

Problem Set 2 - Answers

- 1) Show that for any three events A, B and C with $P(C) > 0$, $P(A \cup B | C) = P(A | C) + P(B | C) - P(A \cap B | C)$.

$$\begin{aligned} P(A \cup B | C) &= \frac{P[(A \cup B) \cap C]}{P(C)} = \frac{P[(A \cap C) \cup (B \cap C)]}{P(C)} \\ &= \frac{P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)}{P(C)} \\ &= P(A | C) + P(B | C) - P(A \cap B | C) \end{aligned}$$

- 2) Let Y be a random variable with $p(y)$ and $p(0)=0.2$, $p(1)=0.3$, $p(2)=0.4$, $p(4)=0.1$. Find $E(Y)$, $E(Y^2)$ and $V(Y)$.

$$E(Y) = 0 \cdot 0.2 + 1 \cdot 0.3 + 2 \cdot 0.4 + 4 \cdot 0.1 = 1.5$$

$$E(Y^2) = 0 \cdot 0.2 + 1 \cdot 0.3 + 4 \cdot 0.4 + 16 \cdot 0.1 = 3.5$$

$$V(Y) = E(Y^2) - (E(Y))^2 = 3.5 - 1.5^2 = 1.25$$

- 3) Let X= the number of days of sick leave taken by a randomly selected employee of a large company during a particular year. Assume the maximum number of allowable sick days per year is 14. Assume also that $F(0)=.58$, $F(1)=.72$, $F(2)=.76$, $F(3)=0.81$, $F(4)=.88$ and $F(5)=0.94$. Calculate the following:

$$a) P(2 \leq X \leq 5) = P(X = 2, 3, 4 \text{ or } 5) = F(5) - F(1) = .22$$

$$b) P(X=3) = F(3) - F(2) = 0.05$$

- 4) A service organization in a large town organizes a raffle each month. One thousand raffle tickets are sold for \$1 each. Each has an equal chance of winning. First prize is \$300, second prize is \$200, and third prize is \$100. Let X denote the net gain from the purchase of one ticket.
- Construct the probability distribution of X .
 - Find the probability of winning any money in purchase of one ticket.
 - Find the expected value of X .

a)

x	299	199	99	-1
$P(x)$	0.001	0.001	0.001	0.997

The net gain of the first prize winner is $300 - 1 = 299$.

b) $P(W) = P(299) + P(199) + P(99) = 0.001 + 0.001 + 0.001 = 0.003$

c) $E(X) = 299 * 0.001 + 199 * 0.001 + 99 * 0.001 + (-1) * 0.997 = -0.4$

- 5) A pair of fair dice is rolled. Let X denote the sum of the number of dots on the top faces.

- Construct the probability distribution of X .
- Find $P(X \geq 9)$.

a)

	1	2	3	4	5	6
1	{1,1}	{1,2}	{1,3}	{1,4}	{1,5}	{1,6}
2	{2,1}	{2,2}	{2,3}	{2,4}	{2,5}	{2,6}
3	{3,1}	{3,2}	{3,3}	{3,4}	{3,5}	{3,6}
4	{4,1}	{4,2}	{4,3}	{4,4}	{4,5}	{4,6}
5	{5,1}	{5,2}	{5,3}	{5,4}	{5,5}	{5,6}
6	{6,1}	{6,2}	{6,3}	{6,4}	{6,5}	{6,6}

The possible values of X belong to [2,12]. X=2 is the event {1,1} and $P(X=2)=1/36$. X=3 is the events {1,2} and {2,1}. Then, $P(X=3)=2/36$

x	2	3	4	5	6	7	8	9	10	11	12
P(x)	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

b) $P(X \geq 9) = 4/36 + 3/36 + 2/36 + 1/36 = 10/36$

6) Let $m(t) = \left(\frac{1}{6}\right)e^t + \left(\frac{2}{6}\right)e^{2t} + \left(\frac{3}{6}\right)e^{3t}$ be the moment-generating function of a random variable Y. Find the following:

a) $E(Y)$

b) $V(Y)$

a) First derivative of the moment generating function:

$$\left(\frac{1}{6}\right)e^t + \left(\frac{2 * 2}{6}\right)e^{2t} + \left(\frac{3 * 3}{6}\right)e^{3t}$$

Let $t=0$.

$$(1/6) + (4/6) + (9/6) = (14/6) = (7/3) = E(Y)$$

b) Second derivative of the moment generating function:

$$\left(\frac{1}{6}\right)e^t + \left(\frac{2 * 2 * 2}{6}\right)e^{2t} + \left(\frac{3 * 3 * 3}{6}\right)e^{3t}$$

$$\text{Let } t=0. E(Y^2) = (1/6) + (8/6) + (27/6) = 6$$

$$V(Y) = E(Y^2) - (E(Y))^2 = 6 - (7/3)^2 = 5/9$$

7) Airlines sometimes overbook flights. Suppose that for a plane with 50 seats, 55 passengers have tickets. Define the random variables Y as the number of tickets passengers who show up for the flight. The probability mass function of Y appears in the accompanying table.

y	45	46	47	48	49	50	51	52	53	54	55
p(y)	.05	.10	.12	.14	.25	.17	.06	.05	.03	.02	.01

a. What is the probability that the flight will accommodate all tickets passengers who show up?

- b. What is the probability that not all ticketed passengers who show up can be accommodated?
- c. If you are the first person on the standby list consisting of people without tickets (which means you will be the first one to get on the plane if there are any seats available after all ticketed passengers have been accommodated), what is the probability that you will be able to take the flight? What is this probability if you are the third person on the standby list?

SOLUTION

- a. In order for the flight to accommodate all the ticketed passengers who show up, no more than 50 can show up. We need $Y \leq 50$.
 $P(Y \leq 50) = .05 + .10 + .12 + .14 + .25 + .17 = .83$
- b. Using the information in a. above, $P(Y > 50) = 1 - P(Y \leq 50) = 1 - .83 = .17$
- c. For you to get on the flight, at most 49 of the ticketed passengers must show up. $P(Y \leq 49) = .05 + .10 + .12 + .14 + .25 = .66$. For the 3rd person on the standby list, at most 47 of the ticketed passengers must show up. $P(Y \leq 47) = .05 + .10 + .12 = .27$

- 8) Suppose that Y possesses the probability mass function:

$$f(y) = \begin{cases} cy, & y = 2, 4, 6, 8, 10 \\ 0, & \text{otherwise} \end{cases}$$

- a) Find the value of c that makes $f(y)$ a probability function.
 b) Find $P(Y \geq 3)$

a)

$$\sum P(Y = y) = 1$$

$$\sum P(Y = y) = \sum_y P(Y = y) = \sum_y cy = 2c + 4c + 6c + 8c + 10c = 30c = 1$$

$$c = 1/30$$

b)

$$P(Y \geq 3) = P(Y = 4) + P(Y = 6) + P(Y = 8) + P(Y = 10) = \frac{1}{30}(4 + 6 + 8 + 10) = \frac{14}{15}$$