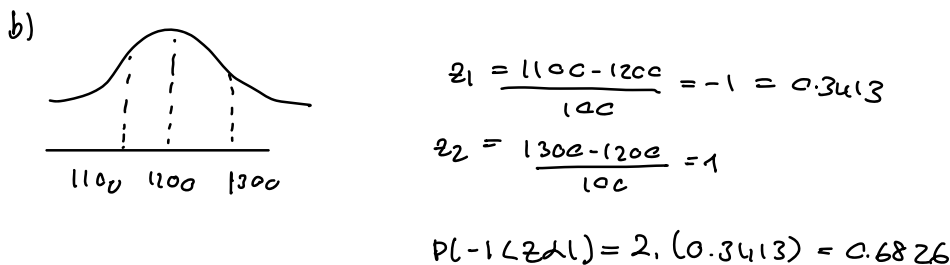
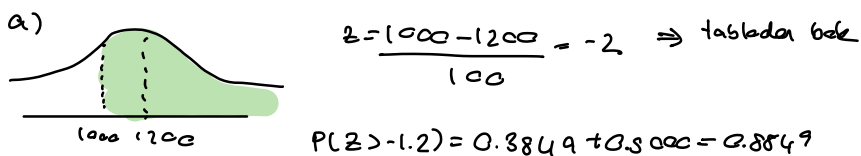


Suppose that hourly wage rate in a certain industrial district is r.v. having a normal distr. with $\mu = 37.6 \$$ $\sigma = 4.6 \$$ per hour. Find probs that a randomly selected worker is paid

- a) at least 44.5 \$ per hour
- b) at most 35 per hour
- c) any wage between 30 - 40 per hour

2) Expected demand for a product in the next year has been represented by a r.v. with $\mu = 1200$, $\sigma = 100$ units

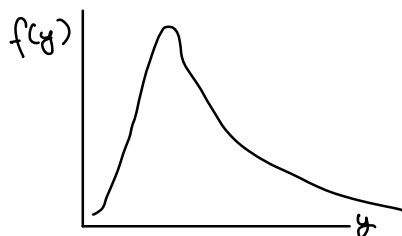
- a) what is prob. that sales will exceed 1000 units
- b) what is prob. that sales will be between 1100 & 1300 units



Gamma Prob. Distribution

Some random variables are always non-negative $\rightarrow$ distributions skewed to right

most are under density function is located near origin & density func. drops gradually as $y \uparrow$



A r.v. Y has a gamma distribution & referred as gamma r.v. iff its pdf is

$$f(y) = \begin{cases} \frac{y^{\alpha-1} e^{-y/\beta}}{\beta^{\alpha} \Gamma(\alpha)} & 0 \leq y < \infty \\ 0 & \text{elsewhere} \end{cases}$$

where parameters α & $\beta > 0$

$$\Gamma(\alpha) = \int_0^{\infty} y^{\alpha-1} e^{-y} dy \Rightarrow \text{gamma function}$$

Gamma func. satisfies recursion formula

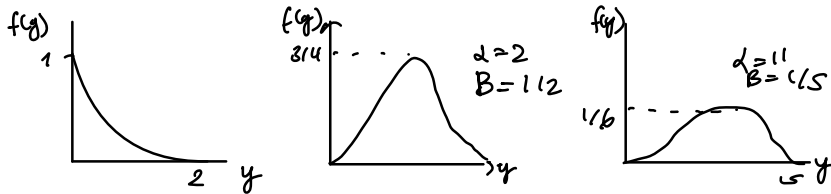
$$\Gamma(\alpha) = (\alpha-1) \Gamma(\alpha-1) \text{ for } \alpha > 1$$

$$\Gamma(1) = \int_0^{\infty} e^{-y} dy = 1$$

by repeated application of recursion formula

$$\Gamma(\alpha) = (\alpha-1)! \text{ when } \alpha \text{ is a positive integer}$$

α is called the shape parameter, β = scale parameter



mean & variance of gamma distr. are

$$\left. \begin{aligned} \mu &= E(Y) = \alpha\beta \\ \sigma^2 &= V(Y) = \alpha\beta^2 \\ E(Y^2) &= \int_0^{\infty} y^2 f(y) dy \\ E(Y) &= \int_0^{\infty} y \cdot f(y) dy \end{aligned} \right\} V(Y) = E(Y^2) - [E(Y)]^2$$

Exponential Distr
Chi-square distr

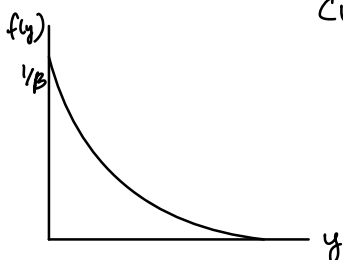
Exponential Distribution

A special case of gamma distr. for $\alpha=1$

A r.v. Y has an exponential distr. & referred as exponential r.v. iff its pdf

$$f(y) = \begin{cases} \frac{1}{\beta} e^{-y/\beta} & \text{for } y \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

★ Sinaida Gilek



where $\beta > 0$

Cumulative distr. func. (cdf) of exponential distr is

$$F(y) = 1 - e^{-y/\beta} \text{ for } y > 0$$

mean & variance of exponential distr. are

$$\mu = E(Y) = \beta, \sigma^2 = V(Y) = \beta^2$$

Ex: Times to make control of an exam paper by students at lecture room follow an exponential distr. with β of 15 minutes. Find prob that for a randomly control that lasts more than 18 minutes

$$P(Y > 18) = 1 - P(Y < 18) = 1 - F(18)$$

$$= 1 - (1 - e^{-18/15}) = e^{-1.2} = 0.301$$

$$\int_{18}^{\infty} \frac{1}{\beta} e^{-y/\beta} \cdot dy$$

Chi-Square Distr.

Special case of gamma distr. $\alpha = \nu/2$, $\beta = 2$

$\nu \Rightarrow$ degrees of freedom

A random variable has a chi-square (χ^2) distr. & referred as a Chi-square r.v. iff its pdf is

$$f(y) = \begin{cases} \frac{1}{2^{\nu/2} \Gamma(\nu/2)} \cdot y^{\frac{\nu-2}{2}} e^{-y/2} & \text{for } y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$\mu = E(Y) = \nu$$

$$\sigma^2 = V(Y) = 2\nu$$

Moments & moment-generating func.

The k^{th} moment about origin of a contin. r.v., μ'_k , is expected value of y^k

$$\mu'_k = E(Y^k) = \int_{-\infty}^{\infty} y^k f(y) \cdot dy, \quad k = 1, 2, \dots$$

$\mu'_1 = \mu \Rightarrow$ mean of a r.v

The k^{th} moment about mean of a contin. r.v., μ_k , is exp. value of $[y - \mu]^k$

$$\mu_k = E[(Y - \mu)^k] = \int_{-\infty}^{\infty} (y - \mu)^k f(y) \cdot dy, \quad k = 1, 2, \dots$$

Ex: $f(y) = \begin{cases} \frac{1}{\ln 3} \cdot \frac{1}{y} & \text{for } 1 < y < 3 \\ 0 & \text{elsewhere} \end{cases}$

$$\mu_2 = V(Y) = \sigma^2 \Rightarrow \text{variance of } Y$$

Find the μ'_k & σ^2

$$\begin{aligned} \mu'_k &= \frac{1}{\ln 3} \int_1^3 y^k \left(\frac{1}{y} \right) dy = \frac{1}{\ln 3} \int_1^3 y^{k-1} \cdot dy = \frac{1}{\ln 3} \left[\frac{y^k}{k} \right]_1^3 = \frac{1}{k \ln 3} \cdot 3^k - 1 \\ &= \frac{3^k - 1}{k(\ln 3)} \end{aligned}$$

1st way. substitute $t=1$

$$\mu_1' = \frac{2}{\ln 3} \quad \text{or} \quad 2^{\text{nd}} \text{ way} \quad \mu_1' = \mu = \frac{1}{\ln 3} \int_1^3 y \cdot \frac{1}{y} dy = \frac{2}{\ln 3}$$

$$\mu_2' = \frac{1}{\ln 3} \int_1^3 y^2 \left(\frac{1}{y}\right) dy = \frac{8}{2\ln 3} = \frac{4}{\ln 3}$$

$$\sigma^2 = \mu_2' - \mu^2 = \frac{4}{\ln 3} - \left(\frac{2}{\ln 3}\right)^2 = \frac{4 \cdot \ln 3 - 1}{(\ln 3)^2}$$

Ex:

Find first four moments about the origin & about the mean for a r.v. Y having pdf

$$f(y) = \begin{cases} 4y(9-y^2)/81 & 0 \leq y \leq 3 \\ 0 & \text{elsewhere} \end{cases}$$

$$\mu_1' = E(Y) = \frac{4}{81} \int_0^3 y^2 (9-y^2) dy = \frac{8}{5} = \mu$$

$$\mu_2' = E(Y^2) = \frac{4}{81} \int_0^3 y^3 (9-y^2) dy = 3$$

$$\mu_3' = E(Y^3) = \frac{4}{81} \int_0^3 y^4 (9-y^2) dy = \frac{216}{35}$$

$$\mu_4' = E(Y^4) = \frac{4}{81} \int_0^3 y^5 (9-y^2) dy = \frac{17}{2}$$

it can be shown that

$$\mu_2 = \mu_2' - \mu^2$$

$$\mu_2 = 11/25$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu + 2\mu^3$$

$$\mu_3 = -32/875$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu + 6\mu_2'\mu^2 - 3\mu^4$$

$$\mu_4 = 3693/8750$$

$$\mu_1 = 0$$

2nd way

$$\int_0^3 (y-\mu)^2 f(y) dy$$

If Y has a variance σ^2 then $\text{Var}(aY+b) = a^2\sigma^2$

$$\text{Ex: } f(y) = \begin{cases} y & \text{for } 0 \leq y \leq 1 \\ 2-y & \text{for } 1 \leq y \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$

Find $\text{Var}(2Y+3)$

$$E(Y) = \mu = \int_0^1 y^2 \cdot dy + \int_1^2 y(2-y) dy = \frac{1}{3} + \left[y^2 - \frac{y^3}{3} \right] \Big|_1^2 = 1$$

$$\mu_2' = \int_0^1 y^3 \cdot dy + \int_1^2 y^2(2-y) \cdot dy = \frac{1}{4} \left[\frac{2y^3}{3} - \frac{y^4}{4} \right] \Big|_1^2 = 7/6$$

$$\sigma^2 = \mu_2' - \mu^2 = 7/6 - 1^2 = \frac{1}{6} \quad \text{Var}(2Y+3) = 4 \text{Var}(Y) = 4 \cdot \frac{1}{6} = \frac{2}{3}$$

The moment generating func. (mgf). of a cont. r.v., where it exists is given by

$$M(t) = E(e^{ty}) = \int_{-\infty}^{\infty} \left[1 + ty + \frac{t^2 y^2}{2!} + \dots + \frac{t^k y^k}{k!} \right] f(y) \cdot dy$$

$$M(t) = E(e^{ty}) = \int_{-\infty}^{\infty} e^{ty} \cdot f(y) \cdot dy$$

→ by using this maclauren's series expansion

$$M(t) = E(e^{ty}) = \int_{-\infty}^{\infty} \left[1 + ty + \frac{t^2 y^2}{2!} + \dots + \frac{t^k y^k}{k!} \right] f(y) \cdot dy$$

$$= \underbrace{\int_{-\infty}^{\infty} f(y) \cdot dy}_1 + t \underbrace{\int_{-\infty}^{\infty} y f(y) \cdot dy}_{\mu} + \frac{t^2}{2!} \underbrace{\int_{-\infty}^{\infty} y^2 f(y) \cdot dy}_{\mu_2'} + \dots + \frac{t^k}{k!} \underbrace{\int_{-\infty}^{\infty} y^k f(y) \cdot dy}_{\mu_k'}$$

$$= 1 + \mu_1 + \mu_2 \frac{t^2}{2!} + \dots + \mu_k' \frac{t^k}{k!}$$

$$\frac{d^k M(t)}{dt^k} = \mu_k'$$

Let Y be a cont. r.v. with density func $f(y)$ & $g(y)$ be a func of Y

Then mgf for $g(Y)$ is

$$E[e^{tg(Y)}] = \int_{-\infty}^{\infty} e^{tg(y)} f(y) \cdot dy$$

Ex: A r.v. Y has pdf

$$f(y) = \begin{cases} 2e^{-2y} & y \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

Find mgf of Y

$$\begin{aligned} M(t) &= E(e^{ty}) = \int_{-\infty}^{\infty} e^{ty} f(y) \cdot dy \\ &= \int_0^{\infty} e^{ty} (2e^{-2y}) \cdot dy = 2 \int_0^{\infty} e^{(t-2)y} \cdot dy \\ &= \frac{2e^{t-2}}{t-2} \Big|_0^{\infty} = \frac{2}{2-t}, \text{ assuming } t < 2 \end{aligned}$$

Tcheby Sheff's Theorem

Let Y be a r.v. with finite mean μ & variance σ^2 for any $k > 0$
 $P(|Y - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}$ or $P(|Y - \mu| \geq k\sigma) \leq \frac{1}{k^2}$

Ex: If we let $k\sigma = c$ in Tcheby Theorem, what does this theorem assert about prob. that a r.v. will take on a value between $\mu - c$ & $\mu + c$?

$$P(|Y - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}, \quad k\sigma = c, \quad k = \frac{c}{\sigma}$$

$$P(|Y - \mu| < c) \geq 1 - \frac{\sigma^2}{c^2} \quad \text{Prob is at least } \frac{1 - \sigma^2}{c^2}$$

Exponentions of Discontinuous Funcs. & Mixed Prob. Distributions

Let Y have a mixed distr. funct

$$F(y) = c_1 F_1(y) + c_2 F_2(y)$$

suppose that x_1 is a discrete r.v. with distr. funct

$F_1(y)$, x_2 is a cont. rv with distr. func $F_2(y)$

Let $g(y)$ denote a func of Y

$$E[g(Y)] = c_1 E[g(x_1)] + c_2 E[g(x_2)]$$

MT1

6 questions, 2 hours

1- basic prob relations (show that)

2- discrete r.v.

3- basic prob. relations (with calculations) ex: find $A \cap B \cap C$

4- prob. with narrative

5- (Continuous r.v. (ask something from (cc) (exponential distr) ^{central tendency}

6- Normal Distr. $\rightarrow$ reading normal distr, asking 2 values or values of $P(z)$

mqf yok $\rightarrow$ finalde olucak

17.40'da