

Spring 2024-
2025
ECON 205

The Answers of the Review Problem Set II

- 1) Three balanced coins are tossed independently. One of the variables of interest is Y_1 , the number of heads. Let Y_2 denote the amount of money won on a side bet in the following manner. If the first head occurs on the first toss, you win \$1. If the first head occurs on toss 2 or on toss 3, you win \$2 or \$3, respectively. If no heads appear, you lose \$1.
- a) Find the joint probability function for Y_1 and Y_2 .
b) Find $F(2,1)$.

SOLUTION

- a. The sample space for the toss of three balanced coins w/ probabilities are below:

Outcome	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
(y_1, y_2)	(3, 1)	(2, 1)	(2, 1)	(1, 1)	(2, 2)	(1, 2)	(1, 3)	(0, -1)
probability	1/8	1/8	1/8	1/8	1/8	1/8	1/8	1/8

		y_1			
		0	1	2	3
y_2	-1	1/8	0	0	0
	1	0	1/8	2/8	1/8
	2	0	1/8	1/8	0
	3	0	1/8	0	0

- b. $F(2, 1) = p(0, -1) + p(1, 1) + p(2, 1) = 1/2$.

- 2) The joint density function of Y_1 and Y_2 is given as:

$$f(y_1, y_2) = \begin{cases} 6y_1^2 y_2, & 0 \leq y_1 \leq y_2, y_1 + y_2 \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

- a) Find the marginal density function for Y_1 .
b) Find the conditional density function of Y_2 given that $Y_1=y_1$.
c) Find $P(Y_2 < 1.1 | Y_1 = 0.6)$.

SOLUTION

$$\text{a. } f_1(y_1) = \int_{y_1}^{2-y_1} 6y_1^2 y_2 dy_2 = 12y_1^2(1-y_1), 0 \leq y_1 \leq 1.$$

b. For $0 < y_1 < 1$:

$$f(y_2 | y_1) = \frac{1}{2} y_2 / (1 - y_1), y_1 \leq y_2 \leq 2 - y_1.$$

c.

$$P(Y_2 < 1.1 | Y_1 = 0.6) = \frac{\int_{0.6}^{1.1} \frac{y_2}{0.4} dy_2}{2} = 0.53$$

- 3) Let Y_1 and Y_2 be independently exponentially distributed random variables, each with mean 1. Find $P(Y_1 > Y_2 | Y_1 < 2Y_2)$.

SOLUTION

$$P(Y_1 > Y_2, Y_1 < 2Y_2) = \int_0^\infty \int_{y_1/2}^{y_1} e^{-(y_1+y_2)} dy_2 dy_1 = \frac{1}{6} \text{ and } P(Y_1 < 2Y_2) = \int_0^\infty \int_{y_1/2}^\infty e^{-(y_1+y_2)} dy_2 dy_1 = \frac{2}{3}. \text{ So, } \\ P(Y_1 > Y_2 | Y_1 < 2Y_2) = 1/4.$$

- 4) The joint density function of Y_1 and Y_2 is given as:

$$f(y_1, y_2) = \begin{cases} e^{-(y_1+y_2)}, & y_1 > 0, y_2 > 0 \\ 0, & \text{elsewhere} \end{cases}$$

- a) Show that Y_1 and Y_2 are independent.
- b) Find $E(Y_1+Y_2)$ and $V(Y_1+Y_2)$.
- c) Find $E(Y_1-Y_2)$ and $V(Y_1-Y_2)$.
- d) Find $P(Y_1-Y_2 > 3)$.

SOLUTION

a)

$$\text{Note that } f_1(y_1) = \int_0^\infty e^{-(y_1+y_2)} dy_2 = e^{-y_1}, y_1 > 0 \text{ and } f_2(y_2) = \int_0^\infty e^{-(y_1+y_2)} dy_1 = e^{-y_2}, y_2 > 0.$$

Thus, $f(y_1, y_2) = f_1(y_1)f_2(y_2)$ so that Y_1 and Y_2 are independent.

- b) The marginal distributions for Y_1 and Y_2 are exponential with $\beta=1$. Also note that Y_1 and Y_2 are independent.

$E(Y_1+Y_2)=E(Y_1)+E(Y_2)=2$, $V(Y_1+Y_2)=2$. Note that $\text{Cov}(Y_1, Y_2)=0$ as Y_1 and Y_2 are independent.

- c) $E(Y_1-Y_2)=E(Y_1)-E(Y_2)=0$, $V(Y_1-Y_2)=2$. Note that $\text{Cov}(Y_1, Y_2)=0$ as Y_1 and Y_2 are independent.

- d) $P(Y_1-Y_2>3)=P(Y_1>3+Y_2)=\int_0^\infty \int_{3+y_2}^\infty e^{-(y_1+y_2)} dy_1 dy_2 = \frac{1}{2}e^{-3} = 0.0249$.

- 5) If c is any constant and Y is a random variable such that $E(Y)$ exists, show that $\text{Cov}(c, Y)=0$.

SOLUTION

$$\text{Since } E(c) = c, \text{Cov}(c, Y) = E[(c - c)(Y - \mu)] = 0.$$

Note that μ denotes $E(Y)$.

- 6) Show that $\text{Cov}(X, X)=\text{Var}(X)$.

SOLUTION

$$\text{Cov}(X, X) = E[(X - \mu_X)(X - \mu_X)] = E[(X - \mu_X)^2] = \text{Var}(X)$$

7)

Suppose we have a bowl with 10 marbles - 2 red marbles, 3 green marbles, and 5 blue marbles. We randomly select 4 marbles from the bowl, with replacement. What is the probability of selecting 2 green marbles and 2 blue marbles?

SOLUTION

Let's define $Y_1 = \#$ of red marbles, $Y_2 = \#$ of green marbles, $Y_3 = \#$ of blue marbles.

Therefore, we have that (Y_1, Y_2, Y_3) is multinomial with $p_1 = 0.2, p_2 = 0.3, p_3 = 0.5$.

$$P(Y_1 = 0, Y_2 = 2, Y_3 = 2) = \frac{4!}{0! * 2! * 2!} (0.2)^0 (0.3)^2 (0.5)^2 \quad (1)$$

$$P(Y_1 = 0, Y_2 = 2, Y_3 = 2) = 0.135 \quad (2)$$

- 8) Let the random variable Y possess a uniform distribution on the interval $(0, 1)$. Find the cumulative distribution function of the random variable $W = \sqrt{Y}$.

SOLUTION

$$F_W(w) = P(W \leq w) = P(\sqrt{Y} \leq w) = P(Y \leq w^2) = F_Y(w^2) = w^2, 0 \leq w \leq 1.$$

$$F_W(w) = \begin{cases} 0, & w < 0 \\ w^2, & 0 \leq w \leq 1 \\ 1, & w > 1 \end{cases}$$

(Remark: The third equality above holds for $0 \leq w \leq 1$ since w is nonnegative. In general, $P(\sqrt{Y} \leq w) + P(w \leq -\sqrt{Y}) = P(Y \leq w^2)$.)

9)

Consider the random variable Y . Its distribution function is given by

$$F(Y) = \begin{cases} 0, & \text{if } y < 0 \\ y(1 - 0.25y), & \text{if } 0 \leq y \leq 2 \\ 1, & \text{if } y > 2 \end{cases}$$

- a.) What is the density of Y ?
b.) Find the density of e^{-Y} .

SOLUTION

- a) Note that $\frac{dF(Y)}{dY} = f(y)$.

$$f(y) = \begin{cases} 1 - 0.5y, & 0 \leq y \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

- b) $u = h(y) = e^{-y}, y = h^{-1}(u) = -\ln u, \left| \frac{dh^{-1}(u)}{du} \right| = \frac{1}{u}$

Note that u is a strictly decreasing function of y .

$$f_u(u) = \begin{cases} \frac{(1 + 0.5 \ln u)}{u} & \text{for } e^{-2} \leq u \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

10) The probability density function of Y is given by

$$f(y) = \begin{cases} \left(\frac{2y}{\theta}\right) e^{\frac{-y^2}{\theta}}, & y \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

Find the density function for $U=Y^2$.

Note that U is a strictly increasing function of Y for $y \geq 0$.

SOLUTION

If $U = Y^2$, $Y = \sqrt{U}$. Thus, $\left|\frac{dy}{du}\right| = \frac{1}{2\sqrt{u}}$ and $f_U(u) = \frac{1}{\theta} e^{-u/\theta}$, $u \geq 0$.

$$f_U(u) = \begin{cases} \frac{1}{\theta} e^{-\frac{u}{\theta}} & \text{for } u \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

11) Let $Y_1, Y_2, \dots, Y_n$ be independent Poisson random variables with means $\lambda_1, \lambda_2, \dots, \lambda_n$, respectively. Find the probability distribution function of $\sum_{i=1}^n Y_i$.

SOLUTION

The moment-generating function for $U = \sum_{i=1}^n Y_i$ is $m_U(t) = e^{(e^t - 1)\sum_i \lambda_i}$ which is the moment-generating function for a Poisson with mean $\sum_i \lambda_i$. Hence the pmf of U is

$$p(u) = \frac{e^{-(\sum_i \lambda_i)} (\sum_i \lambda_i)^u}{u!}, \quad u=0,1,2,\dots$$

12)

Let Y_1 and Y_2 have joint density function

$$f_{Y_1, Y_2}(y_1, y_2) = \begin{cases} 8y_1 y_2, & 0 \leq y_1 < y_2 \leq 1, \\ 0, & \text{otherwise,} \end{cases}$$

and $U_1 = Y_1/Y_2$ and $U_2 = Y_2$

Derive the joint density function for (U_1, U_2) .

SOLUTION

$$f_{Y_1, Y_2}(y_1, y_2) = \begin{cases} 8y_1y_2, & 0 \leq y_1 < y_2 \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

$$u_1 = \frac{y_1}{y_2} \text{ and } u_2 = y_2$$

The given transformation is a one-to-one transformation with the following inverse:

$$y_2 = u_2 \text{ and } y_1 = u_1 u_2.$$

Also, $0 \leq y_1 < y_2 \leq 1$ becomes $0 \leq u_1 u_2 < u_2 \leq 1$. Equivalently, $0 \leq u_1 < 1$ and $0 < u_2 \leq 1$

The Jacobian is:

$$J = \begin{vmatrix} u_2 & u_1 \\ 0 & 1 \end{vmatrix} = u_2$$

$$f_{U_1, U_2}(u_1, u_2) = f_{Y_1, Y_2}(u_1 u_2, u_2) |u_2|$$

$$f_{U_1, U_2}(u_1, u_2) = 8(u_1 u_2) u_2 u_2 = \begin{cases} 8u_1 u_2^3 & \text{for } 0 \leq u_1 < 1, 0 < u_2 \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

13)

An electronic system has one each of two different types of components in joint operation. Let Y_1 and Y_2 denote the random lengths of life of the components of type I and type II, respectively. The joint density function is given by

$$f(y_1, y_2) = \begin{cases} (1/8)y_1 e^{-(y_1+y_2)/2}, & y_1 > 0, y_2 > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

(Measurements are in hundreds of hours.) Find $P(Y_1 > 1, Y_2 > 1)$.

SOLUTION

$$P(Y_1 > 1, Y_2 > 1) = \int_1^\infty \int_1^\infty \frac{1}{8} y_1 e^{-(y_1+y_2)/2} dy_1 dy_2 = \left[\int_1^\infty \frac{1}{4} y_1 e^{-y_1/2} dy_1 \right] \left[\int_1^\infty \frac{1}{2} e^{-y_2/2} dy_2 \right] = \frac{3}{2} e^{-\frac{1}{2}} \left(e^{-\frac{1}{2}} \right) = \frac{3}{2} e^{-1}$$

14) For the following joint probability density function

$$f(y_1, y_2) = \begin{cases} 2, & 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1, 0 \leq y_1 + y_2 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

a Find $P(Y_1 \geq 1/2 | Y_2 \leq 1/4)$.

b Find $P(Y_1 \geq 1/2 | Y_2 = 1/4)$.

c. Are Y_1 and Y_2 are independent?

Note that $f_2(y_2) = \int_0^{1-y_2} 2dy_1 = 2(1 - y_2)$ if $0 \leq y_2 \leq 1$

SOLUTION

a. $P(Y_1 \geq 1/2, Y_2 \leq 1/4) = \int_0^{1/4} \int_{1/2}^{1-y_2} 2dy_1 dy_2 = \frac{3}{16}$. And, $P(Y_2 \leq 1/4) = \int_0^{1/4} 2(1 - y_2) dy_2 = \frac{7}{16}$.

Thus, $P(Y_1 \geq 1/2 | Y_2 \leq 1/4) = \frac{3}{7}$.

b. For $0 \leq y_2 \leq 1$: $f(y_1 | y_2) = \frac{1}{1-y_2}$, $0 \leq y_1 \leq 1 - y_2$. Thus, $f(y_1 | 1/4) = 4/3$, $0 \leq y_1 \leq 3/4$.

Thus, $P(Y_1 \geq 1/2 | Y_2 = 1/4) = \int_{1/2}^{3/4} 4/3 dy_1 = 1/3$.

c.

The ranges of y_1 and y_2 depend on each other so Y_1 and Y_2 cannot be independent.

15)

Suppose that Y_1 and Y_2 are independent χ^2 random variables with ν_1 and ν_2 degrees of freedom, respectively. Find

a $E(Y_1 + Y_2)$.

b $V(Y_1 + Y_2)$.

SOLUTION

Note that a random variable Y has a χ^2 distribution with ν degrees of freedom iff Y is a gamma distributed random variable with parameters $\alpha = \nu/2$ and $\beta = 2$.

Then, $E(Y) = \alpha\beta = \nu$ and $V(Y) = \alpha\beta^2 = 2\nu$.

a. $E(Y_1 + Y_2) = E(Y_1) + E(Y_2) = \nu_1 + \nu_2$.

b. $V(Y_1 + Y_2) = V(Y_1) + V(Y_2) + 2\text{Cov}(Y_1, Y_2)$. As Y_1 and Y_2 are independent, $2\text{Cov}(Y_1, Y_2) = 0$.
 $V(Y_1 + Y_2) = 2\nu_1 + 2\nu_2$.

16) The joint density of Y_1 and Y_2 is given by

$$f(y_1, y_2) = \begin{cases} y_1 + y_2, & 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

Find $E(30Y_1 + 25Y_2)$ and $\text{Var}(30Y_1 + 25Y_2)$.

SOLUTION

$$f_1(y_1) = \int_0^1 (y_1 + y_2) dy_2 = y_1 + \frac{1}{2}, \quad 0 \leq y_1 \leq 1. \quad \text{Similarly } f_2(y_2) = y_2 + \frac{1}{2}, \quad 0 \leq y_2 \leq 1.$$

$$E(Y_1) = 7/12 \text{ and } E(Y_2) = 7/12. \text{ So, } E(30Y_1 + 25Y_2) = 30(7/12) + 25(7/12) = 32.08.$$

We have $E(Y_1) = E(Y_2) = 7/12$. Intermediate calculations give $V(Y_1) = V(Y_2) = 11/144$.

$$\text{Thus, } E(Y_1 Y_2) = \int_0^1 \int_0^1 y_1 y_2 (y_1 + y_2) dy_1 dy_2 = 1/3, \quad \text{Cov}(Y_1, Y_2) = 1/3 - (7/12)^2 = -1/144.$$

$$E(30Y_1 + 25Y_2) = 32.08, \text{ so}$$

$$V(30Y_1 + 25Y_2) = 900V(Y_1) + 625V(Y_2) + 2(30)(25)\text{Cov}(Y_1, Y_2) = 106.08.$$

17)

The amount of flour used per day by a bakery is a random variable Y that has an exponential distribution with mean equal to 4 tons. The cost of the flour is proportional to $U = 3Y + 1$.

- Find the probability density function of U via the method of distribution functions.
- Find the probability density function of U via the method of transformations.

SOLUTION

a)

The distribution function of Y is $F_Y(y) = 1 - e^{-y/4}, \quad 0 \leq y$.

$$F_U(u) = P(U \leq u) = P(3Y + 1 \leq u) = P(Y \leq \frac{u-1}{3}) = F_Y(\frac{u-1}{3}) = 1 - e^{-(u-1)/12}. \quad \text{Thus,}$$

$$f_U(u) = F'_U(u) = \frac{1}{12} e^{-(u-1)/12}, \quad u \geq 1.$$

$$\text{if } \frac{u-1}{3} \geq 0$$

$$F_Y(\frac{u-1}{3}) = 0 \text{ if } \frac{u-1}{3} < 0$$

b) Note that $U = 3Y + 1$ is strictly monotonic.

If $U = 3Y + 1$, then $Y = \frac{U-1}{3}$. Thus, $\frac{dy}{du} = \frac{1}{3}$. With $f_Y(y) = \frac{1}{4} e^{-y/4}$, we have that

$$f_U(u) = \frac{1}{3} \left[\frac{1}{4} e^{-(u-1)/12} \right] = \frac{1}{12} e^{-(u-1)/12}, \quad 1 \leq u.$$

18) For the given joint density function, find the probability density function of $U=Y_1-Y_2$.

$$f(y_1, y_2) = \begin{cases} e^{-y_1}, & 0 \leq y_2 \leq y_1 < \infty \\ 0, & \text{elsewhere} \end{cases}$$

SOLUTION

$$F_U(u) = P(U \leq u) = P(Y_1 - Y_2 \leq u) = P(Y_1 \leq u + Y_2)$$

$$\text{For } u \geq 0, P(Y_1 \leq u + Y_2) = \int_0^\infty \int_{y_2}^{u+y_2} e^{-y_1} dy_1 dy_2 = 1 - e^{-u}$$

$$\text{For } u < 0, P(Y_1 \leq u + Y_2) = 0$$

$$f_U(u) = F'_U(u) = e^{-u}, u \geq 0$$

$$f_U(u) = \begin{cases} e^{-u}, & u \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

19)

The joint probability density function of X and Y is given by

$$f(x, y) = \frac{6}{7} \left(x^2 + \frac{xy}{2} \right), \quad 0 < x < 1, \quad 0 < y < 2$$

- Verify that is indeed a joint density function.
- Compute the marginal density function of X .
- Find $P(X > Y)$.

SOLUTION

a)

By the property of the joint pdf, it must be true that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$$

Let's look at our joint pdf.

$$\int_0^2 \int_0^1 \left(\frac{6}{7} \left(x^2 + \frac{xy}{2} \right) \right) dx dy = \frac{6}{7} \left(\int_0^1 \int_0^2 \left(x^2 + \frac{xy}{2} \right) dy dx \right)$$

$$\frac{6}{7} \left(\int_0^1 \left[x^2 y + \frac{xy^2}{4} \right]_{y=0}^{y=2} dx \right) = \frac{6}{7} \left(\int_0^1 (2x^2 + x) dx \right)$$

$$\frac{6}{7} \left[\frac{2x^3}{3} + \frac{x^2}{2} \right]_{x=0}^{x=1} = \frac{6}{7} \left(\frac{2}{3} + \frac{1}{2} \right) = \frac{6}{7} * \frac{7}{6} = 1$$

b)

To find the marginal pdf of X,

For $0 < x < 1$, $f_X(x) = \int_0^2 f(x, y) \cdot dy = \int_0^2 \left(\frac{6}{7} \left(x^2 + \frac{xy}{2} \right) \right) dy$

$$f_X(x) = \frac{6}{7} \left[x^2 y + \frac{xy^2}{4} \right]_{y=0}^{y=2} = \frac{6}{7} (2x^2 + x) = \frac{12x^2}{7} + \frac{6x}{7} \text{ for } 0 < x < 1$$

So $f_X(x) = \begin{cases} \frac{12x^2}{7} + \frac{6x}{7}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

c)

$$P(X > Y) = \int_0^1 \int_0^x \left(\frac{6}{7} \left(x^2 + \frac{xy}{2} \right) \right) dy dx = \int_0^1 \frac{6}{7} \left[x^2 y + \frac{xy^2}{4} \right]_{y=0}^{y=x} dx$$

$$P(X > Y) = \int_0^1 \left(\frac{6x^3}{7} + \frac{3x^3}{14} \right) dx = \int_0^1 \left(\frac{15x^3}{14} \right) dx$$

$$P(X > Y) = \left[\frac{15x^4}{56} \right]_{x=0}^{x=1} = \frac{15}{56}$$

20)

Assume that (X, Y) has the following joint probability function,

$$f(x, y) = \begin{cases} e^{-y}, & 0 < x < y < \infty \\ 0, & \text{elsewhere} \end{cases}$$

a) Calculate $P(Y > 4 | X = 5)$.

b) Calculate $E(Y | X=x)$.

SOLUTION

a)

First, we need to find the marginal pdf of X.

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy = \int_x^{\infty} e^{-y} dy = [-e^{-y}]_x^{\infty} = e^{-x}$$

$$f_X(x) = \begin{cases} e^{-x}, & 0 < x < \infty \\ 0, & \text{otherwise} \end{cases}$$

For $x > 0$:

$$f_{Y|X}(y|x) = \frac{f(x, y)}{f_X(x)} = \begin{cases} e^{-y+x}, & \text{if } x < y \\ 0, & \text{otherwise} \end{cases}$$

$$P(Y > 4 | X = 5) = \int_5^{\infty} f(y|x=5) dy = \int_5^{\infty} e^{-y+5} dy$$

$$P(Y > 4 | X = 5) = [-e^{-y+5}]_5^{\infty} = e^0 = 1$$

b)

$$E(Y|X=x) = \int_x^{\infty} yf(y|x)dy = \int_x^{\infty} ye^{-y+x}dy$$

Integration by parts: with $u = y$, $du = dy$, $dv = e^{-y+x}dy$, $v = -e^{-y+x}$

$$E(Y|X=x) = y(-e^{-y+x})|_{y=x}^{y=\infty} - \int_x^{\infty} -e^{-y+x}dy$$

$$E(Y|X=x) = x - (e^{-y+x})|_{y=x}^{y=\infty} = x - (0 - 1) = x + 1$$

21)

Let X be such that $\ln(X) \sim N(\mu, \sigma^2)$. Then prove that

a) $E(X) = e^{\mu + (\frac{\sigma^2}{2})}$

b) $E(X^2) = e^{2\mu + 2\sigma^2}$

c) $V(X) = e^{2\mu + \sigma^2} (e^{\sigma^2} - 1)$

SOLUTION

a) $\ln X \sim N(\mu, \sigma^2)$.

Then, show that $E(X) = e^{\mu + \sigma^2/2}$

Suppose that $Y = \ln X$. As Y has a normal distribution by using moment generating function of normal distribution

$$m_Y(t) = E[e^{tY}] = \exp(\mu t + \frac{\sigma^2 t^2}{2})$$

Set $t=1$.

$$E[e^Y] = \exp(\mu + \frac{\sigma^2}{2})$$

$$E[X] = E[e^{\ln X}] = E[e^Y] = \exp(\mu + \frac{\sigma^2}{2})$$

b) $E[X^2] = E[e^{\ln X^2}] = E[e^{2\ln X}] = E[e^{2Y}]$

Let $t=2$.

$$E[e^{2Y}] = \exp(2\mu + 2\sigma^2)$$

c)

$$\begin{aligned} V(X) &= E[X^2] - [E[X]]^2 = \exp(2\mu + 2\sigma^2) - [\exp(\mu + \frac{\sigma^2}{2})]^2 = \exp(2\mu + 2\sigma^2) - \exp(2\mu + \sigma^2) \\ &= \exp(2\mu + \sigma^2) (\exp(\sigma^2) - 1). \end{aligned}$$

22)

Let the probability density function of X_1 and X_2 be given by

$$f(x_1, x_2) = \begin{cases} e^{-(x_1+x_2)}, & \text{for } x_1 > 0, \quad x_2 > 0 \\ 0, & \text{otherwise} \end{cases}$$

Consider two random variables Y_1 and Y_2 are given by

$$Y_1 = X_1 + X_2$$

$$Y_2 = \frac{X_1}{X_1 + X_2}$$

Find the joint density of Y_1 and Y_2

SOLUTION

$$Y_1 = X_1 + X_2$$

$$Y_2 = \frac{X_1}{X_1 + X_2}$$

$$X_1 = Y_1 Y_2$$

$$X_2 = Y_1 - Y_1 Y_2$$

$$J = \begin{vmatrix} y_2 & y_1 \\ 1 - y_2 & -y_1 \end{vmatrix} = -y_1 y_2 - (y_1(1 - y_2)) = -y_1$$

$$|J| = |-y_1| = y_1$$

$$f(y_1, y_2) = \exp(-(y_1 y_2) + y_1 - y_1 y_2) y_1$$

$$f(y_1, y_2) = \begin{cases} e^{-y_1 y_2}, & y_1 > 0 \text{ and } 0 < y_2 < 1 \\ 0, & \text{elsewhere} \end{cases}$$