

Testing Linear Restrictions on Regression Coefficients

$$H_0 : w_0 \beta_0 + w_1 \beta_1 = q$$

$$H_1 : w_0 \beta_0 + w_1 \beta_1 \neq q$$

w_0, w_1 & q are known constants

$$Y_t = \beta_0 + \beta_1 x_t + \beta_2 z_t + u_t$$

$$H_0 : \beta_1 + \beta_2 = 1$$

We want to test $H_0 : \beta_0 = 0$

$H_0 : \beta_1 = 0 \rightarrow$ Joint test of significance

$$H_0 : w_{00} \beta_0 + w_{01} \beta_1 = q_0$$

$$w_{10} \beta_0 + w_{11} \beta_1 = q_1$$

$$H_1 : w_{00} \beta_0 + w_{01} \beta_1 \neq q_0$$

$$w_{10} \beta_0 + w_{11} \beta_1 \neq q_1$$

$$w_{00} = w_{11} = 1, \quad w_{01} = w_{10} = 0, \quad q_0 = 0, q_1 = 0$$

$$H_0 : \beta_0 = \beta_1 = 0$$

$$H_1 : \beta_0 = \beta_1 \neq 0$$

$$Y_t = \beta_0 + \beta_1 x_t + u_t$$

Unconstrained model $\Rightarrow Y_t = \beta_0 + \beta_1 x_t + u_t$ Sum of Square Residuals $\frac{SSR}{S_u}$

Constrained model $\rightarrow Y_t = \beta_1 x_t + u_t$

$$\rightarrow Y_t = \beta_0 + u_t$$

$$\rightarrow Y_t = u_t$$

$$\rightarrow SSR = S_R$$

$$Q = [(S_R - S_U) / S_U] [(T-2) / p] \sim F_{p, T-2}$$

$p \rightarrow$ # of restrictions

We reject H_0 if

$Q > F_{p, T-2}^\alpha$ otherwise not reject H_0

$$H_0: \beta_1 = 0$$

$$Q = [R^2 / (1 - R^2)] (T - 2) \sim F_{1, T-2}$$

Reject H_0 if $Q > F_{1, T-2}$

this test is equivalent to t -test $|\hat{\beta}_1| / \hat{\sigma}_{\hat{\beta}_1} \leq t_{T-2, \alpha/2}$

Prediction

We want to predict values of Y outside the sample by using our estimated model

$$\hat{Y}_t = \hat{\beta}_0 + \hat{\beta}_1 X_t$$

$Y_p \Rightarrow$ observed value of dependent variable to be predicted

$$Y_p = \beta_0 + \beta_1 X_p + U_p$$

$$E(U_p) = 0, \text{ Var}(U_p) = \text{Var}(U_t) = \sigma^2$$

$$E(U_t U_p) = 0$$

Regression coefficient remain same for points to be predicted

Predicted for $Y_p \rightarrow \hat{Y}_p$ would be obtained by extending estimator of $E(Y)$

Such a predictor would be formed by substituting observation x of point x_p in regression function estimated by OLS

$$\text{Then } \hat{Y}_p = \hat{\beta}_0 + \hat{\beta}_1 x_p$$

$$Y_t = \beta_0 + \beta_1 X_t + U_t \quad t = 1990, \dots, 2020$$

$$\begin{bmatrix} Y_t \\ \vdots \\ Y_t \end{bmatrix} \begin{matrix} 1990 \\ \vdots \\ 2020 \end{matrix} \quad \begin{bmatrix} X_t \\ \vdots \\ X_t \end{bmatrix} \begin{matrix} 1990 \\ \vdots \\ 2000 \end{matrix}$$

$$Y_t = 50 + 0.7 X_t$$

$$X_{2021} = 200$$

$$\hat{Y}_{2021} = 50 + 140 = 190 \quad \text{error of 10 units}$$

$$Y_{2021} = 200$$

In order to establish finite sample properties of $\hat{Y}_p$

substitute $\hat{\beta}_0 = \sum_t a_{t0} Y_t$ and

$$\hat{\beta}_1 = \sum_t a_{t1} Y_t$$

$$\textcircled{1} \hat{Y}_p = \sum_t a_{t0} Y_t + x_p \sum_t a_{t1} Y_t = \sum_t a_{tp} Y_t$$

$$\text{where } a_{tp} = a_{t0} + x_p a_{t1}$$

Thus, $\hat{Y}_p$ is a linear predictor Y_p

$$\textcircled{2} \sum a_{tp} = \underbrace{\sum a_{t0}}_1 + x_p \underbrace{\sum a_{t1}}_0 = 1$$

$$\textcircled{3} \sum a_{tp} x_t = \underbrace{\sum a_{t0} x_t}_0 + x_p \underbrace{\sum a_{t1} x_t}_1 = x_p$$

Then substitute $Y_t = \beta_0 + \beta_1 X_t + U_t$ into $\textcircled{1}$

$$\hat{Y}_p = \beta_0 \sum a_{tp} + \beta_1 \sum a_{tp} x_t + \sum a_{tp} U_t$$

By ② & ③

$$\hat{Y}_p = \beta_0 + \beta_1 X_p + \sum a_{tp} U_t$$

$$\textcircled{4} \quad \hat{Y}_p = E(Y_p) + \sum a_{tp} U_t$$

$$E(\hat{Y}_p) = E(E(Y_p)) + \underbrace{E(\sum a_{tp} U_t)}_{0''}$$

$$E(\hat{Y}_p) = E(Y_p)$$

$\hat{Y}_p$ is an unbiased predictor of Y_p

For efficiency, we need variance of $\hat{Y}_p$, $\sigma_{\hat{Y}_p}^2$

By using ④

$$\begin{aligned} \sigma_{\hat{Y}_p}^2 &= E[\hat{Y}_p - E(\hat{Y}_p)]^2 \\ &= E[\sum a_{tp} U_t]^2 \\ &= E(\sum a_{tp}^2) \underbrace{E(U_t)^2}_{\sigma^2} \end{aligned}$$

$$= \sigma^2 \sum a_{tp}^2$$

$$a_{t0} = T^{-1} - \bar{X} a_{t1}$$

$$\sum a_{tp}^2 = \sum (a_{t0} + X_p a_{t1})^2$$

$$\sigma_{\hat{Y}_p}^2 = \sigma^2 \sum (T^{-1} - \bar{X} a_{t1} + X_p a_{t1})^2$$

$$x_p = X_p - \bar{X}$$

$$\begin{aligned} \sigma_{\hat{Y}_p}^2 &= \sigma^2 \sum (T^{-1} + x_p a_{t1})^2 \\ &= \sigma^2 (T^{-1} + x_p^2 \sum a_{t1}^2 + \underbrace{2(T^{-1} x_p \sum a_{t1})}_{0''}) \end{aligned}$$

Since

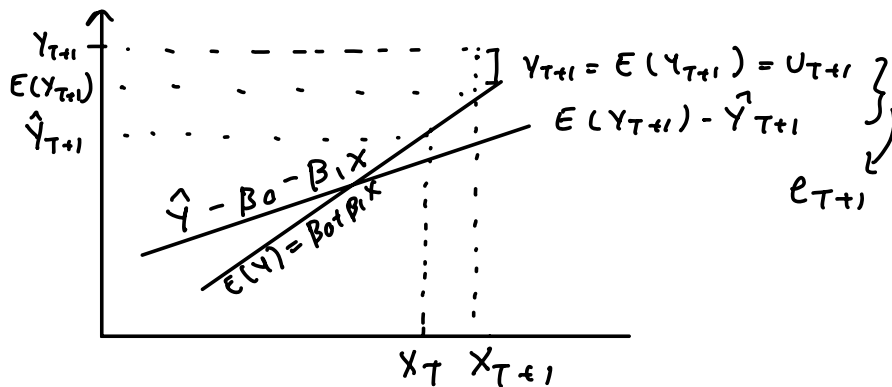
$$\sigma^2 \sum a_{t1}^2 = \frac{\sigma^2}{\sum x_t^2}$$

$$\sigma_{\hat{y}_p}^2 = \sigma^2 \left[T^{-1} + \frac{x_p^2}{\sum x_t^2} \right]$$

$\hat{y}_p$ has a minimum variance it is BLU predictor of y_p

Prediction error is

$$e_p = y_p - \hat{y}_p$$



Then prediction error may be expressed as

$$\begin{aligned} e_p &= y_p - E(y_p) + E(y_p) - \hat{y}_p \\ &= \underbrace{U_p}_{\text{disturbance}} + \underbrace{[E(y_p) - \hat{y}_p]}_{\text{biased } E(y_p)} \end{aligned}$$

If our assumptions hold, then expected value of both components should be zero

This means that $E(e_p) = 0$

Since prediction error is observable it may be used as an estimator of its mean, we test if

$$H_0: E(e_p) = 0$$

$$H_1: E(e_p) \neq 0$$

in order to assess predictive performance of our model if H_0 is rejected, this may be because

1- $E(u_p) \neq 0 \Rightarrow$ model is misspecified

2- $E(y_p) \neq E(\hat{y}_p) \Rightarrow \hat{y}_p$ is a biased estimator of y_p

In order to test $H_0: E(e_p) = 0$

$$\begin{aligned}\sigma_{e_p}^2 &= E(e_p^2) \\ &= E(u_p - (\hat{y}_p - E(y_p)))^2 \\ &= \sigma^2 + \sigma_{\hat{y}_p}^2\end{aligned}$$

$$\begin{aligned}\text{since } E[(\hat{y}_p - E(\hat{y}_p)) u_p] \\ = \sum a_{tp} \underbrace{E(u_t u_p)}_0 = 0\end{aligned}$$

by using

$$\sigma_{\hat{y}_p}^2 = \sigma^2 [T^{-1} + (x_p^2 / \sum_t x_t^2)]$$

$$\sigma_{e_p}^2 = \sigma^2 + \sigma^2 [T^{-1} + (x_p^2 / \sum_t x_t^2)]$$

$$\sigma_{e_p}^2 = \sigma^2 [1 + T^{-1} + (x_p^2 / \sum_t x_t^2)]$$

e_p is a linear function of disturbances

$$e_p = u_p + \sum a_{tp} u_t$$

Assume $u_p \sim N(0, \sigma^2)$

$$e_p \sim N(0, \sigma_{ep}^2)$$

If we estimate σ_{ep}^2 by substituting $\hat{\sigma}^2$ instead of σ^2

$$\hat{\sigma}_{ep}^2 = \hat{\sigma}^2 [1 + T^{-1} + (x_p^2 / \sum_t x_t^2)]$$

$$|e_p / \hat{\sigma}_{ep}| \sim t_{T-2, \alpha/2}$$

If $H_1: E(e_p) \neq 0$, then H_0 is rejected then

$$|e_p / \hat{\sigma}_{ep}| > t_{T-2, \alpha/2}$$

Confidence interval for y_p is

$$P[\hat{y}_p - \hat{\sigma}_{ep} t_{T-2, \alpha/2} \leq y_p \leq \hat{y}_p + \hat{\sigma}_{ep} t_{T-2, \alpha/2}] = 1 - \alpha$$

Model without Intercept Term

$$y_t = \beta x_t + u_t$$

$$\min S = \sum_t \hat{u}_t^2 = \sum_t (y_t - \hat{\beta} x_t)^2$$

$$\partial S / \partial \hat{\beta} = \sum_t (x_t (y_t - \hat{\beta} x_t)) = \sum_t x_t \hat{u}_t = 0$$

$$\hat{\beta} = \sum x_t y_t / \sum x_t^2$$

Two results are important

1 - $\sum u_t$ need not to be equal to zero

Regression line need not pass through $\bar{y}$ & $\bar{x}$

2 - $\hat{\beta}$ is obtained from raw data

Variance of $\hat{\beta}$ will be

$$\sigma_{\hat{\beta}}^2 = \sigma^2 / \sum_t x_t^2$$

$$\hat{\sigma}^2 = \sum \hat{u}_t^2 / (T-1)$$

$$\begin{aligned}\hat{\sigma}_{\hat{\beta}}^2 &= [\sum \hat{u}_t^2 / T-1] / \sum x_t^2 \\ &= \hat{\sigma}^2 / \sum x_t^2\end{aligned}$$

Estimate of $Y_t = \beta x_t + u_t$ may also be used for prediction

Again assuming that Y_p is generated by model by which is generated by T sample observation

$\hat{Y}_p = \hat{\beta} x_p$ may be used as predictor.

This will be BLU predictor and its variance is

$$\begin{aligned}\sigma_{\hat{Y}_p}^2 &= E [\hat{Y}_p - E(\hat{Y}_p)]^2 \\ &= E [(\hat{\beta} - \beta) x_p]^2 \\ &= x_p^2 \sigma_{\hat{\beta}}^2 = \frac{\sigma^2 x_p^2}{\sum_t x_t^2}\end{aligned}$$

Variance of Prediction Error

$e_p = Y - \hat{Y}_p$ is

$$\begin{aligned}\sigma_{e_p}^2 &= \sigma_{\hat{Y}_p}^2 + \sigma^2 \\ &= \hat{\sigma}^2 (1 + (x_p^2 / \sum_t x_t^2))\end{aligned}$$

Estimator of $\sigma_{\hat{Y}_p}^2$ & $\sigma_{e_p}^2$ is found

$$H_0 : E(e_p) = 0$$

$$H_1 : E(e_p) \neq 0$$

H_0 is rejected if

$$|e_p| \hat{\sigma}_{ep} > t_{T-1, \alpha/2}$$

$$Y_t = \beta X_t + u_t$$

Goodness of Fit

Since there is no question of considering common component of distance of each Y_t from origin, there is no question of taking $\bar{Y}$ into account

Consider

$$\begin{aligned}\sum_t Y_t^2 &= \sum (\hat{Y}_t + \hat{u}_t)^2 \\&= \sum \hat{Y}_t^2 + \sum \hat{u}_t^2 + 2 \sum \hat{Y}_t \hat{u}_t \\&= \sum \hat{Y}_t \hat{u}_t = \beta \underbrace{\sum X_t u_t}_{=0} = 0 \\ \sum Y_t^2 &= \sum \hat{Y}_t^2 + \sum \hat{u}_t^2\end{aligned}$$

from here, we may define raw data or uncentered coefficient of determination, R_r^2 , as

$$R_r^2 = \frac{\sum \hat{Y}_t^2}{\sum Y_t^2}$$

$$\text{where } \sum \hat{Y}_t^2 = \sum Y_t^2 - \sum \hat{u}_t^2$$

substitute this into R_r^2 expression, then we have

$$\begin{aligned}R_r^2 &= \frac{\sum Y_t^2}{\sum Y_t^2} - \frac{\sum \hat{u}_t^2}{\sum Y_t^2} \\&= 1 - \left(\frac{\sum \hat{u}_t^2}{\sum Y_t^2} \right)\end{aligned}$$

$0 \leq R_r^2 \leq 1$
 $\downarrow \qquad \qquad \downarrow$
bad fit \qquad \qquad good fit

Test of Hypothesis

$$H_0: \beta = 0$$

$$H_1: \beta \neq 0$$

$$t = \frac{\hat{\beta}}{\hat{\sigma}_{\hat{\beta}}} \sim t_{T-1, \alpha/2}$$

We reject H_0 if

$$|\hat{\beta} / \hat{\sigma}_{\hat{\beta}}| > t_{T-1, \alpha/2}$$

We may use Q-stat

$$Q = \left(\frac{\beta^2 \sum x_t^2}{\sum \hat{u}_t^2} \right) (T-1)$$

$$= \left(\frac{\sum \hat{y}_t^2}{\sum \hat{u}_t^2} \right) (T-1)$$

$$= \left[\frac{R_r^2}{1 - R_r^2} \right] (T-1)$$

If $Q > F_{1, T-1}^{\alpha}$ Reject H_0 .