

— SOLUTIONS FOR EXTRA QUESTIONS —

3.

For each of the utility functions find the Hicksian demand functions, $h_1(p_1, p_2, \bar{u})$ and $h_2(p_1, p_2, \bar{u})$.

(a) $u(x_1, x_2) = x_1^a x_2^{1-a}$

Solution: The Hicksian demands (if both are positive) are the solution of the tangency condition,

$$\left(\frac{a}{1-a} \right) \frac{x_2}{x_1} = \frac{p_1}{p_2}.$$

and the “being-on-the-indifference-curve condition”,

$$x_1^a x_2^{1-a} = \bar{u}. \quad (1)$$

We solve these two equations for x_1 and x_2 to find the Hicksian demands, $h_1(p, \bar{u}) = x_1$ and $h_2(p, \bar{u}) = x_2$. From the tangency condition we again have

$$x_2 = \left(\frac{1-a}{a} \right) \left(\frac{p_1}{p_2} \right) x_1, \quad (2)$$

which we substitute into (1) to yield

$$x_1^a \left(\left(\frac{1-a}{a} \right) \left(\frac{p_1}{p_2} \right) x_1 \right)^{1-a} = \bar{u},$$

which solved for x_1 yields

$$h_1(p_1, p_2, \bar{u}) = x_1 = \left(\left(\frac{a}{1-a} \right) \left(\frac{p_2}{p_1} \right) \right)^{1-a} \bar{u},$$

which substituted back into (2) yields

$$\begin{aligned}
 h_2(p_1, p_2, \bar{u}) &= x_2 = \left(\frac{1-a}{a}\right) \left(\frac{p_1}{p_2}\right) x_1 \\
 &= \left(\frac{1-a}{a}\right) \left(\frac{p_1}{p_2}\right) \left(\left(\frac{a}{1-a}\right) \left(\frac{p_2}{p_1}\right)\right)^{1-a} \bar{u} \\
 &= \left(\left(\frac{a}{1-a}\right) \left(\frac{p_2}{p_1}\right)\right)^{-a} \bar{u}.
 \end{aligned}$$

(b) $u(x_1, x_2) = \frac{x_1^2}{x_2 + 4}$

Solution: The expenditure minimization problem (*EMP*) is given by

$$\min_{x_1, x_2 \geq 0} p_1 x_1 + p_2 x_2 \quad \text{subject to} \quad \frac{x_1^2}{x_2 + 4} \geq \bar{u}$$

Notice that for any $x_1 > 0$, the utility function is strictly decreasing in x_2 , and so $x_2 = 0$ in the solution to (*EMP*), and so the utility constraint becomes

$$\frac{x_1^2}{4} \geq \bar{u},$$

and since the objective is to minimize expenditure we will consume as little of good 1 as needed to satisfy this constraint, meaning it will hold with equality, and so

$$\frac{x_1^2}{4} = \bar{u} \quad \Leftrightarrow \quad x_1^2 = 4\bar{u} \quad \Leftrightarrow \quad x_1 = 2\sqrt{\bar{u}}.$$

The Hicksian demand functions are

$$h_1(p_1, p_2, \bar{u}) = 2\sqrt{\bar{u}},$$

$$h_2(p_1, p_2, \bar{u}) = 0.$$

(c) $u(x_1, x_2) = x_1 + x_2$

Solution: The expenditure minimization problem (*EMP*) is given by

$$\min_{x_1, x_2 \geq 0} p_1 x_1 + p_2 x_2 \quad \text{subject to} \quad x_1 + x_2 \geq \bar{u}$$

Because monotonicity holds, we know that the utility constraint will hold with equality at the solution, so $x_1 + x_2 = \bar{u}$, or $x_2 = \bar{u} - x_1$. Plugging this into the expenditure function, we see that consumer is trying to minimize the expression

$$p_1 x_1 + p_2 (\bar{u} - x_1),$$

which can be rewritten as

$$(p_1 - p_2) x_1 + p_2 \bar{u},$$

which is strictly increasing in x_1 for $p_1 > p_2$ and strictly decreasing in x_1 for $p_1 < p_2$. Therefore, the solution is such that consumer chooses the maximum possible x_1 (i.e. $x_1 = \bar{u}$, since anything more would make $x_2 = \bar{u} - x_1$ infeasible) when $p_1 < p_2$, and the minimum possible x_1 (i.e. $x_1 = 0$) when $p_1 > p_2$. Consumer's Hicksian demand functions must satisfy

$$h_1(p_1, p_2, \bar{u}) = \begin{cases} 0 & \text{if } p_1 > p_2 \\ \bar{u} & \text{if } p_1 < p_2, \end{cases}$$

$$h_2(p_1, p_2, \bar{u}) = \begin{cases} \bar{u} & \text{if } p_1 > p_2 \\ 0 & \text{if } p_1 < p_2, \end{cases}$$

and can be anything so long as $h_1(p_1, p_2, \bar{u}) + h_2(p_1, p_2, \bar{u}) = \bar{u}$ in the exact case when $p_1 = p_2$.

① (a) Seung maximizes:

$$\max_{x_1, x_2, x_3} u(x_1, x_2, x_3) = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}$$

$$\text{s.t. } p_1 x_1 + p_2 x_2 + p_3 x_3 \leq M$$

$$\mathcal{L}(x_1, x_2, x_3, \lambda) = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} + \lambda (M - p_1 x_1 - p_2 x_2 - p_3 x_3)$$

$$\text{FOCs: } \partial \mathcal{L} / \partial x_1: \alpha_1 x_1^{\alpha_1-1} x_2^{\alpha_2} x_3^{\alpha_3} - \lambda p_1 = 0 \quad (i)$$

$$\partial \mathcal{L} / \partial x_2: \alpha_2 x_1^{\alpha_1} x_2^{\alpha_2-1} x_3^{\alpha_3} - \lambda p_2 = 0 \quad (ii)$$

$$\partial \mathcal{L} / \partial x_3: \alpha_3 x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3-1} - \lambda p_3 = 0 \quad (iii)$$

$$\cdot \text{ From (i) \& (ii) } \Rightarrow \frac{\alpha_1}{\alpha_2} \cdot \frac{x_2}{x_1} = \frac{p_1}{p_2} \text{ which implies } x_2 = \frac{p_1}{p_2} \frac{\alpha_2}{\alpha_1} x_1$$

$$(i) \& (iii) \Rightarrow \frac{\alpha_1}{\alpha_3} \frac{x_3}{x_1} = \frac{p_1}{p_3} \text{ which implies } x_3 = \frac{p_1}{p_3} \frac{\alpha_3}{\alpha_1} x_1$$

• Substituting the solutions for x_2 and x_3 in the budget constraint we obtain:

$$p_1 x_1 + p_2 \left(\frac{p_1}{p_2} \frac{\alpha_2}{\alpha_1} x_1 \right) + p_3 \left(\frac{p_1}{p_3} \frac{\alpha_3}{\alpha_1} x_1 \right) = M$$

$$x_1^* = \frac{\alpha_1}{\alpha_1 + \alpha_2 + \alpha_3} \frac{M}{p_1}$$

$$x_2^* = \frac{\alpha_2}{\alpha_1 + \alpha_2 + \alpha_3} \frac{M}{p_2}$$

$$\text{and } x_3^* = \frac{\alpha_3}{\alpha_1 + \alpha_2 + \alpha_3} \frac{M}{p_3}$$

the share of money spent on (good i) constant

$$\left\{ \begin{array}{l} \frac{x_i^* p_i}{M} = \frac{\alpha_i}{\alpha_1 + \alpha_2 + \alpha_3} \\ \text{for } i=1, 2, \text{ and } 3. \end{array} \right.$$

(b) In all of the expressions above for x_1^* , x_2^* and x_3^* , the conditions $x_i^* \geq 0$ are satisfied, and hence the boundary constraints are satisfied. ($x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$)

(c) we can compute

$$\partial x_1^* / \partial M = \frac{\alpha_1}{\alpha_1 + \alpha_2 + \alpha_3} \frac{1}{p_1}$$

which is positive for all levels of the parameters, and hence the good is normal.

Note:

Cobb-Douglas preferences (represented by a Cobb-Douglas utility func.)

$$u(x_1, x_2) = x_1^a x_2^b$$

If $a+b=1 \Rightarrow$ normalized Cobb-Douglas utility func.

$a > 0$, $b > 0$ — x_1 & x_2 goods

$a = 0$ — x_1 : neutral

$a < 0$ — x_1 : bad

② $u(x, y) = \sqrt{x} + \sqrt{y}$, $m > 0$, $x > 0$, $y > 0$

(a) $\frac{\partial u(x, y)}{\partial x} = \frac{1}{2\sqrt{x}} > 0$ $\frac{\partial^2 u(x, y)}{\partial x^2} = -\frac{1}{4} x^{-3/2} < 0$, $x > 0$
 it is increasing/so monotonic utility function is concave in x .

(b) $\max_{x, y} \sqrt{x} + \sqrt{y}$ s.t. $p_x \cdot x + p_y \cdot y \leq m$

Since utility function is strictly increasing both in x and y , so budget constraint is satisfied.

(c) $\mathcal{L} = \sqrt{x} + \sqrt{y} + \lambda (m - p_x \cdot x - p_y \cdot y)$

(d) $\frac{\partial \mathcal{L}}{\partial x} : \frac{1}{2} x^{-1/2} - \lambda p_x = 0$ $\frac{\partial \mathcal{L}}{\partial y} : \frac{1}{2} y^{-1/2} - \lambda p_y = 0$ $\left\{ \begin{array}{l} \frac{y^{1/2}}{x^{1/2}} = \frac{p_x}{p_y} \Rightarrow y^* = \left(\frac{p_x}{p_y} \right)^2 x^* \end{array} \right.$

$\frac{\partial \mathcal{L}}{\partial \lambda} : m - p_x \cdot x - p_y \cdot y = 0$

(e) $x^* = \frac{m}{p_x} \frac{1}{1 + \frac{p_x}{p_y}} = \frac{m p_y}{p_x (p_x + p_y)}$ $y^* = \frac{m}{p_y} \frac{1}{1 + \frac{p_y}{p_x}} = \frac{m p_x}{p_y (p_x + p_y)}$

(f)

$\det \bar{H} = \begin{vmatrix} \mathcal{L}_{xx} & \mathcal{L}_{xy} & \mathcal{L}_{x\lambda} \\ \mathcal{L}_{xy} & \mathcal{L}_{yy} & \mathcal{L}_{y\lambda} \\ \mathcal{L}_{x\lambda} & \mathcal{L}_{y\lambda} & \mathcal{L}_{\lambda\lambda} \end{vmatrix} = \begin{vmatrix} 0 & -p_x & -p_y \\ -p_x & -\frac{1}{4} x^{-3/4} & 0 \\ -p_y & 0 & -\frac{1}{4} y^{-3/4} \end{vmatrix}$

$= 0 - (-p_x) \left(\frac{1}{4} p_x y^{-3/4} \right) + (-p_y) \left(0 - \frac{1}{4} x^{-3/4} p_y \right) > 0$

Since $x^* > 0$ and $y^* > 0 \Rightarrow \det \bar{H} > 0$ negative definite.

which means that there exists a maximum.

Please note that in class the bordered Hessian is

$\begin{vmatrix} 0 & p_1 & p_2 \\ p_1 & u_{11} & u_{12} \\ p_2 & u_{21} & u_{22} \end{vmatrix}$

In the first row & column, we take the derivative of the budget constraint & then we take the second derivatives of the utility w.r.t 1st & 2nd goods (shown by the blue color).

so the $\det = 0 - p_1 (p_1 u_{22} - p_2 u_{21}) + p_2 (p_1 u_{12} - p_2 u_{11}) = -\Omega$

the sign of Ω determines u is positive definite or negative definite.

In 2nd question we take the derivative of Lagrange so we directly check the sign of the determinant.