

Middle East Technical University
Department of Economics
Econ 206:
Statistics for Economists-II

Güneş A. Aşık

November 20, 2025

Recitation

1. A binomial experiment consisting of n trials resulted in observations $y_1, y_2, \dots, y_n$, where $y_i = 1$ if the i th trial was a success and $y_i = 0$ otherwise. Since Each Y_i is a Bernoulli random variable, the probability mass function is given by:

$$P(Y_i = y_i) = p^{y_i}(1 - p)^{1-y_i}, \quad y_i = 0, 1$$

Find the MLE of p , the probability of a success.

Solution:

Write the likelihood function:

Since the trials are independent, the likelihood function is:

$$L(p) = \prod_{i=1}^n P(Y_i = y_i) = \prod_{i=1}^n p^{y_i}(1 - p)^{1-y_i}$$

Simplify the likelihood function

$$L(p) = p^{\sum_{i=1}^n y_i} (1 - p)^{n - \sum_{i=1}^n y_i}$$

Let $S = \sum_{i=1}^n y_i$ be the total number of successes. Then:

$$L(p) = p^S (1 - p)^{n-S}$$

Write the log-likelihood function

$$\ell(p) = \ln L(p) = S \ln p + (n - S) \ln(1 - p)$$

Find the MLE by differentiation

Differentiate with respect to p :

$$\frac{\partial \ell(p)}{\partial p} = \frac{S}{p} - \frac{n - S}{1 - p}$$

Set the derivative equal to zero:

$$\frac{S}{p} - \frac{n - S}{1 - p} = 0$$

$$\frac{S}{p} = \frac{n - S}{1 - p}$$

$$S(1 - p) = p(n - S)$$

$$S - Sp = pn - Sp$$

$$S = pn$$

$$\hat{p}_{\text{MLE}} = \frac{S}{n} = \frac{\sum_{i=1}^n y_i}{n}$$

Verify it's a maximum

The second derivative is:

$$\frac{\partial^2 \ell(p)}{\partial p^2} = -\frac{S}{p^2} - \frac{n - S}{(1 - p)^2} < 0 \quad \text{for } 0 < p < 1$$

Since the second derivative is negative, we have found a maximum.

$$\hat{p}_{\text{MLE}} = \frac{1}{n} \sum_{i=1}^n y_i$$

Interpretation: The maximum likelihood estimator of p is simply the sample proportion of successes.

2. Let $X_1, X_2, \dots, X_n$ be a random sample from a normal distribution with unknown mean μ and unknown variance σ^2 .
 - (a) Find the maximum likelihood estimators for both μ and σ^2
 - (b) Show that the MLE for σ^2 is biased
 - (c) Suggest an unbiased estimator for σ^2 and verify its unbiasedness

Solution:

The probability density function for a normal distribution is:

$$f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

The likelihood function is:

$$L(\mu, \sigma^2) = \prod_{i=1}^n f(x_i|\mu, \sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2\right)$$

The log-likelihood function is:

$$\ell(\mu, \sigma^2) = -\frac{n}{2} \ln(2\pi) - \frac{n}{2} \ln(\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2$$

Taking partial derivatives:

For μ :

$$\frac{\partial \ell}{\partial \mu} = \frac{1}{\sigma^2} \sum_{i=1}^n (x_i - \mu)$$

Setting equal to zero:

$$\frac{1}{\sigma^2} \sum_{i=1}^n (x_i - \mu) = 0 \Rightarrow \sum_{i=1}^n (x_i - \mu) = 0$$

$$\sum_{i=1}^n x_i - n\mu = 0 \Rightarrow \hat{\mu}_{MLE} = \frac{1}{n} \sum_{i=1}^n x_i = \bar{X}$$

For σ^2 , it's easier to work with $\theta = \sigma^2$:

$$\frac{\partial \ell}{\partial \theta} = -\frac{n}{2\theta} + \frac{1}{2\theta^2} \sum_{i=1}^n (x_i - \mu)^2$$

Setting equal to zero:

$$-\frac{n}{2\theta} + \frac{1}{2\theta^2} \sum_{i=1}^n (x_i - \mu)^2 = 0$$

$$\frac{1}{\theta^2} \sum_{i=1}^n (x_i - \mu)^2 = \frac{n}{\theta}$$

$$\hat{\sigma}_{MLE}^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \mu)^2$$

Since we don't know μ , we substitute $\hat{\mu}_{MLE} = \bar{X}$:

$$\hat{\sigma}_{MLE}^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{X})^2$$

Showing the MLE for σ^2 is biased

We need to show that $E[\hat{\sigma}_{MLE}^2] \neq \sigma^2$.

Recall that:

$$\hat{\sigma}_{MLE}^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2$$

We know that:

$$\sum_{i=1}^n (X_i - \bar{X})^2 = \sum_{i=1}^n (X_i - \mu)^2 - n(\bar{X} - \mu)^2$$

Taking expectations:

$$E \left[\sum_{i=1}^n (X_i - \bar{X})^2 \right] = E \left[\sum_{i=1}^n (X_i - \mu)^2 \right] - nE[(\bar{X} - \mu)^2]$$

Since $E[(X_i - \mu)^2] = \sigma^2$ and $E[(\bar{X} - \mu)^2] = \frac{\sigma^2}{n}$, we have:

$$E \left[\sum_{i=1}^n (X_i - \bar{X})^2 \right] = n\sigma^2 - n \cdot \frac{\sigma^2}{n} = (n-1)\sigma^2$$

Therefore:

$$E[\hat{\sigma}_{MLE}^2] = E \left[\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2 \right] = \frac{1}{n}(n-1)\sigma^2 = \frac{n-1}{n}\sigma^2 \neq \sigma^2$$

Thus, $\hat{\sigma}_{MLE}^2$ is biased.

Finding an unbiased estimator for σ^2

From the bias calculation above, we can see that:

$$E \left[\frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2 \right] = \frac{1}{n-1} E \left[\sum_{i=1}^n (X_i - \bar{X})^2 \right] = \frac{1}{n-1} (n-1)\sigma^2 = \sigma^2$$

Therefore, the unbiased estimator is:

$$S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$$

Verification of unbiasedness:

$$E[S^2] = E \left[\frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2 \right] = \frac{1}{n-1} E \left[\sum_{i=1}^n (X_i - \bar{X})^2 \right] = \frac{1}{n-1} (n-1)\sigma^2 = \sigma^2$$

3. Suppose that a company requests 541 customers to rate their services from. The average rating was estimated as 5.68. From earlier years, the standard deviation of the customer population is known to be 1.21. Construct
- (a) 90% confidence interval,
 - (b) 95% confidence interval,
 - (c) 99% confidence interval,
 - (d) discuss which one is larger and why.

Solution

We are given:

- Sample size: $n = 541$
- Sample mean: $\bar{x} = 5.68$
- Population standard deviation: $\sigma = 1.21$
- Confidence levels: 90%, 95%, 99%

Since we know the population standard deviation and have a large sample size, we use the **z-confidence interval**.

General Formula for Confidence Interval

$$CI = \bar{x} \pm z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$$

where:

- $\bar{x} = 5.68$
- $\sigma = 1.21$
- $n = 541$
- $z_{\alpha/2}$ is the critical value

Standard Error Calculation

$$SE = \frac{\sigma}{\sqrt{n}} = \frac{1.21}{\sqrt{541}} = \frac{1.21}{23.259} = 0.0520$$

a) 90% Confidence Interval

Critical value: $z_{0.05} = 1.645$

$$\text{Margin of Error} = 1.645 \times 0.0520 = 0.0855$$

$$CI = 5.68 \pm 0.0855 = (5.5945, 5.7655)$$

b) 95% Confidence Interval

Critical value: $z_{0.025} = 1.96$

$$\text{Margin of Error} = 1.96 \times 0.0520 = 0.1019$$

$$CI = 5.68 \pm 0.1019 = (5.5781, 5.7819)$$

c) 99% Confidence Interval

Critical value: $z_{0.005} = 2.576$

$$\text{Margin of Error} = 2.576 \times 0.0520 = 0.1339$$

$$CI = 5.68 \pm 0.1339 = (5.5461, 5.8139)$$

d) Discussion

Which confidence interval is larger and why?

The **99% confidence interval** is the largest, followed by the 95% interval, and then the 90% interval.

Why this occurs:

- Higher confidence levels require wider intervals to capture the true population parameter with greater certainty
- The critical value $z_{\alpha/2}$ increases as the confidence level increases:
 - 90%: $z = 1.645$
 - 95%: $z = 1.96$
 - 99%: $z = 2.576$
- Since the margin of error is directly proportional to the critical value, higher confidence levels result in larger margins of error and wider intervals

Trade-off: There is an inverse relationship between precision (interval width) and confidence level. We can be more confident that our interval contains the true parameter, but at the cost of having a less precise estimate.

Confidence Level	Critical Value	Margin of Error	Confidence Interval
90%	1.645	0.0855	(5.5945, 5.7655)
95%	1.960	0.1019	(5.5781, 5.7819)
99%	2.576	0.1339	(5.5461, 5.8139)

All intervals suggest the true population mean rating is significantly above 5, indicating generally positive customer satisfaction with the company's services.

- Suppose, for a sample selected from a normally distributed population, $\bar{x} = 65.5$. The population standard deviation is known to be 8.6.
 - Construct a 95 percent confidence interval for μ assuming $n = 16$.
 - Construct a 90 percent confidence interval for μ assuming $n = 16$. Is the width of the 90 percent confidence interval smaller or larger than the width of the 95 percent confidence interval in part (a)? Explain why.

(c) Construct a 95 percent confidence interval for μ assuming $n = 25$. Is it larger or smaller than the one you found in part (b)? Explain why.

Solution

- Sample mean: $\bar{x} = 65.5$
- Population standard deviation: $\sigma = 8.6$
- Population is normally distributed
- We use z-distribution since population standard deviation is known

Formula for Confidence Interval

$$CI = \bar{x} \pm z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$$

where:

- $\bar{x} = 65.5$
- $\sigma = 8.6$
- $z_{\alpha/2}$ is the critical value from standard normal distribution

(a) 95% Confidence Interval with $n = 16$

Critical value: $z_{0.025} = 1.96$

Standard error:

$$SE = \frac{\sigma}{\sqrt{n}} = \frac{8.6}{\sqrt{16}} = \frac{8.6}{4} = 2.15$$

Margin of error:

$$ME = 1.96 \times 2.15 = 4.214$$

Confidence interval:

$$CI = 65.5 \pm 4.214 = (61.286, 69.714)$$

(b) 90% Confidence Interval with $n = 16$

Critical value: $z_{0.05} = 1.645$

Standard error: $SE = 2.15$ (same as part a)

Margin of error:

$$ME = 1.645 \times 2.15 = 3.5368$$

Confidence interval:

$$CI = 65.5 \pm 3.5368 = (61.9632, 69.0368)$$

Comparison with part (a):

- **95% CI width:** $69.714 - 61.286 = 8.428$

- **90% CI width:** $69.0368 - 61.9632 = 7.0736$

The **90% confidence interval is narrower** than the 95% confidence interval.

Explanation: A lower confidence level means we require less certainty that the interval contains the true population mean. Therefore, we can use a smaller critical value ($z_{0.05} = 1.645$ vs $z_{0.025} = 1.96$), which results in a smaller margin of error and a narrower interval. With 90% confidence, we're willing to accept a higher risk (10% vs 5%) that our interval does not contain the true mean, which allows for a more precise estimate.

(c) 95% Confidence Interval with $n = 25$

Critical value: $z_{0.025} = 1.96$ (same as part a)

Standard error:

$$SE = \frac{\sigma}{\sqrt{n}} = \frac{8.6}{\sqrt{25}} = \frac{8.6}{5} = 1.72$$

Margin of error:

$$ME = 1.96 \times 1.72 = 3.3712$$

Confidence interval:

$$CI = 65.5 \pm 3.3712 = (62.1288, 68.8712)$$

Comparison with part (b):

- **Part (b) 90% CI, $n = 16$ width:** 7.0736
- **Part (c) 95% CI, $n = 25$ width:** $68.8712 - 62.1288 = 6.7424$

The **95% confidence interval with $n = 25$ is narrower** than the 90% confidence interval with $n = 16$.

Explanation: Although we increased the confidence level from 90% to 95% (which would normally widen the interval by requiring a larger critical value), we also increased the sample size from 16 to 25. The larger sample size reduces the standard error (2.15 to 1.72), and this effect dominates, resulting in a narrower interval overall. This demonstrates that increasing sample size can compensate for higher confidence requirements while still providing a more precise estimate.

Part	Confidence Level	Sample Size	Margin of Error	Confidence Interval
(a)	95%	16	4.214	(61.286, 69.714)
(b)	90%	16	3.537	(61.963, 69.037)
(c)	95%	25	3.371	(62.129, 68.871)

Summary

- **Confidence level effect:** Higher confidence levels require wider intervals for greater certainty, as seen when comparing parts (a) and (b) with the same sample size.

- **Sample size effect:** Larger sample sizes reduce standard error, leading to narrower intervals, as demonstrated by the narrower interval in part (c) compared to part (a).
- **Interaction effect:** The combination of increased sample size can overcome the widening effect of increased confidence level, as shown by the narrower interval in part (c) compared to part (b).