[\frac{1}{2}^2 + 1^2 + \frac{3}{2}^2 + 2^2\right] = \frac{15}{4}$$

$Q2$: *calculate the upper and lower riemann sum for the funcation* $f(x) = \cos x$, *corresponding to*

the partition P_n *of* $[0,2\pi]$ *into 4 subintervals of equal lenght.*



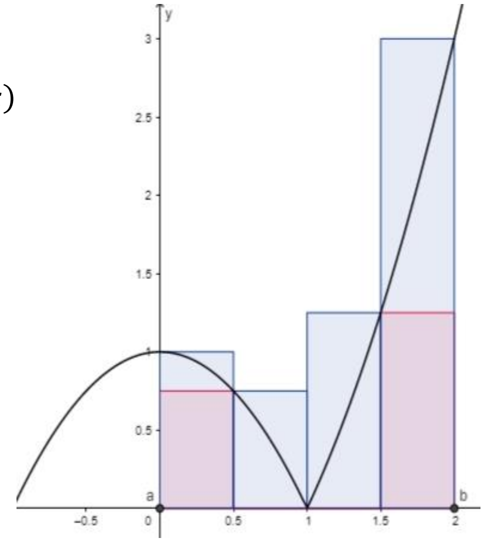
$$\Delta x = \frac{b-a}{n} = \frac{2\pi - 0}{4} = \frac{\pi}{2}$$

$$L(f,p) = \Delta x\,[f(l_1) + f(l_2) + f(l_3) + f(l_4)]$$

$$= \frac{\pi}{2}\left[f\left(\frac{\pi}{2}\right) + f(\pi) + f(\pi) + f\left(\frac{3\pi}{2}\right)\right] = \frac{\pi}{2}[0 - 1 - 1 + 0] = -\pi$$

$$U(f,p) = \Delta x\,[f(u_1) + f(u_2) + f(u_3) + f(u_4)]$$

$$= \frac{\pi}{2}\left[f(0) + f\left(\frac{\pi}{2}\right) + f\left(\frac{3\pi}{2}\right) + f(2\pi)\right] = \frac{\pi}{2}[1 + 0 + 0 + 1] = \pi$$



$Q3$: *calculate the upper and lower riemann sum for the funcation* $f(x) = |x^2 - 1|$ *, corresponding to*

the partition P_n *of* $[0,2]$ *into* 4 *subintervals of equal lenght.*

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$$

$$L(f,p) = \Delta x\,[f(l_1) + f(l_2) + f(l_3) + f(l_4)]$$

$$= \frac{1}{2}\left[f\left(\frac{1}{2}\right) + f(1) + f(1) + f\left(\frac{3}{2}\right)\right] = \frac{1}{2}\left[\frac{3}{4} + 0 + 0 + \frac{5}{4}\right] = 1$$

$$U(f,p) = \Delta x\,[f(u_1) + f(u_2) + f(u_3) + f(u_4)]$$

$$= \frac{1}{2}\left[f(0) + f\left(\frac{1}{2}\right) + f\left(\frac{3}{2}\right) + f(2)\right] = \frac{1}{2}\left[1 + \frac{3}{4} + \frac{5}{4} + 3\right] = 3$$

The definite integral

Q4: *By using* $Q1$, Prove that $f(x) = x^2$ *is integrable on the interval* $[0,2]$

Let $I = \int_0^2 x^2\,dx$. such that for every partition P of $[0,2]$ we have $L(f,P) \le I \le U(f,P) \rightarrow by\ Q1 \rightarrow \frac{7}{4} \le I \le \frac{15}{4}$

Then the function $f(x)$ is integrable on $[0,2]$, and we call I the definite integral of function f on $[0,2]$.

Q5: *By using* $Q2$, Prove that $\int_0^{2\pi} cosx\,dx \le \pi$

Since the function $f(x) = cosx$ is integrable on $[0,2\pi]$ then $L(f,P) \le \int_0^{2\pi} cosx\,dx \le U(f,P) \therefore \int_0^{2\pi} cosx\,dx \le \pi$

Q6: *By using* $Q3$, *Prove* $1 \le I \le 3$ *where* $I = \int_0^2 |x^2 - 1|dx,$

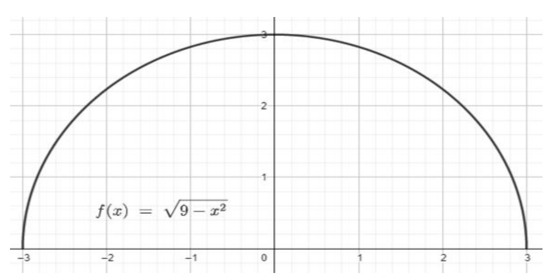
Since the function $f(x) = |x^2 - 1|$ is integrable on $[0,2]$ then $L(f,P) \le I \le U(f,P) \therefore 1 \le I \le 3$

Section 13: Problems session

1) $\int_{-2}^{2}(2+5x)\,dx = 2\int_{-2}^{2} dx + 5\int_{-2}^{2} x\,dx = 2x|_{-2}^{2} + 0 = 2\big(2-(-2)\big) = 8$ *since x is odd function*

2) $$\int_{-\pi}^{\pi} 1 + \sin^3 x\sqrt{9x^2 + 4cos^2 x}\,dx$$
$$= \int_{-\pi}^{\pi} dx + \int_{-\pi}^{\pi} \sin^3 x\sqrt{9x^2 + 4cos^2 x}\,dx = x|_{-\pi}^{\pi} + 0$$
$$= 2\pi$$

since $\sin^3 x\sqrt{9x^2 + 4cos^2 x}$ *is odd function*



3) $\int_{-1}^{1} \frac{\sin x}{1+x^2}\,dx = 0$ *since* $\frac{\sin x}{1+x^2}$ *is odd function*

4) $\int_{-3}^{3}\sqrt{9-x^2}\,dx = \frac{\pi\, r^2}{2}(half\ circle) = \frac{\pi\,(3)^2}{2} = \frac{9}{2}\pi$

$$\begin{Bmatrix} let\ y=\sqrt{9-x^2} \quad \to \quad y^2 = 9-x^2 \quad \to \quad y^2+x^2 = 9 = 3^2 \\ using\ circle\ eqn: y^2+x^2=r^2 \quad \therefore we\ have\ r = 3 \end{Bmatrix}$$

Section 14: Problems session

2) $G(x) = x^2\int_{-4}^{5x} e^{-t^2}\,dt \quad find\ \frac{d}{dx}\,G(x)\ at\ x = -\frac{4}{5}$

$$\frac{d}{dx}\,G(x) = 2x\int_{-4}^{5x} e^{-t^2}\,dt + 5x^2\left(e^{-25x^2}\right)\ \therefore G'\left(-\frac{4}{5}\right) = 2\left(-\frac{4}{5}\right)\int_{-4}^{5\left(-\frac{4}{5}\right)} e^{-t^2}\,dt + 5\left(-\frac{4}{5}\right)^2\left(e^{-25\left(-\frac{4}{5}\right)^2}\right) = \frac{16}{5}e^{-\frac{16}{5}}$$

3) $H(x) = \int_{x^2}^{x^3} e^{-t^2}\,dt \quad \therefore \frac{d}{dx}\,H(x) = 3x^2e^{-x^6} - 2xe^{-x^4} \ \therefore\ H'(1) = 3e^{-1} - 2e^{-1} = e^{-1} = \frac{1}{e}$

4) $\lim_{x\to 0}\frac{1}{x^3}\int_0^x tan(t^2)\,dt = \lim_{x\to 0}\frac{\int_0^x tan(t^2)\,dt}{x^3}\left[\frac{0}{0}\right] = (L`H)\lim_{x\to 0}\frac{tan(x^2)}{3x^2} = \frac{1}{3}\lim_{x\to 0}\frac{sin(x^2)}{x^2} * \frac{1}{cos(x^2)} = \frac{1}{3} * 1 * 1 = \frac{1}{3}$

5) $\lim_{x\to\pi/2}\frac{\int_x^{\pi/2} ln\,(\sin t)\ dt}{sinx - 1}\left[\frac{0}{0}\right] = (L`H)\lim_{x\to\pi/2}\frac{-ln\,(\sin x)}{\cos x}\left[\frac{0}{0}\right] = (L`H)\lim_{x\to\pi/2}\frac{-\frac{\cos x}{\sin x}}{-\sin x} = \lim_{x\to\pi/2}\frac{\cos x}{\sin^2 x} = \frac{0}{1^2} = 0$

6) $\lim_{x\to 0^+}\frac{238x + x^{2019}}{\int_0^x(\cos(t^3) + cos^3 t)\,dt}\left[\frac{0}{0}\right] = (L`H)\lim_{x\to 0^+}\frac{238 + 2019x^{2018}}{\cos(x^3) + cos^3 x} = \frac{238+0}{1+1^3} = \frac{238}{2} = 119$

Section 15.2: Problems session

(1) $\int_0^{\frac{\pi}{3}}\frac{tan\,x\,\sqrt{sec\,x} + sec\,x\sqrt{tan\,x}}{cos\,x}dx = \int_0^{\frac{\pi}{3}}\sec x\,tan\,x\ \sqrt{sec\,x}\,dx + \int_0^{\frac{\pi}{3}} sec^2\,x\,\sqrt{tan\,x}\,dx = (*)\int_1^2\sqrt{u}\,du + \int_0^{\sqrt{3}}\sqrt{v}\,dv$

$= \frac{2}{3}u^{3/2}\Big|_1^2 + \frac{2}{3}v^{3/2}\Big|_0^{\sqrt{3}} = \frac{2^{5/2}}{3} - \frac{2}{3} + 2\left(3^{-1/4}\right)$

$$(*)\begin{cases} \int_0^{\frac{\pi}{3}}\sec x\,tan\,x\ \sqrt{sec\,x}\,dx \quad let\ u = \sec x \quad then\quad du = \sec x\tan x\,dx \quad as\ x\to 0 \therefore\ u\to 1 \quad as\ x\to\frac{\pi}{3}\therefore\ u\to 2 \\ \int_0^{\frac{\pi}{3}} sec^2\,x\,\sqrt{tan\,x}\,dx \quad let\ v = \tan x \quad then\quad dv = \sec^2 x\,dx \quad as\ x\to 0 \therefore\ v\to 0 \quad as\ x\to\frac{\pi}{3}\therefore\ v\to\sqrt{3} \end{cases}$$

(2) $\int_{\frac{\pi}{2}}^{\pi}\cos^5 x\sin^2 x\,dx = \int_{\frac{\pi}{2}}^{\pi}(\cos^2 x)^2\sin^2 x\cos x\,dx = \int_{\frac{\pi}{2}}^{\pi}(1-\sin^2 x)^2\sin^2 x\cos x\,dx$

$= (*)\int_1^0(1-u^2)^2u^2du = \int_1^0 u^2 - 2u^4 + u^6du = \left[\frac{u^3}{3} - 2\frac{u^5}{5} + \frac{u^7}{7}\right]\Big|_1^0 = -\frac{1}{3} + \frac{2}{5} - \frac{1}{7} = \frac{-35+42-15}{105} = \frac{-8}{105}$

$(*)$ $let\ u = \sin x \quad then\ du = \cos x\ dx \quad as\ x\to\frac{\pi}{2}\therefore\ u\to 1,\ as\ x\to\pi\therefore\ u\to 0$

(3) $\int\frac{\sin^3 x}{\sqrt{\cos x}}dx = \int\frac{\sin^2 x}{\sqrt{\cos x}}\sin x\,dx = \int\frac{(1-\cos^2 x)}{\sqrt{\cos x}}\sin x\,dx = (*) - \int\frac{(1-u^2)}{\sqrt{u}}du = \int u^{3/2} - u^{-1/2}\,du$

$\frac{2}{5}u^{5/2} - 2u^{1/2} + c = \frac{2}{5}\cos^{5/2} x - 2\cos^{1/2} x + c \qquad (*)\ let\ u = \cos x \quad then\ du = -\sin x\ dx$

(4) $\int_1^2 \sqrt{\frac{1+\sqrt{x}}{x}}\,dx = \int_1^2 \frac{1}{\sqrt{x}}\sqrt{1+\sqrt{x}}dx = 2\int_2^{1+\sqrt{2}} \sqrt{u}\,du = (*)\, 2\,\frac{2}{3}u^{3/2}\Big|_2^{1+\sqrt{2}} = \frac{4}{3}\left[\left(1+\sqrt{2}\right)^{3/2} - 2^{3/2}\right]$

$(*)$ *let* $u = 1+\sqrt{x}$ *then* $du = \dfrac{1}{2\sqrt{x}}\,dx$

(5) $\int \frac{\ln(\ln x)}{x\ln x}dx = (*)\int \frac{\ln u}{u}du = (**)\int t\,dt = \frac{t^2}{2} + c = \frac{\ln^2 u}{2} + c = \frac{\ln^2(\ln x)}{2} + c$

$(*)$ *let* $u = \ln x$ *then* $du = \dfrac{1}{x}\,dx$ $(**)$ *let* $t = \ln u$ *then* $dt = \dfrac{1}{u}\,du$

(6) $\int \tan^5 x \sec x\,dx = \int \tan^4 x \sec x \tan x\,dx = \int (\sec^2 x - 1)^2 \sec x \tan x\,dx = (*)\int (u^2-1)^2 du$

$= \int (u^4 - 2u^2 + 1)du = \frac{u^5}{5} - \frac{2}{3}u^3 + u + c = \frac{\sec^5 x}{5} - \frac{2}{3}\sec^3 x + \sec x + c$

$(*)$ *let* $u = \sec x$ *then* $du = \sec x \tan x\,dx$

(7) $\displaystyle\int \frac{2x\,dx}{x^2+2x+3}dx = \int \frac{2(x+1)-2}{(x+1)^2+2}dx = 2\int \frac{x+1}{(x+1)^2+2}dx - 2\int \frac{1}{(x+1)^2+2}dx$

$(*)\int \frac{du}{u} - \int \frac{1}{\left(\frac{x+1}{\sqrt{2}}\right)^2+1}dx = \ln((x+1)^2+2) - \sqrt{2}\arctan\left(\frac{x+1}{\sqrt{2}}\right) + c$ $(*)$ *let* $u = (x+1)^2 + 2$ *then* $du = 2(x+1)\,dx$

(8) $\int \frac{x^2+x+1\,dx}{\sqrt{x}} = (*)\, 2\int (u^4+u^2+1)\,du = 2\left(\frac{x^{5/2}}{5} + \frac{x^{3/2}}{3} + \sqrt{x}\right) + c$ $(*)$ *let* $u = \sqrt{x}$ *then* $du = \frac{1}{2\sqrt{x}}\,dx$

(9) $\int_0^{\frac{1}{2}} \frac{\arcsin x}{\sqrt{1-x^2}}dx = (*)\int_0^{\frac{\pi}{6}} u\,du = \frac{u^2}{2}\Big|_0^{\frac{\pi}{6}} = \frac{\pi^2}{72}$

$(*)$ *let* $u = \arcsin x$ *then* $du = \dfrac{1}{\sqrt{1-x^2}}\,dx$ *as* $x \to 0 \therefore u \to 0$, *as* $x \to \dfrac{1}{2} \therefore u \to \dfrac{\pi}{6}$

(10) $\int \frac{1+x}{1+\sqrt{x}}dx = (*)\int \frac{1+u^2}{1+u}2u\,du = 2\int \frac{u^3+u}{1+u}du = 2\int \left(u^2 - u + 2 - \frac{2}{1+u}\right)du$ $(*)$ *let* $u^2 = x$ *then* $2u\,du = dx$

$= 2\left(\frac{u^3}{3} - \frac{u^2}{2} + 2u - 2\ln|u+1|\right) + c = 2\left(\frac{x^{3/2}}{3} - \frac{x}{2} + 2\sqrt{x} - 2\ln\left|\sqrt{x}+1\right|\right) + c$

(11) $\int \frac{x^3}{\sqrt{1+x^2}}dx = (*)\int (u^2-1)\,du = \frac{u^3}{3} - u + c = \frac{(1+x^2)^{3/2}}{3} - \sqrt{1+x^2} + c$

$(*)$ *let* $u = \sqrt{1+x^2}$ *then* $du = \dfrac{x}{\sqrt{1+x^2}}dx$

(12) $\int_0^{\frac{\pi}{4}} \frac{\sin x - \cos x}{\sin x + \cos x}dx = (*)\int_1^{\sqrt{2}} \frac{-du}{u} = -\ln|u|\,\Big|_1^{\sqrt{2}} = -\ln\sqrt{2} = -\frac{\ln 2}{2}$

$(*)$ *let* $u = \sin x + \cos x$ *then* $du = (\cos x - \sin x)\,dx = -(\sin x - \cos x)\,dx$ *as* $x \to 0 \therefore u \to 1$, *as* $x \to \dfrac{\pi}{4} \therefore u \to \sqrt{2}$

Section 15.3: Problems session

(1) $\int e^{6x}\sin(e^{3x})\,dx = (*) -\frac{1}{3}e^{3x}\cos(e^{3x}) + \int e^{3x}\cos(e^{3x})\,dx = -\frac{1}{3}e^{3x}\cos(e^{3x}) + \frac{1}{3}\sin(e^{3x}) + c$

$(*)\begin{pmatrix} u = e^{3x} & dv = e^{3x}\sin(e^{3x})\,dx \\ du = 3e^{3x}dx & v = -\dfrac{1}{3}\cos(e^{3x}) \end{pmatrix} \to u\,v - \displaystyle\int v\,du$

(2) $\int_1^4 \sqrt{x}\,e^{\sqrt{x}}\,dx = \int_1^4 \sqrt{x} * e^{\sqrt{x}} * 2\sqrt{x} * \frac{1}{2\sqrt{x}}\,dx = (*)\int_1^2 t * e^t * 2t\,dt = (**)2\int_1^2 t^2 e^t\,dt$

$$= 2\left[t^2e^t|_1^2 - (***)\ 2\int_1^2 te^t\,dt\right] = 2(4e^2 - e) - 4\left[(te^t)|_1^2 - \int_1^2 e^t\,dt\right] = 8e^2 - 2e - 4[2e^2 - e - (e^t)|_1^2]$$

$$= 8e^2 - 2e - 8e^2 + 4e + 4e^2 - 4e = 4e^2 - 2e$$

$(*)$ *let* $t = \sqrt{x}$ *then* $dt = \dfrac{1}{2\sqrt{x}}\,dx$ *as* $x \to 1 \therefore t \to 1$, *as* $x \to 4 \therefore t \to 2$

$(**)\begin{pmatrix} u = t^2 & dv = e^t\,dt \\ du = 2t\,dt & v = e^t \end{pmatrix} \quad (***)\begin{pmatrix} u = t & dv = e^t\,dt \\ du = dt & v = e^t \end{pmatrix} \to u\,v - \int v\,du$

$$(3) \int x \arctan(x^2)\,dx = (*)\left[\frac{x^2}{2}\arctan(x^2) - (**)\int \frac{x^3}{1+x^4}\,dx\right] = \frac{x^2}{2}\arctan(x^2) - \frac{1}{4}\ln(1+x^4) + c$$

$(*)\begin{pmatrix} u = \arctan(x^2) & dv = x dx \\ du = \dfrac{2x}{1+x^4}dx & v = \dfrac{x^2}{2} \end{pmatrix} \to u\,v - \int v\,du \quad (**)$ *let* $u = 1 + x^4$ *then* $du = 4x^3\,dx \to \int \dfrac{x^3}{1+x^4}dx = \dfrac{1}{4}\int \dfrac{1}{u}\,du$

$$(4) \int x^2\,e^x\,dx = (*)\left[x^2e^x - 2(**)\int xe^x\,dx\right] = x^2e^x - 2\left(xe^x - \int e^x\,dx\right) = x^2e^x - 2xe^x + 2e^x + c$$

$$= e^x(x^2 - 2x + 2) + c$$

$(*)\begin{pmatrix} u = x^2 & dv = e^x dx \\ du = 2x\,dx & v = e^x \end{pmatrix} \quad (**)\begin{pmatrix} u = x & dv = e^x\,dx \\ du = dx & v = e^x \end{pmatrix} \to u\,v - \int v\,du$

$$(5) \int (x-1)^2 \ln(x-1)\ dx = (*)\left[\tfrac{(x-1)^3}{3}\ln(x-1) - \tfrac{1}{3}\int (x-1)^2\,dx\right] = \tfrac{(x-1)^3}{3}\ln(x-1) - \tfrac{(x-1)^3}{9} + c$$

$(*)\begin{pmatrix} u = \ln(x-1) & dv = (x-1)^2dx \\ du = \dfrac{dx}{x-1}dx & v = \dfrac{(x-1)^3}{3} \end{pmatrix} \to u\,v - \int v\,du$

$$(6) \int e^{\sqrt[3]{x}}\ dx = (*)\int e^y 3y^2\,dy = (**)3\int y^2e^y\,dy = 3\left[y^2e^y - 2(***)\int ye^y\,dy\right]$$

$$= 3\left[y^2e^y - 2\left(ye^y - \int e^y\,dy\right)\right] = 3y^2e^y - 6ye^y + 2e^y + c = 3x^{2/3}e^{\sqrt[3]{x}} - 6\sqrt[3]{x}e^{\sqrt[3]{x}} + 2e^{\sqrt[3]{x}} + c$$

$(*)$ *let* $y = \sqrt[3]{x}$ *then* $dy = \dfrac{1}{3}x^{-2/3}\,dx$ *i.e* $dx = 3x^{2/3}\,dy = 3y^2\,dy$

$(**)\begin{pmatrix} u = y^2 & dv = e^y dy \\ du = 2y\,dy & v = e^y \end{pmatrix} \quad (***)\begin{pmatrix} u = y & dv = e^y\,dy \\ du = dy & v = e^y \end{pmatrix} \to u\,v - \int v\,du$

$$(7) \int_1^3 \frac{\arctan\sqrt{x}}{\sqrt{x}}\,dx = (*)2\int_1^{\sqrt{3}} \arctan t\ dt = (**)2\left[t\arctan t|_1^{\sqrt{3}} - (***)\int_1^{\sqrt{3}} \frac{t}{1+t^2}\,dt\right]$$

$$= 2\left[\sqrt{3} * \tfrac{\pi}{3} - \tfrac{\pi}{4}\right] - [\ln 4 - \ln 2] = \tfrac{2\sqrt{3}\pi}{3} - \tfrac{\pi}{2} - \ln 2$$

$(*)$ *let* $t = \sqrt{x}$ *then* $dt = \dfrac{1}{2\sqrt{x}}\,dx$ *as* $x \to 1 \therefore t \to 1$, *as* $x \to 3 \therefore t \to \sqrt{3}$

$(**)\begin{pmatrix} u=\arctan t & dv=dt \\ du=\frac{1}{1+t^2}\,dy & v=t\,dt \end{pmatrix} \quad \to u\,v-\int v\,du \quad (***)\ let\ w=1+t^2\ \ then\ dw=2t\,dt\ \ i.e\ \int_1^{\sqrt{3}}\frac{t}{1+t^2}\,dt=\frac{1}{2}\int_2^4\frac{1}{w}\,dw$

(8) $\int \frac{x^3e^{x^2}}{(x^2+1)^2}dx=\int\left(x^2e^{x^2}\right)\frac{x}{(x^2+1)^2}dx=(*)\frac{-x^2e^{x^2}}{2(x^2+1)}+\int xe^{x^2}\,dx=\frac{-x^2e^{x^2}}{2(x^2+1)}+\frac{1}{2}e^{x^2}+c$

$(*)\begin{pmatrix} u=x^2e^{x^2} & dv=\frac{x}{(x^2+1)^2}\,dx \\ du=2xe^{x^2}(1+x^2)\,dx & v=-\frac{1}{2(x^2+1)} \end{pmatrix} \quad \to u\,v-\int v\,du$

Section 15.4: Problems session

(1) $\int \frac{x+1}{x^2(x^2+1)}dx.$

$$\frac{x+1}{x^2(x^2+1)}=\frac{Ax+B}{x^2}+\frac{Cx+D}{(x^2+1)}\Rightarrow x+1=(A+C)x^3+(B+D)x^2+Ax+B\ \therefore A=B=1, C=D=-1$$

$$\int\frac{x+1}{x^2(x^2+1)}dx=\int\frac{x+1}{x^2}dx+\int\frac{-x-1}{(x^2+1)}dx=\int\frac{1}{x}dx+\int\frac{1}{x^2}dx-\int\frac{x}{(x^2+1)}dx-\int\frac{1}{(x^2+1)}dx$$

$$=\ln|x|-\frac{1}{x}-\frac{1}{2}\ln(x^2+1)-\arctan x+c$$

(2) $\int \frac{3x^3-3x^2+x-1}{x^2(2x^2+1)}dx.$

$$\frac{3x^3-3x^2+x-1}{x^2(2x^2+1)}=\frac{Ax+B}{x^2}+\frac{Cx+D}{(2x^2+1)}\Rightarrow 3x^3-3x^2+x-1=(2A+C)x^3+(2B+D)x^2+Ax+B$$

$2A+C=3 \quad 2B+D=1 \quad \therefore A=C=1 \quad D=B=-1$

$$\int\frac{3x^3-3x^2+x-1}{x^2(2x^2+1)}dx=\int\frac{x-1}{x^2}dx+\int\frac{x-1}{(2x^2+1)}dx=\int\frac{1}{x}dx-\int\frac{1}{x^2}dx+\int\frac{x}{(2x^2+1)}dx-\int\frac{1}{(2x^2+1)}dx$$

$$=\ln|x|+\frac{1}{x}+\frac{1}{4}\ln(x^2+1)-\frac{1}{\sqrt{2}}\arctan\left(\sqrt{2}\,x\right)+c$$

(3) $\int \frac{x^2+11x+10}{x^3+4x^2+5x}dx=\int\frac{x^2+11x+10}{x(x^2+4x+5)}dx$ (Assume that ($x\neq 0$))

$$\frac{x^2+11x+10}{x(x^2+4x+5)}=\frac{A}{x}+\frac{Bx+C}{(x^2+4x+5)}\Rightarrow x^2+11x+10=(A+B)x^2+(4A+C)x+5A$$

$\therefore A=2 \quad B=-1 \quad C=3$

$$\int\frac{x^2+11x+10}{x^3+4x^2+5x}dx=\int\frac{2}{x}dx+\int\frac{-x+3}{(x^2+4x+5)}dx=2\ln|x|+\int\frac{-x+3}{((x+2)^2+1)}dx\begin{cases}u=x+2\\ du=dx\end{cases}$$

$$=2\ln|x|+\int\frac{5-u}{(u^2+1)}du=2\ln|x|+\int\frac{5}{(u^2+1)}du-\int\frac{u}{(u^2+1)}du=2\ln|x|+5\arctan u-\frac{1}{2}\ln(u^2+1)+c$$

$$=2\ln|x|+5\arctan(x+2)-\frac{1}{2}\ln((x+2)^2+1)+c$$

(4) $\int \frac{8x+1}{x^2(x^2+4)}dx.$

$$\frac{8x+1}{x^2(x^2+4)}=\frac{Ax+B}{x^2}+\frac{Cx+D}{(x^2+4)} \;\Rightarrow 8x+1=(A+C)x^3+(B+D)x^2+4Ax+4B$$

$$\therefore A=2 \quad B=\frac{1}{4} \quad C=-2 \quad D=-\frac{1}{4}$$

$$\int \frac{8x+1}{x^2(x^2+4)}dx=\int \frac{2x+\frac{1}{4}}{x^2}dx+\int \frac{-2x-\frac{1}{4}}{(x^2+4)}dx=2\int \frac{1}{x}dx+\frac{1}{4}\int \frac{1}{x^2}dx-\int \frac{2x}{(x^2+4)}dx-\frac{1}{4}\int \frac{1}{(x^2+4)}dx$$

$$=2\ln|x|+\frac{1}{4x}-\ln(x^2+4)-\frac{1}{8}\arctan\left(\frac{x}{2}\right)+c$$

(5) $\int \frac{10}{(x-1)(x^2+9)}dx.$

$$\frac{10}{(x-1)(x^2+9)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+9} \Rightarrow 10=(A+B)x^2+(-B+C)x+9A-C \quad \therefore A=1 \quad B=-1 \quad C=-1$$

$$\int \frac{10}{(x-1)(x^2+9)}dx=\int \frac{1}{x-1}dx-\int \frac{x+1}{x^2+9}dx=\int \frac{1}{x-1}dx-\int \frac{x}{x^2+9}dx-\int \frac{1}{x^2+9}dx$$

$$=\ln|x-1|-\frac{1}{2}\ln(x^2+9)-\frac{1}{3}\arctan\left(\frac{x}{3}\right)+c=\ln\frac{|x-1|}{\sqrt{x^2+9}}-\frac{1}{3}\arctan\left(\frac{x}{3}\right)+c$$

(6) $\int \frac{x^3+1}{x^3-x^2}dx=\int 1\,dx+\int \frac{x^2+1}{x^3-x^2}dx=\int 1\,dx+\int \frac{x^2}{x^3-x^2}dx+\int \frac{1}{x^3-x^2}dx=\int 1\,dx+\int \frac{1}{x-1}dx+\int \frac{1}{x^2(x-1)}dx$

$$=x+\ln|x-1|+(*)\int \frac{1}{x^2(x-1)}dx=x+\ln|x-1|-\ln|x|+\frac{1}{x}+c=x+\frac{1}{x}+\ln\frac{(x-1)^2}{x}+c$$

$$(*)\frac{1}{x^2(x-1)}=\frac{Ax+B}{x^2}+\frac{C}{x-1} \;\Rightarrow 1=(A+C)x^2+(B-A)x-B \quad \therefore A=-1 \quad B=-1 \quad C=1$$

$$(*)\int \frac{1}{x^2(x-1)}dx=-\int \frac{x+1}{x^2}dx+\int \frac{1}{x-1}dx=-\int \frac{1}{x}dx-\int \frac{1}{x^2}dx+\int \frac{1}{x-1}dx=-\ln|x|+\frac{1}{x}+c$$

(7) $\int \frac{1}{x^3(x-1)}dx.$

$$\frac{1}{x^3(x-1)}=\frac{Ax^2+Bx+C}{x^3}+\frac{D}{x-1} \;\Rightarrow\; 1=(A+D)x^3+(-A+B)x^2+(-B+C)x+(-C)$$

$$\therefore A=-1 \quad B=-1 \quad C=-1 \quad D=1$$

$$\int \frac{1}{x^3(x-1)}dx=-\int \frac{x^2+x+1}{x^3}dx+\int \frac{1}{x-1}dx=-\int \frac{1}{x}dx-\int \frac{1}{x^2}dx-\int \frac{1}{x^3}dx+\int \frac{1}{x-1}dx$$

$$=-\ln|x|+\frac{1}{x}+\frac{1}{2x^2}-\ln|x-1|+c$$

(8) $\int \frac{1}{(1+x^{1/3})(1+x^{2/3})}dx=\int \frac{3u^2}{(1+u)(1+u^2)}du \;\; where \begin{pmatrix} u=x^{1/3}\rightarrow u^3=x \\ dx=3u^2du \end{pmatrix}$

$$\frac{3u^2}{(1+u)(1+u^2)}=\frac{A}{1+u}+\frac{Bu+C}{1+u^2} \Rightarrow 3u^2=(A+B)u^2+(B+C)u-(A+C) \quad \therefore A=\frac{3}{2} \quad B=\frac{3}{2} \quad C=-\frac{3}{2}$$

$$\int \frac{3u^2}{(1+u)(1+u^2)}du = \frac{3}{2}\int \frac{1}{1+u}du + \frac{3}{2}\int \frac{u-1}{1+u^2}du = \frac{3}{2}\int \frac{1}{1+u}du + \frac{3}{2}\int \frac{u}{1+u^2}du - \frac{3}{2}\int \frac{1}{1+u^2}du$$

$$= \frac{3}{2}\ln|u+1| + \frac{3}{4}\ln|u^2+1| - \frac{3}{2}\arctan(u) + c = \frac{3}{2}\ln\left|x^{1/3}+1\right| + \frac{3}{4}\ln\left|x^{2/3}+1\right| - \frac{3}{2}\arctan\left(x^{1/3}\right) + c$$

(9) $\int \frac{dx}{x(1+x^{119})} = \int \frac{du}{119\,u\,(u-1)}\, where \begin{pmatrix} u = 1+x^{119} \to du = 119x^{118}dx \\ dx = \frac{du}{119\,x^{118}} \to \frac{dx}{x} = \frac{du}{119\,x^{119}} = \frac{du}{119\,(u-1)} \end{pmatrix}$

$$= \frac{1}{u\,(u-1)} = \frac{A}{u-1} + \frac{B}{u} \Rightarrow 1 = (A+B)u - B \quad \therefore A = 1 \quad B = -1$$

$$\frac{1}{119}\int \frac{du}{u\,(u-1)} = \frac{1}{119}\left[\int \frac{du}{u-1} - \int \frac{du}{u}\right] = \frac{1}{119}[\ln|u-1| - \ln|u|] + c = \frac{1}{119}\ln\left|\frac{u-1}{u}\right| + c$$

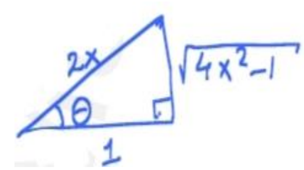
$$= \frac{1}{119}\ln\left|\frac{x^{119}}{1+x^{119}}\right| + c$$

Section 15.5: Problems session

(1) $\int \frac{x^2}{(4x^2-1)^{3/2}}dx$. (Assume that $(x > 1/2)$)

$$\int \frac{x^2}{(4x^2-1)^{3/2}}dx = (*)\int \frac{\frac{1}{4}\sec^2\theta}{\tan^3\theta}\frac{1}{2}\sec\theta\tan\theta\ d\theta = \frac{1}{8}\int \frac{\sec^3\theta}{\tan^2\theta}d\theta = \frac{1}{8}\int \frac{1}{\cos^3\theta}\frac{\cos^2\theta}{\sin^2\theta}d\theta$$

$$= \frac{1}{8}\int \frac{1}{\cos\theta\sin^2\theta}d\theta = \frac{1}{8}\int \frac{\sin^2\theta+\cos^2\theta}{\cos\theta\sin^2\theta}d\theta = \frac{1}{8}\left[\int \frac{1}{\cos\theta}d\theta + (**)\int \frac{\cos\theta}{\sin^2\theta}d\theta\right]$$



$$(**)\begin{cases} u = \sin\theta \\ du = \cos\theta\, d\theta \end{cases}$$

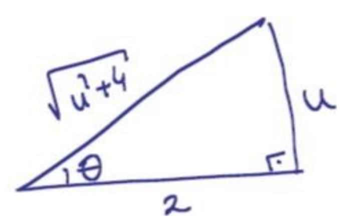
$$= \frac{1}{8}\left[\int \sec\theta\, d\theta + \int \frac{1}{u^2}du\right] = \frac{1}{8}\left[\ln|\sec\theta+\tan\theta| - \frac{1}{\sin\theta}\right] + c = \frac{1}{8}\left[\ln\left|2x+\sqrt{4x^2-1}\right| - \frac{2x}{\sqrt{4x^2-1}}\right] + c$$

$$(*) let \sec\theta = \frac{2x}{1} \ \to\ x = \frac{1}{2}\sec\theta \qquad so \quad dx = \frac{1}{2}\sec\theta\tan\theta\, d\theta$$

(2) $\int \frac{1}{\sqrt{t^2-6t+13}}dt = \int \frac{1}{\sqrt{(t-3)^2+4}}dt = (*)\int \frac{1}{\sqrt{u^2+4}}du \quad (*)\begin{cases} u = t-3 \\ du = dt \end{cases}$

$$= \int \frac{1}{\sqrt{u^2+4}}du = (**)\int \frac{\cos\theta}{2}2\sec^2\theta\, d\theta = \int \cos\theta\frac{1}{\cos^2\theta}\ d\theta = \int \sec\theta\, d\theta$$

$$= \ln|\sec\theta+\tan\theta| + c$$



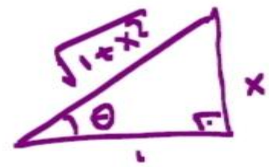
$$= \ln\left|\frac{\sqrt{u^2+4}}{2}+\frac{u}{2}\right| + c = \ln\left|\frac{\sqrt{(t-3)^2+4}}{2}+\frac{t-3}{2}\right| + c \qquad (**) let \tan\theta = \frac{u}{2}\ \to\ u = 2\tan\theta \quad so \quad du = 2\sec^2\theta\, d\theta$$

(3) $\int \frac{x^3}{\sqrt{1+x^2}}dx = (*)\int \tan^3\theta\cos\theta\sec^2\theta\, d\theta = \int \frac{\sin^3\theta}{\cos^3\theta}\cos\theta\frac{1}{\cos^2\theta}d\theta = \int \frac{1-\cos^2\theta}{\cos^4\theta}\sin\theta\, d\theta$

$$= \int \frac{1-\cos^2\theta}{\cos^4\theta}\sin\theta\, d\theta \left(\begin{matrix} u=\cos\theta \\ du=-\sin\theta\, d\theta\end{matrix}\right) = -\int \frac{1-u^2}{u^4}du = \int (u^{-2}-u^{-4})\,du = -\frac{1}{u}+\frac{1}{3u^3}+c$$

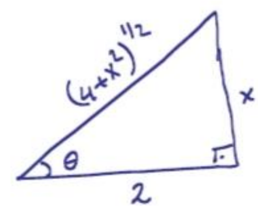
$$= -\frac{1}{\cos\theta}+\frac{1}{3\cos^3\theta}+c = -\sqrt{1+x^2}+\frac{(1+x^2\,)^{\frac{3}{2}}}{3}+c$$

$(*) let \tan\theta = x \quad so \quad dx = \sec^2\theta\, d\theta$



$$(4) \int \frac{dx}{(4+x^2)^{\frac{3}{2}}} = (*) \int \frac{1}{8}\cos^3\theta\, 2\sec^2\theta\, d\theta = \frac{1}{4}\int \cos\theta\, d\theta = \frac{1}{4}\sin\theta + c = \frac{1}{4}\frac{x}{\sqrt{4+x^2}}+c$$

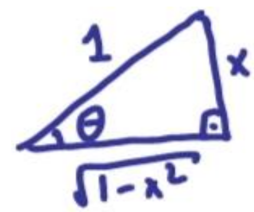
$(*) let \tan\theta = \frac{x}{2} \ \rightarrow\ x = 2\tan\theta \quad so \quad dx = 2\sec^2\theta\, d\theta$



$$(5) \int \frac{x^3}{\sqrt{1-x^2}}\, dx = (*) \int \frac{\sin^2\theta}{\cos\theta}\cos\theta\, d\theta = \int \sin^2\theta\, d\theta = \frac{1}{2}\int [1-\cos 2\theta]\, d\theta$$

$$= \frac{1}{2}\int [1-\cos 2\theta]\, d\theta = \frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]+c = \frac{1}{2}[\theta - \sin\theta\cos\theta]+c = \frac{1}{2}\left[\arcsin x - x\sqrt{1-x^2}\right]+c$$

$(*) let \sin\theta = \frac{x}{1} \ \rightarrow\ x = \sin\theta \quad so \quad dx = \cos\theta\, d\theta$



Section 16: Problems session

Evaluate the improper integral (1 to 7):

1) $\int_{-\infty}^{0} xe^{x}\ dx$ *this integral is improper at* $x=-\infty$ $\quad \left(\begin{matrix} u=x & dv=e^x\,dx \\ du=dx & v=e^x\end{matrix}\right) \rightarrow uv-\int v\,du$

$$\lim_{R\to-\infty}\int_R^0 xe^x\,dx = \lim_{R\to-\infty}\left[xe^x|_R^0 - \int_R^0 e^x\,dx\right] = \lim_{R\to-\infty}[xe^x - e^x|_R^0] = \lim_{R\to-\infty}(-1-Re^R+e^R)$$

$$= -1-0(*)+0 = -1 \therefore \int_{-\infty}^{0} xe^x\ dx \text{ is convergent} \quad (*)\lim_{R\to-\infty} Re^R = \lim_{R\to-\infty}\frac{R}{e^{-R}}\left[\frac{-\infty}{\infty}\right] = (L`H)\lim_{R\to-\infty}\frac{1}{-e^{-R}} = 0$$

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2) $\int_0^{\infty} xe^{-x^2}\ dx$ *this integral is improper at* $x=\infty$

$$\lim_{R\to\infty}\int_0^R xe^{-x^2}\,dx = \lim_{R\to\infty}\left[-\frac{1}{2}e^{-x^2}\Big|_0^R\right] = \lim_{R\to\infty}\left[-\frac{1}{2}e^{-x^2}\Big|_0^R\right] = \lim_{R\to\infty}\left[-\frac{1}{2}e^{-R^2}+\frac{1}{2}\right] = \frac{1}{2} \quad (\,convergent)$$

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3) $\int_1^{\infty}\frac{1}{x\,(\ln x)^2}dx = \int_0^{\infty}\frac{1}{u^2}du$ *this integral is improper at* $u=0$ *and* $u=\infty$

$$so, \int_0^{\infty}\frac{1}{u^2}du = \int_0^{1}\frac{1}{u^2}du + \int_1^{\infty}\frac{1}{u^2}du = \lim_{R\to\infty}\left[-\frac{1}{u}\Big|_R^0\right] + \lim_{R\to\infty}\left[-\frac{1}{u}\Big|_1^R\right] = \lim_{R\to\infty}\left[\infty+\frac{1}{R}-\frac{1}{R}+1\right] = \infty$$

$$\int_1^{\infty}\frac{1}{x\,(\ln x)^2}dx \text{ is divergent}$$

4) $\int_{2019}^{\infty}\frac{\ln x}{x^2}dx$ *this integral is improper at* $x=\infty$

$$\begin{pmatrix} u=\ln x & dv=\dfrac{1}{x^2}\,dx \\ du=\dfrac{1}{x}\,dx & v=\dfrac{-1}{x} \end{pmatrix} \to u\,v-\int v\,du$$

$$\lim_{R\to\infty}\left[\int_{2019}^{R}\frac{\ln x}{x^2}dx\right]=\lim_{R\to\infty}\left[-\frac{\ln x}{x}\bigg|_{2019}^{R}+\int_{2019}^{R}\frac{1}{x^2}dx\right]=\lim_{R\to\infty}\left[-\frac{\ln x}{x}\bigg|_{2019}^{R}-\frac{1}{x}\bigg|_{2019}^{R}\right]$$

$$=\lim_{R\to\infty}\left[-\frac{\ln R}{R}\left[\frac{\infty}{\infty}\right](L`H)+\frac{\ln 2019}{2019}-\frac{1}{R}+\frac{1}{2019}\right]=\lim_{R\to\infty}\left[-\frac{1}{R}+\frac{\ln 2019}{2019}-\frac{1}{R}+\frac{1}{2019}\right]=\frac{\ln 2019+1}{2019}$$

$$\therefore \int_{2019}^{\infty}\frac{\ln x}{x^2}dx \;\; is\ convergent$$

5) $\int_{120}^{\infty}\frac{1}{x\,lnx\ln(\ln x)}\,dx$ *this integral is improper at* $x=\infty$

$$\lim_{R\to\infty}\int_{120}^{R}\frac{1}{x\,lnx\ln(lnx)}dx$$

$$let\ u=ln(lnx)\ \to du=\frac{1}{x\,lnx}dx\ \ as\ x=R\ \to u=\ln(lnR)\ and\ as\ x=120\ \to u=\ln(ln120)$$

$$\lim_{R\to\infty}\int_{\ln(ln120)}^{\ln(lnR)}\frac{1}{u}du=\lim_{R\to\infty}\left[\ln u|_{\ln(ln120)}^{\ln(lnR)}\right]=\lim_{R\to\infty}[\ln(\ln(lnR))-\ln(\ln(ln120))]=\infty \qquad (divergent)$$

6) $\int_{0}^{3}\frac{x\,dx}{(9-x^2)^{3/2}}$ *this integral is improper at* $x=3$

$$\lim_{R\to3^-}\left[\int_0^R\frac{x\,dx}{(9-x^2)^{3/2}}\right]=(*)\lim_{R\to3^-}\left[\int_{x=0}^{x=R}\frac{(3\sin\theta)(3\cos\theta)\,d\theta}{(3\cos\theta)^3}\right]=\lim_{R\to3^-}\left[\int_{x=0}^{x=R}\frac{\sin\theta\ d\theta}{3\cos^2\theta}\right]=\lim_{R\to3^-}\left[\frac{1}{3}\int_{x=0}^{x=R}\sec\theta\ \tan\theta\ d\theta\right]$$

$$=\lim_{R\to3^-}\left[\frac{1}{3}\sec\theta|_{x=0}^{x=R}\right]=\lim_{R\to3^-}\left[\frac{1}{3}\frac{3}{\sqrt{9-x^2}}\bigg|_0^R\right]=\lim_{R\to3^-}\left[\frac{1}{\sqrt{9-R^2}}-\frac{1}{3}\right]=\infty \quad (*)\begin{cases}\sin\theta=\dfrac{x}{3}\ \ and\ \cos\theta=\dfrac{\sqrt{9-x^2}}{3}\\ \cos\theta\ d\theta=\dfrac{1}{3}\,dx\end{cases}$$

$$\therefore \int_0^3\frac{x\,dx}{(9-x^2)^{3/2}}\ \ is\ divergent$$

7) $\int_0^{\infty}\frac{x\,dx}{1+x^4}$ *this integral is improper at* $x=\infty$

$$\lim_{R\to\infty}\left[\int_0^R\frac{x\,dx}{1+x^4}\right]=\lim_{R\to\infty}\left[\frac{1}{2}\arctan(x^2)\bigg|_0^R\right]=\lim_{R\to\infty}\left[\frac{1}{2}\arctan(R^2)-0\right]=\frac{1}{2}\left(\frac{\pi}{2}\right)=\frac{\pi}{4}\quad (convergent)$$

8) $\int_0^{\infty}\frac{e^x}{x^{2/3}}dx$ *this integral is improper at* $x=0$ *and* $x=\infty$

$$\int_0^\infty \frac{e^x}{x^{2/3}}dx = \int_0^1 \frac{e^x}{x^{2/3}}dx + \int_1^\infty \frac{e^x}{x^{2/3}}dx = I_1 + I_2 = divergent$$

$(0,1]$ $f(x) = \frac{e^x}{x^{2/3}} \geq 0$ $let\ g(x) = \frac{1}{x^2} \geq 0$ $f(x) \geq g(x) \geq 0$ We can use comparison test $\int_0^1 \frac{e^x}{x^{2/3}}dx \geq \int_0^1 \frac{1}{x^2}dx \geq 0$ since $\int_0^1 \frac{1}{x^2}dx$ is divergent by p-test so $\int_0^1 \frac{e^x}{x^{2/3}}dx$ is divergent by comparison test $\rightarrow I_1$	$[1,\infty)$ $f(x) = \frac{e^x}{x^{2/3}} \geq 0$ $let\ g(x) = \frac{1}{x^{2/3}} \geq 0$ $f(x) \geq g(x) \geq 0$ We can use comparison test $\int_1^\infty \frac{e^x}{x^{2/3}}dx \geq \int_1^\infty \frac{1}{x^{2/3}}dx \geq 0$ since $\int_1^\infty \frac{1}{x^2}dx$ is divergent by p-test so $\int_1^\infty \frac{e^x}{x^{2/3}}dx$ is divergent by comparison test $\rightarrow I_2$

9) $\int_0^{119} \frac{\sin^2 x}{x^{5/2}}\ dx$ $\ this\ integral\ is\ improper\ at\ x = 0$

$(0,1]$ $f(x) = \frac{\sin^2 x}{x^{5/2}} \geq 0$ $let\ g(x) = \frac{1}{x^{1/2}} \geq 0$

$g(x) \geq f(x) \geq 0$ We can use comparison test $\int_0^{119} \frac{1}{x^{1/2}}\ dx \geq \int_0^{119} \frac{\sin^2 x}{x^{5/2}}\ dx \geq 0$

since $\int_0^{119} \frac{1}{x^{1/2}}dx$ is convergent by p-test. Thus, $\int_0^{119} \frac{\sin^2 x}{x^{5/2}}\ dx$ is convergent by comparison test

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10) $\int_0^\infty \frac{dx}{\sqrt{x}+x^3}$ $\ this\ integral\ is\ improper\ at\ x = 0\ and\ x = \infty$

$$\int_0^\infty \frac{dx}{\sqrt{x}+x^3} = \int_0^1 \frac{dx}{\sqrt{x}+x^3} + \int_1^\infty \frac{dx}{\sqrt{x}+x^3} = I_1 + I_2 = convergent$$

$(0,1]$ $f(x) = \frac{1}{\sqrt{x}+x^3} \geq 0$ $let\ g(x) = \frac{1}{x^{1/2}} \geq 0$ $f(x) \geq g(x) \geq 0$ We can use comparison test $\int_0^1 \frac{1}{\sqrt{x}+x^3}dx \geq \int_0^1 \frac{1}{x^{1/2}}dx \geq 0$ since $\int_0^1 \frac{1}{x^{1/2}}dx$ is convergent by p-test so $\int_0^1 \frac{1}{\sqrt{x}+x^3}dx$ is convergent by comparison test $\rightarrow I_1$	$[1,\infty)$ $f(x) = \frac{1}{\sqrt{x}+x^3} \geq 0$ $let\ g(x) = \frac{1}{x^3} \geq 0$ $f(x) \geq g(x) \geq 0$ We can use comparison test $\int_1^\infty \frac{1}{\sqrt{x}+x^3}dx \geq \int_1^\infty \frac{1}{x^3}dx \geq 0$ since $\int_1^\infty \frac{1}{x^3}dx$ is convergent by p-test so $\int_1^\infty \frac{1}{\sqrt{x}+x^3}dx$ is convergent by comparison test $\rightarrow I_2$

11) $\int_0^\infty \frac{dx}{x^7+x^{1/3}}$ $\ this\ integral\ is\ improper\ at\ x = 0\ and\ x = \infty$

$$\int_0^\infty \frac{dx}{x^7+x^{1/3}} = \int_0^1 \frac{dx}{x^7+x^{1/3}} + \int_1^\infty \frac{dx}{x^7+x^{1/3}} = I_1 + I_2 = convergent$$

$(0,1]$ $f(x) = \frac{dx}{x^7+x^{1/3}} \geq 0$ $let\ g(x) = \frac{1}{x^{1/3}} \geq 0$ $f(x) \geq g(x) \geq 0$ We can use comparison test $\int_0^1 \frac{dx}{x^7+x^{1/3}}dx \geq \int_0^1 \frac{1}{x^{1/3}}dx \geq 0$ since $\int_0^1 \frac{1}{x^{1/3}}dx$ is convergent by p-test so $\int_0^1 \frac{dx}{x^7+x^{1/3}}dx$ is convergent by comparison test $\rightarrow I_1$	$[1,\infty)$ $f(x) = \frac{dx}{x^7+x^{1/3}} \geq 0$ $let\ g(x) = \frac{1}{x^2} \geq 0$ $f(x) \geq g(x) \geq 0$ We can use comparison test $\int_1^\infty \frac{dx}{x^7+x^{1/3}}dx \geq \int_1^\infty \frac{1}{x^2}dx \geq 0$ since $\int_1^\infty \frac{1}{x^2}dx$ is convergent by p-test so $\int_1^\infty \frac{dx}{x^7+x^{1/3}}dx$ is convergent by comparison test $\rightarrow I_2$

12) $\int_0^\infty \frac{dx}{x^{1/2}+x^2}$ $\ this\ integral\ is\ improper\ at\ x = 0\ and\ x = \infty$

$$\int_0^\infty \frac{dx}{x^{1/2}+x^2} = \int_0^1 \frac{dx}{x^{1/2}+x^2} + \int_1^\infty \frac{dx}{x^{1/2}+x^2} = I_1 + I_2 = convergent$$

$(0,1]$ $f(x) = \frac{dx}{x^{1/2}+x^2} \geq 0$ $let\ g(x) = \frac{1}{x^{1/2}} \geq 0$ $f(x) \geq g(x) \geq 0$ We can use comparison test $\int_0^1 \frac{dx}{x^{1/2}+x^2} dx \geq \int_0^1 \frac{1}{x^{1/2}} dx \geq 0$ since $\int_0^1 \frac{1}{x^{1/2}} dx$ is convergent by p-test so $\int_0^1 \frac{dx}{x^{1/2}+x^2} dx$ is convergent by comparison test $\rightarrow I_1$	$[1,\infty)$ $f(x) = \frac{dx}{x^{1/2}+x^2} \geq 0$ $let\ g(x) = \frac{1}{x^2} \geq 0$ $f(x) \geq g(x) \geq 0$ We can use comparison test $\int_1^\infty \frac{dx}{x^{1/2}+x^2} dx \geq \int_1^\infty \frac{1}{x^2} dx \geq 0$ since $\int_1^\infty \frac{1}{x^2} dx$ is convergent by p-test so $\int_1^\infty \frac{dx}{x^{1/2}+x^2} dx$ is convergent by comparison test $\rightarrow I_2$

13) $\int_0^{\pi/2} \frac{dx}{x \sin x}$ $\quad this\ integral\ is\ improper\ at\ x = 0$

$$0 \leq \sin x \leq 1 \quad \rightarrow \quad 0 \leq x\sin x \leq x \quad \rightarrow \quad \frac{1}{x\sin x} \geq \frac{1}{x} \geq 0$$

$f(x) = \frac{1}{x \sin x} \geq 0$ $let\ g(x) = \frac{1}{x} \geq 0$ We can use comparison test:

$\int_0^{\pi/2} \frac{dx}{x \sin x} \geq \int_0^{\pi/2} \frac{dx}{x} \geq 0$ since $\int_0^{\pi/2} \frac{dx}{x}$ is divergent by p-test. Thus, $\int_0^{\pi/2} \frac{dx}{x \sin x}$ is divergent by comparison test.

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14) $\int_0^4 \frac{dx}{x^2+x-6} = \int_0^4 \frac{dx}{(x-2)(x+3)} = \int_0^2 \frac{dx}{(x-2)(x+3)} + \int_2^4 \frac{dx}{(x-2)(x+3)} =$ $\quad this\ integral\ is\ improper\ at\ x = 2$

$f(x) = \frac{1}{x^2+x-6} \geq 0$ $let\ g(x) = \frac{1}{(x-2)} \geq 0$ We can use comparison test:

$\int_0^2 \frac{dx}{x^2+x-6} \geq \int_0^2 \frac{dx}{(x-2)} \geq 0$ since $\int_0^2 \frac{dx}{(x-2)} = \lim_{R\to\infty} \int_0^R \frac{dx}{(x-2)} = \lim_{R\to\infty} [\ln|x-2| |_0^R] = \lim_{R\to\infty} [\ln|R-2| - \ln|-2|] = \infty$

So $\int_0^2 \frac{dx}{(x-2)}$ is divergent by direct-test. Thus, $\int_0^2 \frac{dx}{x^2+x-6}$ is divergent by comparison test. Similarly, $\int_2^4 \frac{dx}{(x-2)(x+3)}$ is divergent Thus, $\int_0^4 \frac{dx}{x^2+x-6}$ is divergent by comparison test.

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15) $\int_1^\infty \frac{1-\cos x}{x^{5/2}+x+119} dx$ $\ this\ integral\ is\ improper\ at\ x = \infty$

$$0 \leq \cos x \leq 1 \quad \rightarrow \quad \frac{1}{x^{5/2}} \geq \frac{1}{x^{5/2}+x+119} \geq \frac{1-\cos x}{x^{5/2}+x+119} \geq 0$$

$f(x) = \frac{1-\cos x}{x^{5/2}+x+119} \geq 0$ $\quad let\ g(x) = \frac{1}{x^{5/2}} \geq 0$ We can use comparison test:

$\int_1^\infty \frac{dx}{x^{5/2}} \geq \int_1^\infty \frac{1-\cos x}{x^{5/2}+x+119} dx \geq 0$ $\quad$ since $\int_1^\infty \frac{dx}{x^{5/2}}$ is convergent by p-test.

Thus, $\int_1^\infty \frac{1-\cos x}{x^{5/2}+x+119} dx$ is convergent by comparison test.

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16) $\int_0^3 \frac{(1+e^{-x})(x^3+x+1)}{x^5} dx$ $\quad this\ integral\ is\ improper\ at\ x = 0$

$f(x) = \frac{(1+e^{-x})(x^3+x+1)}{x^5} \geq 0$ $\quad let\ g(x) = \frac{1}{x^5} \geq 0$ We can use comparison test:

$\int_0^3 \frac{(1+e^{-x})(x^3+x+1)}{x^5} dx \geq \int_0^3 \frac{1}{x^5} dx \geq 0$ $\quad$ since $\int_0^3 \frac{1}{x^5} dx$ is divergent by p-test.

Thus, $\int_0^3 \frac{(1+e^{-x})(x^3+x+1)}{x^5} dx$ is divergent by comparison test.

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17) $\int_{\sqrt{3}}^{\infty} x^{-\arctan x} dx$ *this integral is improper at* $x = \infty$

Since $arctan$ is an increasing function, $arctan(x) \geq arctan(\sqrt{3}) = \frac{\pi}{3}$ $\;for\ x \geq \sqrt{3}$

Thus $0 \leq \frac{1}{x^{\arctan x}} \leq \frac{1}{x^{\frac{\pi}{3}}}$ $\;for\ all\ x \geq \sqrt{3}\;$ *as the improper integral* $\int_{\sqrt{3}}^{\infty} \frac{1}{x^{\frac{\pi}{3}}} dx$ *converges by* $p-test$ *where*

$p = \frac{\pi}{3} > 1$. Consequently, $\int_{\sqrt{3}}^{\infty} x^{-\arctan x} dx$ converges by comparison test.

Question: *Let* f *be differentiable function on* $\mathbb{R}$ *such that* $\lim\limits_{x\to+\infty} f(x) = 120$ *and* $\lim\limits_{x\to-\infty} f(x) = 1$. *Find the* ***value*** *of the improper integral* $\int\limits_{-\infty}^{\infty} f`(x)\,dx$. *Explain and show your work*.

$$\int\limits_{-\infty}^{\infty} f`(x)\,dx = \int\limits_{-\infty}^{0} f`(x)\,dx + \int\limits_{0}^{\infty} f`(x)\,dx \;\; if\ both\ integrals\ exist$$

$$\int\limits_{-\infty}^{0} f`(x)\,dx = \lim_{R\to-\infty}\left[\int\limits_{R}^{0} f`(x)\,dx\right] = \lim_{R\to-\infty}\left[f(x)|_R^0\right] = \lim_{R\to-\infty}\left[f(0) - f(R)\right] = f(0) - 1 \quad and$$

$$\int\limits_{0}^{\infty} f`(x)\,dx = \lim_{R\to\infty}\left[\int\limits_{0}^{R} f`(x)\,dx\right] = \lim_{R\to\infty}\left[f(x)|_0^R\right] = \lim_{R\to\infty}\left[f(R) - f(0)\right] = 120 - f(0) \quad and$$

$$Thus,\ \int\limits_{-\infty}^{\infty} f`(x)\,dx = f(0) - 1\ + 120 - f(0) = 119$$

Section 17: Problems session

Q1: *Sketch and find the area of the finite region bounded by the curves* $y = x^3$ *and* $y = 4x$.

$x^3 - 4x = 0 \;\therefore\; x(x^2 - 4) = x(x+2)(x-2) = 0$

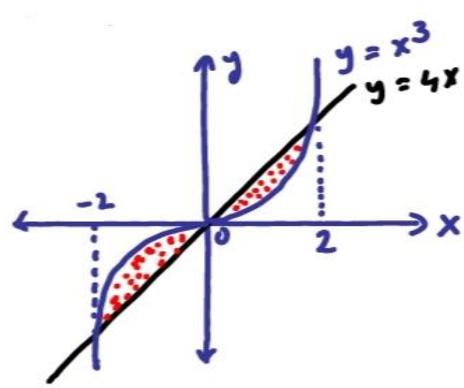
$\therefore\; x = 0 \;\therefore\; x = -2 \;\therefore\; x = 2$

Intersect points are $(0,0), (-2,-8)$ *and* $(2,8)$

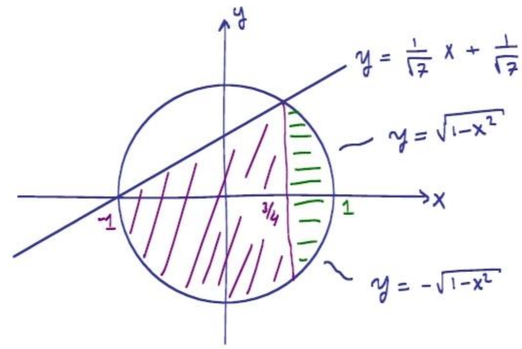
$$A = \int\limits_{-2}^{0} (x^3 - 4x)\,dx + \int\limits_{0}^{2} (4x - x^3)\,dx$$

$$= \left(\frac{x^4}{4} - 2x^2\right)\Big|_{-2}^{0} + \left(2x^2 - \frac{x^4}{4}\right)\Big|_{0}^{2}$$

$A = -4 + 8 + 8 - 4 = 8\ unit^2$



$Q2$: *Write* (***DO NOT EVALUATE***) *an integral* (*or sum of the ingrals*) *which gives the area of the region lying below the line* $\sqrt{7}\, y = x + 1$ *and inside the circle* $x^2 + y^2 = 1$.



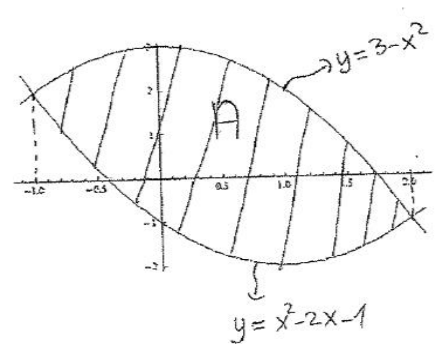
$$x^2 + y^2 = 1 \;\; and \;\; y^2 = \frac{(x+1)^2}{7} \quad so\; x^2 + \frac{(x+1)^2}{7} = 1$$

$$7x^2 + (x+1)^2 = 7 \quad so\; 8x^2 + 2x - 6 = 0 \quad \therefore 2(4x-3)(x+1) = 0$$

$x = \frac{3}{4}$ $\therefore$ $x = -1$ Intersect points are $\left(\frac{3}{4}, \frac{\sqrt{7}}{4}\right), (-1,0)$

$$A = \int_{-1}^{3/4} \frac{1}{\sqrt{7}}(x+1) - (-\sqrt{1-x^2})\, dx + 2\int_{3/4}^{1} \sqrt{1-x^2}\, dx$$

$Q3$: *Sketch the region between the parabolas* $y = 3 - x^2$ *and* $y = x^2 - 2x - 1$ *and find the area of this region.*



$$3 - x^2 = x^2 - 2x - 1 \quad \therefore x^2 - x - 2 = 0 \quad \therefore x = -1 \; and \; x = 2$$

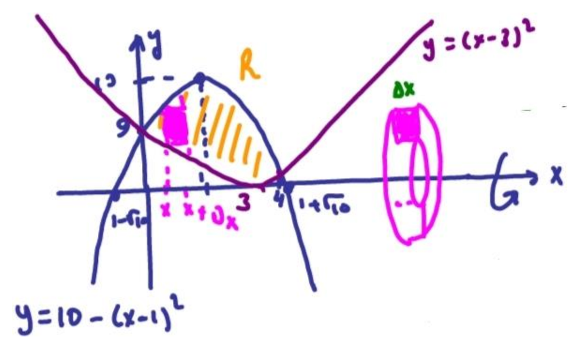
$$A = \int_{-1}^{2} [3 - x^2 - (x^2 - 2x - 1)]\, dx = \int_{-1}^{2} (2x^2 + 2x + 4)\, dx$$

$$= \left(-\frac{2x^3}{3} + x^2 + 4x\right)\Bigg|_{-1}^{2} = -\frac{16}{3} + 4 + 8 - \left(\frac{2}{3} + 1 - 4\right) = 9$$

Section 18: Problems session

$Q1$: *Let R the region bounded by the curves* $y = 10 - (x-1)^2$ *and* $y = (x-3)^2$

a) *Write a difinite integral which is equal to the volume of the solid obtained by rotating the region R about x–axis. DO NOT EVALUATE.*



$$10 - (x-1)^2 = (x-3)^2 \rightarrow 10 - x^2 - 1 + 2x = x^2 + 6x + 9$$

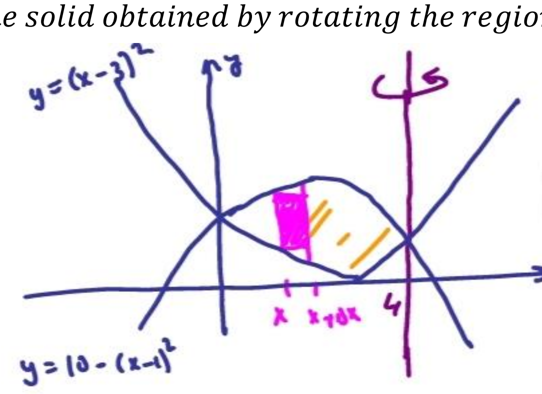
$$\therefore 2x^2 - 8x = 2x(x-4) = 0 \quad \therefore x = 0 \; and \; x = 4$$

the disc (*slicing, washer*) *method*

$$V = \pi \int_{0}^{4} [(10 - (x-1)^2)^2 - (x-3)^4]\, dx$$

b) *Write a difinite integral which is equal to the volume of the solid obtained by rotating the region R about the line* $x = 4$. *DO NOT EVALUATE.*

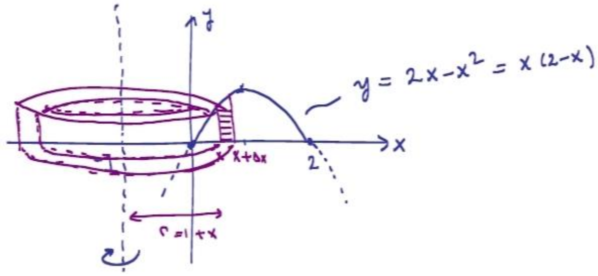
the cylindrical shell method



$$V = 2\pi \int_{0}^{4} (4 - x)\,[10 - (x-1)^2 - (x-3)^2]\, dx$$

Q2: Let R the region bounded from above the x–axis and from below by the curve $y = 2x - x^2$. *R is rotated about* $x = -1$ *to generate a solid. Write (**DO NOT EVALUATE**) integrals giving the volume V of the resulting solid of the revolution by using following methods;*

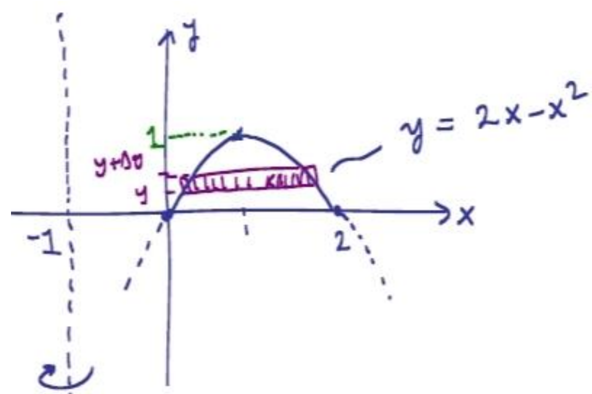
a) the cylindrical shell method.



$$V = 2\pi \int_0^2 (1+x)\,[2x - x^2]\,dx$$

b) the disc (slicing, washer) method.

$$y = 2x - x^2 \;\rightarrow\; x^2 - 2x + y = 0 \;\therefore x = \frac{2 \pm \sqrt{4-4y}}{2} = 1 \pm \sqrt{1-y}$$



$$V = \pi \int_0^1 \left[\left(1+\sqrt{1-y}+1\right)^2 - \left(1-\sqrt{1-y}+1\right)^2\right] dy$$

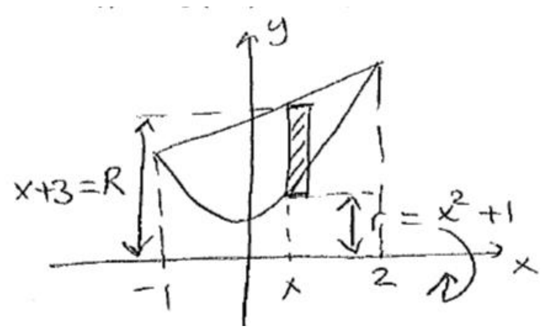
$$V = \pi \int_0^1 \left[\left(2+\sqrt{1-y}\right)^2 - \left(2-\sqrt{1-y}\right)^2\right] dy$$

$$V = 8\pi \int_0^1 \sqrt{1-y}\,dy$$

Q3: Let R be the region bounded by the parabola $y = x^2 + 1$ and the line $y = x + 3$.

Let S the solid obtained by rotated about R about the x–axis.In each part sketch the region R,draw a typical rectangle (area element dA) and a typical volume (volume element dV) and use them to write a difinte integral (or sum of the ingrals,DO NOT EVALUATE) which gives the volume of S using

a) the slicing (disc) method.



$$x^2 + 1 = x + 3 \;\rightarrow x^2 - x - 2 = (x-2)(x+1) = 0 \;\therefore x = 2 \;\&\; x = -1$$

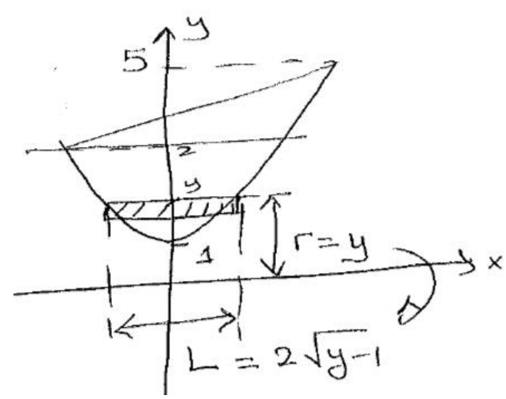
$$V = \pi \int_{-1}^{2} [(x+3)^2 - (x^2+1)^2]\,dx$$

b) the cylindrical shell method.

$$y = x^2 + 1 \;\rightarrow x = \pm\sqrt{y-1} \quad and\; x = y - 3$$

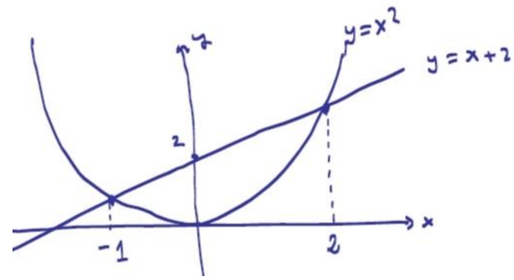
$$y - 1 = (y-3)^2 \rightarrow y - 1 = y^2 - 6y + 9$$

$$\therefore y^2 - 7y + 10 = (y-5)(y-2) = 0 \;\therefore y = 5 \;and\; y = 2$$



$$V = 4\pi \int_1^2 y\sqrt{y-1}\,dy + 2\pi \int_2^5 y\left(\sqrt{y-1} - y + 3\right) dy$$

$Q4$: *Let R the finite region bounded by the curves* $y = x^2$ *and* $y = x + 2$



$a)$ *Sketch the region R.*

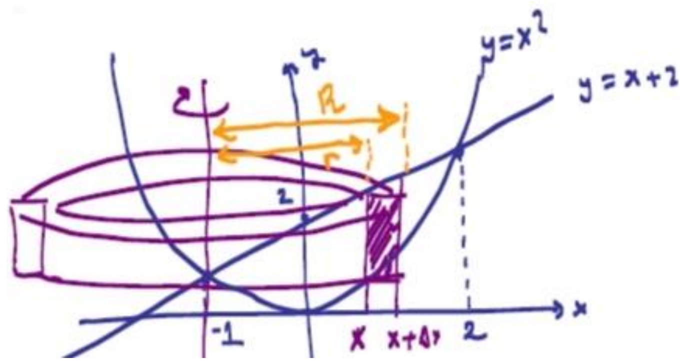
$x^2 = x + 2 \quad \therefore x^2 - x - 2 = (x + 1)(x - 2) = 0$

$\therefore x = 2$ *and* $x = -1$

$b)$ *Set up a difinite integral which gives the* ***volume*** *of the solid obtained by rotating the region R about x–axis. DO NOT THE EVALUATE*

the disc (slicing, washer) method

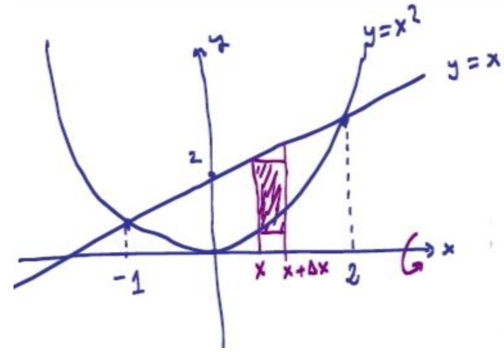
$$V = \pi \int_{-1}^{2} [(x + 2)^2 - x^4]\, dx$$



$c)$ *Set up a difinite integral which gives the* ***volume*** *of the solid obtained by rotating the region R about the* ***line*** $x = -1$*. DO NOT THE EVALUATE*

the cylindrical shell method

$$V = 2\pi \int_{-1}^{2} (1 + x)\, [x + 2 - x^2]\, dx$$



Section 19: Problems session

$Q1$: *Write a difinite integral which is equal to the* ***arclengh*** *of the curve* $y = \ln\left(\frac{x+3}{2-x}\right)$ *between the point* $\left(0, \ln\left(\frac{3}{2}\right)\right)$ *and* $(1, \ln(4))$ *. DO NOT EVALUATE.*

$s = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\, dx \quad$ *where* $y = \ln\left(\frac{x+3}{2-x}\right) = \ln(x + 3) - \ln(2 - x) \qquad \therefore \begin{cases} f'(x) = \frac{1}{x+3} + \frac{1}{2-x} \\ a = 0\,,\ \ b = 1 \end{cases}$

$$\therefore arclengh = \int_0^1 \sqrt{1 + \left(\frac{1}{x + 3} + \frac{1}{2 - x}\right)^2}\, dx$$

$Q2$: *Write the* ***arclengh*** *of the curve* given $y = f(x) = \int_1^x \sqrt{t^3 - 1}\, dt \quad$ *from* $x = 1$ *to* $x = 2$

$s = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\, dx$ *where* $\begin{cases} f'(x) = \sqrt{x^3 - 1} \\ a = 1\,,\ \ b = 2 \end{cases} \quad \therefore arclengh = \int_1^2 \sqrt{1 + \left(\sqrt{x^3 - 1}\right)^2}\, dx$

$$\therefore arclengh = \int_1^2 \sqrt{x^3}\, dx = \frac{2}{5}\left(x^{5/2}\right)\Big|_1^2 = \frac{2}{5}\left(\sqrt{32} - 1\right)$$

$Q1$: *Find the Cartesian equation of the curve represented by the polar eqation* $r = \sin\theta + \cos\theta$

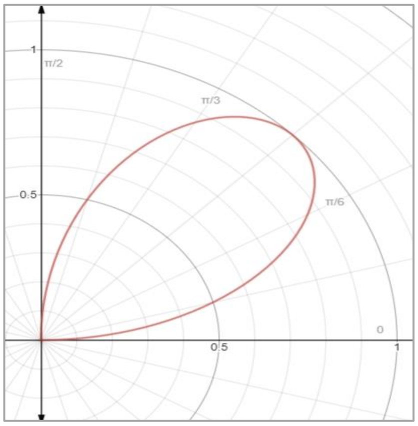
$r^2 = r\sin\theta + r\cos\theta$ *then we have* $x^2 + y^2 = y + x$

$Q2$: *Find the Cartesian equation of the curve represented by the polar eqation* $r = \frac{2}{2 - cos\theta}$

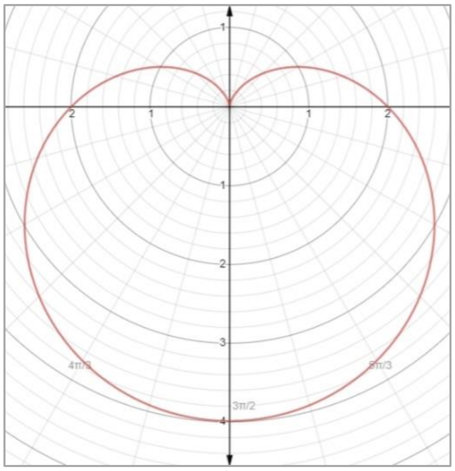
$2r - rcos\theta = 2$ *then we have* $2\sqrt{x^2 + y^2} - x = 2$

$Q3$: *Let* C *the curve given by the polar equation* $r = \sin 2\theta$. *Sketch the graph of the part the curve* C *in the first quadrant* (*this is, where* $x \geq 0,\ y \geq 0$).

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
$\sin 2\theta$	$\sin 0$	$\sin\frac{\pi}{2}$	$\sin \pi$
r	0	1	0



$Q4$: *Let* C *the curve given by the polar equation* $r = 2(1 - \sin\theta)$. *Sketch the graph of the curve* C.



θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	$\sin 0$	$\sin\frac{\pi}{2}$	$\sin\pi$	$\sin\frac{3\pi}{2}$	$\sin 2\pi$
r	2	0	2	4	2