

Continuous Random Variables

Distribution func.

Let y be any random var. distr. func. of Y denoted $f(y)$, is given by

$$f(y) = P(Y \leq y) \text{ for } -\infty < y < \infty$$

Nature of a distr. func of a random var. determines whether variable is continuous or discrete

Ex: Find distr. function of total # of heads obtained in four tosses of a coin

$$\text{given } f(0) = 1/16 \quad f(1) = 4/16 \quad f(2) = 6/16 \quad f(3) = 4/16 \\ f(4) = 1/16$$

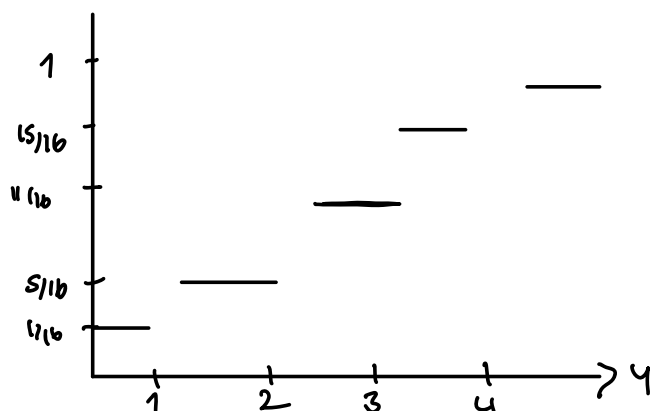
$$F(0) = f(0) = 1/16$$

$$F(1) = f(0) + f(1) = 5/16$$

$$F(2) = f(0) + f(1) + f(2) = 11/16$$

$$F(3) = f(0) + f(1) + f(2) + f(3) = 15/16$$

$$F(4) = f(0) + f(1) + f(2) + f(3) + f(4) = 1$$



$$F(y) = \begin{cases} 0 & \text{for } y < 0 \\ 1/16 & \text{for } 0 \leq y < 1 \\ 5/16 & \text{for } 1 \leq y < 2 \\ 11/16 & \text{for } 2 \leq y < 3 \\ 15/16 & \text{for } 3 \leq y < 4 \\ 1 & \text{for } y \geq 4 \end{cases}$$

Ex: $F(y) = \begin{cases} 0 & \text{for } y \leq 0 \\ 1 - e^{-3y} & \text{for } y > 0 \end{cases}$

Properties of Distr. Func.

① $F(-\infty) = \lim_{y \rightarrow -\infty} F(y) = 0$

② $F(\infty) = \lim_{y \rightarrow \infty} F(y) = 1$

③ $F(y)$ is a nondecreasing function of y . If y_1 & y_2 are any values such that $y_1 < y_2$
 $F(y_1) \leq F(y_2)$

Let Y be a random var. with distr. func. $F(y)$. Y is said to be continuous, if distr. func $F(y)$ is continuous for $-\infty < y < \infty$

Let $F(y)$ be distr. func. for a cont. rv. Y . Then $f(y)$ is given by

$$f(y) = \frac{d F(y)}{dy} = F'(y) \quad \text{whenever this derivative exists}$$

↳ $f(y)$ is called probability density function (pdf)

cdf $\rightarrow$ cumulative density function

$$F(y) = \int_{-\infty}^y f(t) \cdot dt$$

Properties of Density func.

If $f(y)$ is a density func,

① $f(y) \geq 0$ for $-\infty < y < \infty$

② $\int_{-\infty}^{\infty} f(y) \cdot dy = 1$

Ex: If Y has a pdf.

$$f(y) = \begin{cases} ke^{-3y} & y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Find k & $P(0.5 \leq Y \leq 1)$

$$\int_{-\infty}^{\infty} f(y) \cdot dy = \int_0^{\infty} ke^{-3y} \cdot dy = k \lim_{t \rightarrow \infty} \frac{e^{-3y}}{-3} \Big|_0^y = \frac{k}{3} = 1$$

$$f(y) = \begin{cases} 3e^{-3y} & y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$P(0.5 \leq Y \leq 1) = \int_{0.5}^1 3e^{-3y} \cdot dy = -e^{-3y} \Big|_{0.5}^1 = e^{-3} + e^{-1.5} = 0.173$$

Ex: Find distr. func. of rv Y of

$$f(y) = \begin{cases} 3e^{-3y} & \text{for } y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

& use it to re-evaluate $P(0.5 \leq Y \leq 1)$

for $y > 0$

$$F(y) = \int_{-\infty}^y f(y) dy = \int_0^y 3e^{-3y} dy = -e^{-3y} \Big|_0^y = 1 - e^{-3y}$$

$$F(y) = \begin{cases} 0 & \text{for } y \leq 0 \\ 1 - e^{-3y} & \text{for } y > 0 \end{cases}$$

$$P(0.5 \leq Y \leq 1) = F(1) - F(0.5) = (1 - e^{-3}) - (1 - e^{-1.5}) = 0.173$$

Ex: The pdf of rv Y is

$$f(y) = \begin{cases} \frac{c}{\sqrt{y}} & \text{for } 0 < y < 4 \\ 0 & \text{elsewhere} \end{cases}$$

Find value of c & $P(Y < 1/4)$, $P(Y > 1)$

$$1 = \int_0^4 \frac{c}{\sqrt{y}} dy = c \int_0^4 y^{-1/2} dy = \frac{c \cdot y^{1/2}}{1/2} \Big|_0^4 = 2c \cdot 2 = 4c \Rightarrow c = 1/4$$

$$f(y) = \begin{cases} 1/4\sqrt{y} & \text{for } 0 < y < 4 \\ 0 & \text{elsewhere} \end{cases}$$

$$P(1/4) = \int_0^{1/4} \frac{1}{4\sqrt{y}} \cdot dy = \frac{1}{4} \int_0^{1/4} y^{-1/2} \cdot dy = \frac{1}{4} \cdot \frac{\sqrt{y}}{1/2} \Big|_0^{1/4} = 1/4$$

$$P(y > 1) = 1 - \int_0^1 \frac{1}{4\sqrt{y}} \cdot dy = 1 - \frac{1}{2} \sqrt{y} \Big|_0^1 = 1/2$$

Ex:

Find distr. func. of Y whose pdf is

$$f(y) = \begin{cases} 6y(1-y) & \text{for } 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$F(y) = \int_0^y 6y(1-y) dy = 3y^2 - 2y^3$$

$$F(y) = \begin{cases} 0 & y \leq 0 \\ 3y^2 - 2y^3 & 0 < y < 1 \\ 1 & y \geq 1 \end{cases}$$

Expected Value for a continuous r.v.

If Y is a continuous r.v. & $f(y)$ is value of its prob. density of y , expected value of Y is

$$E(Y) = \int_{-\infty}^{\infty} y f(y) dy, \text{ provided that integral exists}$$

Ex:

For following pdf, find exp. value of Y

$$f(y) = \begin{cases} \frac{4}{\pi(1+y^2)} & \text{for } 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$E(Y) = \int_0^1 y \cdot \frac{4}{\pi(1+y^2)} dy = \frac{4}{\pi} \int_0^1 \frac{y}{1+y^2} dy = \frac{\ln 4}{\pi} = 0.4413$$

If Y is a continuous random variable, $f(Y)$ is value of its prob. density of Y , exp. value of $g(Y)$ is

$$E[g(Y)] = \int_{-\infty}^{\infty} g(y) \cdot f(y) \cdot dy$$

Ex: If Y has pdf

$$f(y) = \begin{cases} e^{-y} & \text{for } y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Find exp. value of $g(y) = e^{3y/4}$

$$E[e^{3y/4}] = \int_0^{\infty} e^{3y/4} \cdot e^{-y} \cdot dy = \int_0^{\infty} e^{-y/4} = 4$$

Let c be a constant & let $g(y), g_1(y), g_2(y) \dots g_k(y)$ be functions of r.v. Y

Then

$$\textcircled{1} E(c) = c$$

$$\textcircled{2} E[c \cdot g(y)] = c E[g(y)]$$

$$\textcircled{3} E[g_1(y) + g_2(y) + \dots + g_k(y)] = E[g_1(y)] + E[g_2(y)] + \dots + E[g_k(y)]$$

Let Y be a continuous r.v. with pdf $f(y)$, then

$$V(Y) = \sigma^2 = E[(Y - \mu)^2] = E(Y^2) - \mu^2$$

if a & b are constants.

$$E(aY + b) = aE(Y) + b$$

Proof:

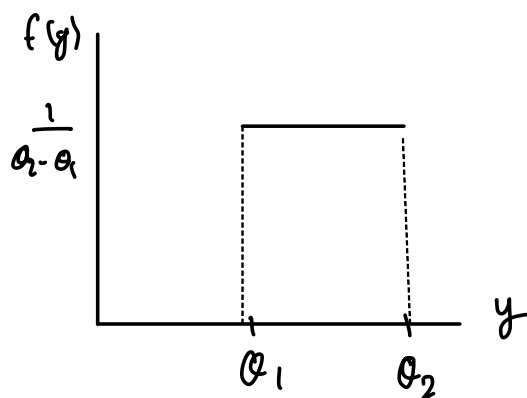
$$\begin{aligned} E(aY + b) &= \int_{-\infty}^{\infty} (aY + b) f(y) dy = a \int_{-\infty}^{\infty} y f(y) dy + b \int_{-\infty}^{\infty} f(y) dy \\ &= aE(Y) + b \end{aligned}$$

Uniform Prob. Distribution

A r.v. has a uniform distribution & referred as cont. uniform r.v. iff its pdf is

$$f(y) = \begin{cases} \frac{1}{a_2 - a_1} & \text{for } a_1 < y < a_2 \\ 0 & \text{elsewhere} \end{cases}$$

Parameters θ_1 & θ_2 are real constants, $\theta_1 < \theta_2$



mean & variance of uniform distr. are given by

$$\mu = E(Y) = \frac{\theta_1 + \theta_2}{2}$$

$$\sigma^2 = V(Y) = \frac{1}{12} (\theta_2 - \theta_1)^2$$

Normal Prob. Distr.

errors of measurement $\rightarrow$ continuous curves

$\hookrightarrow$ normal curves of errors

Mathematical properties of these curves $\rightarrow$ Abraham de Moivre (1667-1745)

Pierre Laplace

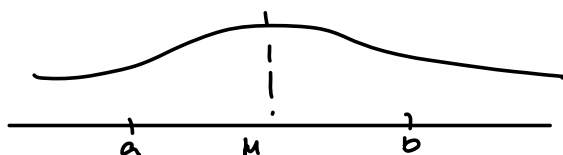
Karl Gauss

A random variable Y has a normal distribution & referred as a normal r.v. i.f.f. its pdf is

$$f(y) = \frac{1}{\sigma \sqrt{2\pi}} e^{-(y - \mu)^2 / 2\sigma^2}$$

There are two parameters $\rightarrow \sigma, \mu$

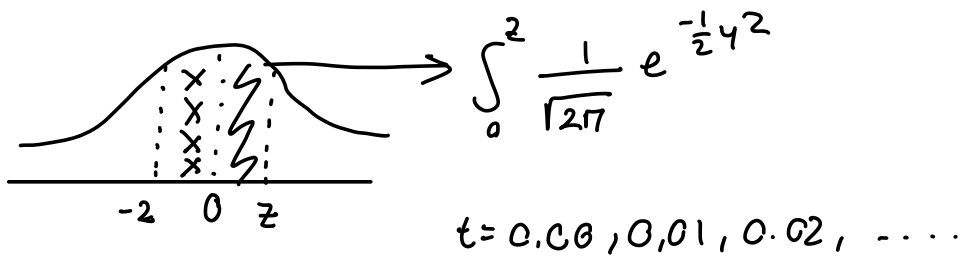
$$E(Y) = \mu, V(Y) = \sigma^2$$



$$P(a \leq Y \leq b) = \int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(Y-\mu)^2}{2\sigma^2}} \cdot dY$$

for special case $\mu=0$, $\sigma=1$ $Y \sim (0,1)$

↳ standard normal distribution



If Y has a normal distr. with mean μ , & st. dev σ , then

$$Z = \frac{Y - \mu}{\sigma} \Rightarrow \text{has st normal distr.}$$

Probability Distributions
APP

Ex: Find probs that a r.v. having st. normal distr will take on a value

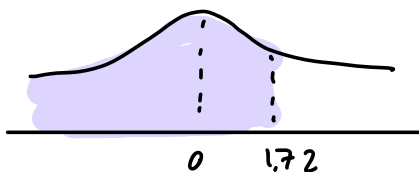
a) less than 1.72

b) less than -0.88

c) between 1.3 & 1.75

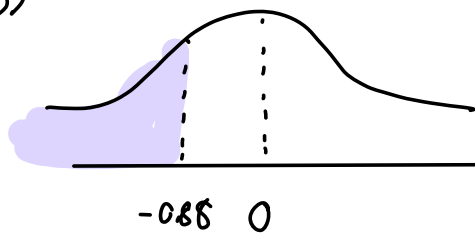
d) between -0.25 & 0.45

a)



$$P(Z < 1.72) = 0.5000 + 0.4573 = 0.9573$$

b)

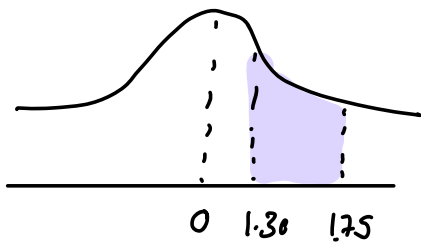


$$z = 0.88 \rightarrow 0.3106$$

$$P(z < -0.88)$$

$$= 0.5000 - 0.3106 = 0.1894$$

c)



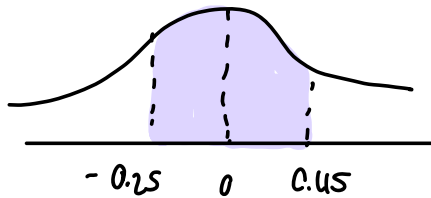
$$z = 1.30 \rightarrow 0.4032$$

$$z = 1.75 \rightarrow 0.4592$$

$$P(1.30 < z < 1.75)$$

$$= 0.4598 - 0.4032 = 0.0567$$

d)



$$z = 0.25 \rightarrow 0.0987$$

$$z = 0.45 \rightarrow 0.1736$$

$$P(-0.25 < z < 0.45) = 0.0987 + 0.1736$$

$$= 0.2723$$

Ex: With reference to table find values of z corresponding to entries a) 0.3512 b) 0.2533

a) 0.3512 falls between 0.3508 & 0.3531

$z = 1.04$, $z = 1.05$ since 0.3512 is closer to 0.3508

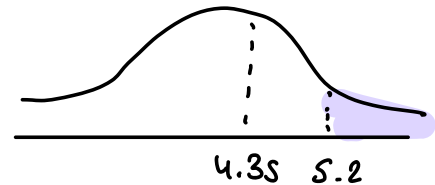
Choose $z = 1.04$

b) 0.2533 falls midway bw 0.2517 & 0.2549

$z = 0.68$ $z = 0.69$, choose $z = 0.685$

Ex: Suppose that amount of defective items, produced by a factory is a rr. having a normal distribution with mean 4.35 & st dev 0.59 per day. What is the prob. that number of defective items produced in this factory is more than 5.20 unit.

$$\frac{y - \mu}{\sigma} = z = \frac{5.20 - 4.35}{0.59} = 1.44$$



$$z = 1.44 \rightarrow 0.4251$$

$$P(Z > 1.44) = 0.5000 - 0.4251 = 0.0749$$