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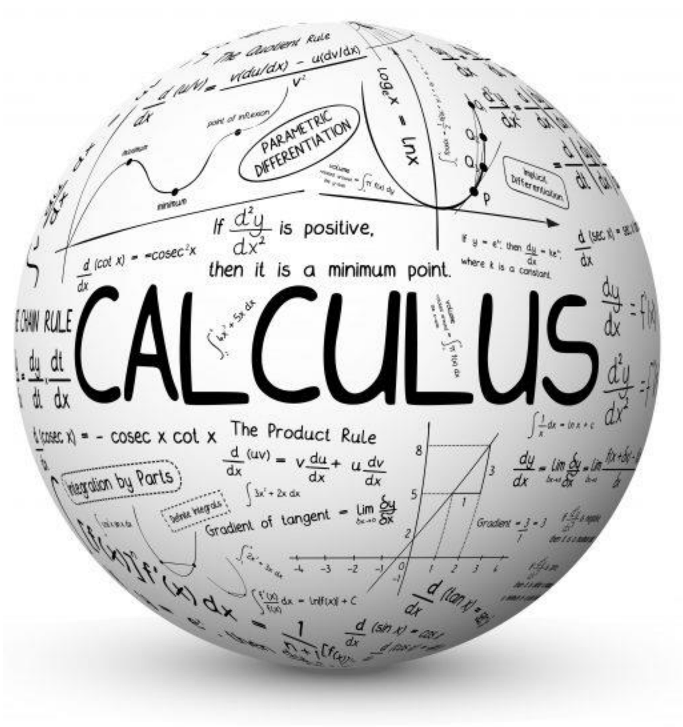
Calculus with Analytic Geometry

[MATH 119 BOOKLET]



Notebook

Road to success

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INTRODUCTION

This booklet is for those who are taking Math119 (Calculus with Analytic Geometry).

What does the booklet offer?

1. Sufficient way to **read** the calculus subjects.
2. Organized way to **understand** calculus theorems and some proves as well as some necessary rules.
3. Sample examples to **understand** how to solve the related problems
4. Some suggested problems to **practice**
5. Appendix which contains functions and their graphs Geometric formulas
6. Short answer key of the suggested problems at the end of the booklet as well as full solution in the answer key booklet.

$$Read \dashrightarrow Understand \dashrightarrow Practice \dashrightarrow AA$$

What are the external booklets offered?

1. Notebook Booklet (This booklet)
2. Answer key Booklet
3. Sample midterms and Final exam Booklet
4. Brief Booklet

Note: Math 119 booklets can be found online via *https://mathbooklet.weebly.com*

To have chance of having mock/sample exams as well as other online extra recourses and support, you may join *https://mathbooklet.weebly.com*

If you have any inquiries, please do not hesitate to contact me via *bookletmath@gmail.com*

I hope such a work helps you to reach AA letter grade.

Best Regards,

Risha

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Strategy to Calculate Limits

To compute $\lim_{x \to \boldsymbol{a}} f(x) = L \quad or \quad \lim_{x \to \boldsymbol{a}} g(x) = M$

I. You need to know the limit rules:

- Limit of a sum or difference: $\lim_{x \to \boldsymbol{a}} [f(x) \pm g(x)] = L \pm M$
- Limit of a product: $\lim_{x \to \boldsymbol{a}} [f(x)g(x)] = LM$
- Limit of a quotient: $\lim_{x \to \boldsymbol{a}} \frac{f(x)}{g(x)} = \frac{L}{M} \qquad where \;\; M \neq 0$
- Limit of a power: $\lim_{x \to \boldsymbol{a}} [\, f(x)]^n = L^n \;\; , n \in \mathbb{Z}\,(integer)$

II. How to solve the limit:

Step 1: Try to plug the value of $\boldsymbol{a}$ directly into the function.

- If we get a number ($L \; and \; M \in \mathbb{R}$), then we are done!

$$Example: \lim_{x \to 1} \frac{x^2 - 2x - 3}{x + 1} = \frac{-4}{2} = -2$$

- However, the limit/value is undefined, having the form

$\left(\left[\frac{0}{0}\right] , \left[\frac{\infty}{\infty}\right], or \; [0 * \infty] \right)$ $(\boldsymbol{read\ hint\ 1})$ $\xrightarrow{then}$ Follow step 2

Step 2: If the limit/its value is undefined, then we need to simplify the expression. Simplification can involve any number of techniques including but certainly not limited to the 7 given techniques:

Hint 1:
(Very Important!)
l'Hˆopital's theorem (section 7) should not be used at this part since it involves taking derivatives. In the exam, if you are NOT asked to use this theorem, then Instructors will NOT give any credit for any answer that includes such a method.

Hint 2:

$$-1 \leq \sin(x) \leq 1$$
$$-1 \leq \cos\,(x) \leq 1$$
$$0 \leq |\sin(x)| \leq 1$$
$$0 \leq |\cos\,(x)| \leq 1$$
$$0 \leq \sin^2(x) \leq 1$$
$$0 \leq \cos^2\,(x) \leq 1$$

Hint 3:

$$\lim_{x \to \infty} \frac{\sin x}{x} = \lim_{x \to \infty} \frac{\cos x}{x} = 0$$
$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

Hint 4:

$$\lim_{x \to \infty} \frac{1}{x} = 0 \qquad \lim_{x \to 0^+} \frac{1}{x} = +\infty$$
$$\lim_{x \to 0^-} \frac{1}{x} = -\infty \qquad \lim_{x \to 0} \frac{1}{x} = D.N.E$$

Hint 5:

$e \approx 2.7 \qquad \ln 1 = 0 \qquad \ln e = 1$

$\ln 0 = -\infty \qquad \ln \infty = \infty$

$e^{\infty} = \infty \qquad e^{-\infty} = 0 \qquad Check\ Note\ VI$

1. **Factoring *(chapter 1.2)***

$$\boldsymbol{ex}: \lim_{x \to 1} \frac{x^2+2x-3}{x-1} = \lim_{x \to 1} \frac{(x+3)\cancel{(x-1)}}{\cancel{x-1}} = \lim_{x \to 1} x + 3 = 4 \qquad factoring\ the\ limit\ to\ get\ rid\ of\ (x-1)$$

2. **Use Two-sided Limits** (Use of the absolute functions in limits) ***(chapter 1.2)***

Theorem: A function $f(x)$ has limit L at $x = a$ if and only if it has both the right and left limits there and these one-sided limits are both equal to L:

$$\lim_{x \to \boldsymbol{a}} f(x) = L \iff \lim_{x \to \boldsymbol{a}^-} f(x) = \lim_{x \to \boldsymbol{a}^+} f(x) = L$$

$$\boldsymbol{ex}: \lim_{x \to -2} \frac{x^2-4}{x\,|x+2|} = \begin{cases} \lim_{x \to -2^+} \dfrac{x^2-4}{x(x+2)} = \lim_{x \to -\boldsymbol{2}^+} \dfrac{(x-2)\cancel{(x+2)}}{x\cancel{(x+2)}} = \lim_{x \to -2^+} \dfrac{(x-2)}{x} = \dfrac{-4}{-2} = 2 & \to (1) \\ \lim_{x \to -2^-} \dfrac{x^2-4}{-x\,(x+2)} = \lim_{x \to -\boldsymbol{2}^-} \dfrac{(x-2)\cancel{(x+2)}}{-x\cancel{(x+2)}} = \lim_{x \to -2^-} \dfrac{(x-2)}{-x} = \dfrac{-4}{2} = -2 & \to (2) \end{cases}$$

$From\ (1)\ and\ (2),\ we\ can\ conculde\ that\ \lim_{x \to -2} \frac{x^2-4}{x\,|x+2|}$ $\boldsymbol{Does\ Not\ Exist}\ (\boldsymbol{D.N.E})$ $Since\ both\ right\ and\ left\ side\ are\ NOT\ equal.$

Check general note III (3rd one) to learn more about the common rules of the absolute functions in limit

3. Using Squeeze Theorem *(chapter 1.2)*

<u>**Theorem**</u>: suppose that $g(x) \le f(x) \le h(x)$ holds for all x in some open interval containing a, except possibly at $x = a$ itself. Suppose also that $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$ Then, $\lim_{x\to a} f(x) = L$

$Aslo, similar\ statements\ hold\ for\ left\ and\ right\ limits.$ **(*read hint 2)***

$\boldsymbol{ex}$: $\lim_{x\to\infty} \frac{2\sin(x^3)}{x^3+1}$ $\quad since\ -1 \le \sin(x^3) \le 1 \quad \therefore \quad \lim_{x\to\infty} \frac{-2}{x^3+1} \le \lim_{x\to\infty} \frac{2\sin(x^3)}{x^3+1} \le \lim_{x\to\infty} \frac{2}{x^3+1}$

$$since \quad \lim_{x\to\infty} \frac{2}{x^3+1} = \lim_{x\to\infty} \frac{-2}{x^3+1} = 0 \qquad \therefore \lim_{x\to\infty} \frac{2\sin(x^3)}{x^3+1} = 0 \quad By\ Squeeze\ Theorem$$

4. Use of the limit rules Finding a common denominator (chapter 1.3)

$$\boldsymbol{ex}: \lim_{x\to\infty} \frac{2x + \sqrt[4]{81x^6 + x^5 - 3x + 7}}{5x + 8 + \sqrt{x^3 + x - 2}} = \lim_{x\to\infty} \frac{\frac{2x}{x^{\frac{3}{2}}} + \sqrt[4]{\frac{81x^6}{x^6} + \frac{x^5}{x^6} - \frac{3x}{x^6} + \frac{7}{x^6}}}{\frac{5x}{x^{\frac{3}{2}}} + \frac{8}{x^{\frac{3}{2}}} + \sqrt{\frac{x^3}{x^3} + \frac{x}{x^3} - \frac{2}{x^3}}}$$

$$\lim_{x\to\infty} \frac{\left(2\left(\frac{1}{x^{\frac{1}{2}}}\right) + \sqrt[4]{81 + \frac{1}{x} - 3\left(\frac{1}{x^5}\right) + 7\left(\frac{1}{x^6}\right)}\right)}{\left(5\left(\frac{1}{x^{\frac{1}{2}}}\right) + 8\left(\frac{1}{x^{\frac{3}{2}}}\right) + \sqrt{1 + \frac{1}{x^2} - 2\left(\frac{1}{x^3}\right)}\right)} = \frac{\left(0 + \sqrt[4]{81 + 0 - 3(0) + 7(0)}\right)}{\left(0 + 8(0) + \sqrt{1 + 0 - 2(0)}\right)} = \frac{\sqrt[4]{81}}{1} = 3$$

$\left(where \lim_{x\to\infty} \frac{1}{x} = 0\right)$

- Check the highest power in the denominator then factor it out from both denominator and nominator $\left(x^{\frac{3}{2}}\right)$
- The general formula of $\lim_{x\to\infty} \frac{1}{x}$ $is \lim_{x\to\infty} \frac{1}{x^p} = 0 \quad where \quad 0 < p < \infty$

5. Multiplying by the conjugate (chapter 1.3)

$$\boldsymbol{ex}: \lim_{x\to\infty} \left(x - \sqrt{x^2 + x + 2}\right)\left(\frac{x + \sqrt{x^2 + x + 2}}{x + \sqrt{x^2 + x + 2}}\right) = \lim_{x\to\infty} \frac{x^2 - (x^2 + x + 2)}{x + \sqrt{x^2 + x + 2}}$$

$$= \lim_{x\to\infty} \frac{-x-2}{x+\sqrt{x^2+x+2}} = \lim_{x\to\infty} \frac{-\frac{x}{x}-\frac{2}{x}}{\frac{x}{x}+\sqrt{\frac{x^2}{x^2}+\frac{x}{x^2}+\frac{2}{x^2}}} = \lim_{x\to\infty} \frac{-1-2\left(\frac{1}{x}\right)}{1+\sqrt{1+\frac{1}{x}+2\left(\frac{1}{x^2}\right)}} = \frac{-1}{2} \qquad Since \lim_{x\to\infty} \frac{1}{x} = 0$$

6. Applying some memorized limit (chapter 2.5) (*read hint 3)*

$$\boldsymbol{ex}: \lim_{x\to 0} \frac{\tan 5x}{\sin 3x} = \left(\lim_{x\to 0} \frac{\sin 5x}{\cos 5x}\right)\left(\lim_{x\to 0} \frac{1}{\sin 3x}\right) = \left(\lim_{x\to 0} \frac{\sin 5x}{5x}\right)\left(\lim_{x\to 0} \frac{1}{\cos 5x}\right)\left(\lim_{x\to 0} \frac{3x}{\sin 3x}\right)\left(\frac{5}{3}\right)$$

$$= (1)(1)(1)\left(\frac{5}{3}\right) = \frac{5}{3} \qquad \Longrightarrow Since \lim_{x\to 0} \frac{\sin x}{x} = 1$$

7. Using the definition of the Derivative (chapter 2.7)

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

$$\boldsymbol{ex}: \lim_{x\to 0} \frac{\cos(x) - 1}{x} = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = f'(0) \quad where\ f(x) = \cos(x) \qquad \therefore f'(x) = -sin\ x \qquad \therefore f'(0) = 0$$

General Notes about limits:

I. $\sqrt{x^2} = |x| \rightarrow \begin{cases} x & x \rightarrow \infty \\ -x & x \rightarrow -\infty \end{cases}$ $\qquad \sqrt{x^4} = |x^2| \rightarrow \begin{cases} x^2 & x \rightarrow \infty \\ -x^2 & x \rightarrow -\infty \end{cases}$

II. Squeeze theorem is applicable for $\lim_{x \to \infty} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} = 0$, Not for $\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to \infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} = 1$

III. For absolute questions, you need to know when you are going to use positive or negative sign. (example)

$\lim_{x \to 2} f(x)$	Hints	$\lim_{x \to 2^+} f(x)$ $x > 2$	$\lim_{x \to 2^-} f(x)$ $x < 2$
$\lvert x-2 \rvert$	$\begin{cases} x-2 & x > 2 \\ -(x-2) & x < 2 \end{cases} \Rightarrow x = 2$	$x-2$	$-(x-2)$
$\lvert x-4 \rvert$	$\begin{cases} x-4 & x > 4 \\ -(x-4) & x < 4 \end{cases} \Rightarrow x = 4$	$-(x-4)$	$-(x-4)$
$\lvert x+2 \rvert$	$\begin{cases} x+2 & x > -2 \\ -(x+2) & x < -2 \end{cases} \Rightarrow x = -2$	$x+2$	$x+2$

IV. Look how to take the factor out from the equation: (this is just one way of factorizing)

$$5x + 8 - \sqrt{x^3 + x - 2} = x^{\frac{3}{2}}\left(5x^{-\frac{1}{2}} + 8x^{-\frac{3}{2}} - \sqrt{1 + x^{-2} - 2x^{-3}}\right)$$

Before factoring	$5x$	8	$\sqrt{x^3 + \cdots - \cdots}$	$\sqrt{\ldots + x - \cdots}$	$\sqrt{\ldots + \cdots - 2}$
After factoring	$5x^{-\frac{1}{2}}$	$8x^{-\frac{3}{2}}$	$\sqrt{1 + \cdots - \cdots}$	$\sqrt{\ldots + x^{-2} - \cdots}$	$\sqrt{\ldots + \cdots - 2x^{-3}}$
Power of x after factoring	$1 - \frac{3}{2} = -\frac{1}{2}$	$0 - \frac{3}{2} = -\frac{3}{2}$	$3 - 3 = 0$	$1 - 3 = -2$	$0 - 3 = -3$

V. Polynomial Function Degree

$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \infty$	$f(x) > g(x)$	ex: $\lim_{x \to \infty} \frac{7x^5 + x^3 + 1}{5x^4 - 1} = \infty$	Large Degree Polynomial Function
$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L$ $0 < L < \infty$	$f(x) = g(x)$	ex: $\lim_{x \to \infty} \frac{7x^5 + x^3 + 1}{5x^5 - 1} = \frac{7}{5}$	Equal Degree Polynomial Functions
$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$	$f(x) < g(x)$	ex: $\lim_{x \to \infty} \frac{7x^4 + x^3 + 1}{5x^5 - 1} = 0$	Small Degree Polynomial Function

VI. $a^{\infty} = \begin{cases} \infty & a > 1 \\ 0 & a < 1 \end{cases} \quad \Rightarrow \quad ex: 5^{\infty} = \infty, \quad \left(\frac{1}{5}\right)^{\infty} = 0$

$a^{-\infty} = \begin{cases} 0 & a > 1 \\ \infty & a < 1 \end{cases} \quad \Rightarrow \quad ex: 5^{-\infty} = 0, \quad \left(\frac{1}{5}\right)^{-\infty} = \infty$

Section 1: Problems session

$\boldsymbol{Evaluate}\ the\ following\ limits.\ If\ the\ limit\ does\ not\ exist, is\ it\ \infty, -\infty\ or\ neither?\ (Do\ not\ use\ L'Hôpital's\ Rule)$

Part 1 (Technique 1)

1. $\lim_{x\to 1} \frac{x^2-2x+1}{x-1}$

2. $\lim_{x\to 1} \frac{\tan x - x}{x^3}$

3. $\lim_{x\to \frac{\pi}{2}^-} \frac{e^x+3x+1}{\sin x - 1}$

4. $\lim_{x\to 3} \frac{(x^2+x-12)\sqrt{x+3}}{\sqrt{5}(x^2-9)}$

5. $\lim_{x\to 0} \frac{1-\cos x}{\sin^2 x}$

6. $\lim_{x\to 0} x\sin x$

7. $\lim_{x\to \infty} \frac{e^x}{\sinh x}$

$\leftarrow Review\ Section\ 3$

8. $\lim_{x\to 0} \frac{\arcsin x + 3\ln(x+1)}{x+\sin x-\cos x}$

← *Review Section* 3

Part 2 (Technique 2)

9. $\lim_{x\to 2} \frac{x^2-4}{3x|x-2|}$

10. $\lim_{x\to 1} \frac{2x-2}{|x^3-x^2|}$

11. $\lim_{x\to -2} \frac{|x+2|}{|x|-2}$

12. $\lim_{x\to 1} \frac{|x^2-5|-|x+3|}{x-1}$

13. $\lim_{x \to 0} \frac{|2x-3|-|x-3|}{x}$

14. $\lim_{x \to 0} \frac{|\sin(x)|}{x}$

15. $\lim_{x \to 3} \frac{|x^2-9|}{\sqrt[3]{x-3}}$

16. $\lim_{x \to \frac{\pi}{2}} \frac{\arctan(\tan(x))}{\tan(\arctan(x))}$

$\leftarrow$ *Review Section* 3

Part 3 (Technique 3)

17. $\lim_{x \to \infty} \frac{2\sin(x^5)}{x^4+1}$

18. $\lim_{x \to \infty} \frac{cos\ (x \sin x)}{x^2}$

19. $\lim_{x \to \infty} \frac{sin\ (x^2+1)}{x^2}$

20. $\lim_{x \to \infty} \frac{sin\ (x^{119})}{\ln(x)}$

21. $\lim\limits_{x\to\infty} \frac{3x + |sin\,(2x)|}{x^2}$

22. $\lim\limits_{x\to\infty} \frac{x^2+\sin x}{x^2+\sin^2 x}$

23. $\lim\limits_{x\to\infty} \frac{\cos\left(\sqrt{x^2+1}\right)}{e^x+119}$

24. $\lim\limits_{x\to\infty} \frac{\ln\left(1+\sin^2 x\right)}{\arctan x+e^x}$

$\leftarrow$ *Review Section* 3

25. $\lim\limits_{x\to\infty} \frac{4x^2-\cos(2x)}{x^2+1}$

26. $\lim_{x \to 0} x^2 \, e^{\sin\left(\frac{\pi}{x}\right)}$

27. $\lim_{x \to 0^+} e^{\sin\left(\frac{1}{x}\right)} \ln(1+x)$

<u>Part 4</u> (Technique 4)

28. $\lim_{x \to \infty} \frac{x+\sqrt[4]{16x^6+x^5-3x+7}}{5x+8-\sqrt{x^3+x-2}}$

29. $\lim_{x \to \infty} \frac{x^3+\sqrt{x^6+2x^5+1}}{x^3+2x^2+x+1}$

30. $\lim_{x \to \infty} \frac{\sqrt{x^4+x+1}}{\sqrt[3]{x^6+1}}$

31. $\lim_{x\to\infty} \frac{\sqrt{x^2+e}}{x^2+7}$

32. $\lim_{x\to\infty} [\ln(x^2+1) - \ln(x^2-1)]$ $\leftarrow$ *Review Section* 3

Part 5 (Techniques 4 and 5)

33. $\lim_{x\to\infty} \left(\sqrt{x+\sqrt{x}} - \sqrt{x}\right)$

34. $\lim_{x\to-\infty} \left(x - \sqrt{x^2+4x}\right)$

35. $\lim_{x\to-\infty} \left(x + \sqrt{x^2-x-4}\right)$

36. $\lim_{x\to\infty} x - \sqrt{x^2+x+2}$

37. $\lim_{x\to\infty} x - \sqrt{x^2-4x+1}$

38. $\lim_{x\to\infty} \sqrt{x^2+4x} - \sqrt{x^2-4x}$

39. $\lim_{x\to-\infty} \sqrt{2x^2-x+1} - \sqrt{x^2-x+5}$

40. $\lim_{x\to\infty} \sqrt{x^2+x} - \sqrt{x}$

41. $\lim_{x\to-\infty} \sqrt{x^{10/3} - 5x^2} - \sqrt{x^{10/3} + 5x^2}$

42. $\lim_{x\to-\infty} \ln\left(\sqrt{x^2 + 1} + x\right)$

43. $\lim_{x\to 0} \frac{1-\cos x}{x^2}$

44. $\lim_{x\to 0} \frac{1-\cos(x^2)}{x^2 \sin^2(x)}$

45. $\lim_{x\to 0} \frac{\sqrt{1+\tan x}-\sqrt{1+\sin x}}{x^3}$

Part 6 (Technique 6)

46. $\lim\limits_{x\to\infty} \frac{2-x+\sin(x)}{x+\cos(x)}$

47. $\lim\limits_{x\to\infty} \frac{2x^2+\sin^2 x}{x^2}$

48. $\lim\limits_{x\to 0} \frac{\sin 5x-\sin 3x}{\sin x}$

49. $\lim\limits_{x\to 0} \frac{x^3 \sin\left(\frac{1}{x}\right)}{\sin(x)}$

50. $\lim\limits_{x\to 0^+} \frac{(\tan\sqrt{x})(\sqrt{2x+1})}{x}$

Part 7 (Technique 7)

$\leftarrow$ *Review Section 3 before solving part 7*

51. $\lim\limits_{x\to\frac{1}{2}} \frac{\arccos(x)-\frac{\pi}{3}}{x-\frac{1}{2}}$

52. $\lim_{x \to 1} \frac{\arcsin(x^3-1)}{x-1}$

53. $\lim_{x \to 0} \frac{\cos x - e^x}{x}$

54. $\lim_{x \to 0} \frac{3^{(2+x)^2}-81}{x}$

55. $\lim_{x \to 0} \frac{2^x - 1}{\sin 4x}$

56. $\lim_{x \to 0} \frac{\cos[(x+2)^2] - \cos 4}{x}$

57. $\lim_{x \to 1} \frac{e^{\sqrt{x}}-e}{x^3-1}$

Continuity:

Rule: $If\ f(x)\ is\ continous\ at\ x=a \ \overset{Then,}{\rightleftarrows}\ \lim_{x\to a} f(x) = f(a) \quad OR \quad \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a)$

Examples:

1- **Example**: $f(x) = \begin{cases} -x, & x<0 \\ x^2, & x\geq 0 \end{cases}$ $\quad show\ that\ the\ function\ f\ is\ continous\ at\ x=0$

Answer: $\because \lim_{x\to 0^+} x^2 = f(0) = 0 \quad and \quad \lim_{x\to 0^-} -x = 0 \quad \therefore\ the\ function\ f\ is\ continous\ at\ x=0$

2- **Example**: $f(x) = \begin{cases} -x, & x\neq 0 \\ x^2, & x = 0 \end{cases}$ $\quad show\ that\ the\ function\ f\ is\ continous\ at\ x=0$

Answer: $\because \lim_{x\to 0} x^2 = 0 \quad and \quad f(0) = 0 \quad \therefore\ the\ function\ f\ is\ continous\ at\ x=0$

Differentiation: (Rules)

(Formal) Definition of the Derivative	**Definition of the Derivative**
$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$	$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$
$If\ f(x)\ is\ differentiable\ at\ x=a\ \overset{Then,}{\rightleftarrows}$ $f'(x) = \lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h}$ $OR \quad f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$	$If\ f(x)\ is\ differentiable\ at\ x=a\ \overset{Then,}{\rightleftarrows}$ $f'(a) = \lim_{x\to a^+} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a^+} \frac{f(x)-f(a)}{x-a}$ $OR \quad f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$

Important: $If\ f(x)\ is\ differentiable\ at\ x=a\ \ Then,\ \ f(x)\ is\ continous\ at\ x=a$ (NOT the inverse)

3- **Example**: $f(x) = \begin{cases} -x, & x<0 \\ x^2, & x\geq 0 \end{cases}$ $\quad Find\ f'(0)\ if\ it\ exists$

Answer using **(Formal) Definition of the Derivative**	Answer using **Definition of the Derivative**
$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ $f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h}$ $\lim_{h\to 0^+} \frac{f(h)}{h} = \lim_{h\to 0^+} \frac{h^2}{h} = h = 0$, $\lim_{h\to 0^-} \frac{f(h)}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1$ $\because \lim_{h\to 0^+} \frac{f(h)}{h} \neq \lim_{h\to 0^-} \frac{f(h)}{h}$ $\therefore\ the\ function\ f\ is\ NOT\ differentiable\ at\ x=0$	$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ $f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{f(x)}{x}$ $\lim_{x\to 0^+} \frac{f(x)}{x} = \lim_{x\to 0^+} \frac{x^2}{x} = x = 0$, $\lim_{x\to 0^-} \frac{f(x)}{x} = \lim_{x\to 0^-} \frac{-x}{x} = -1$ $\because \lim_{x\to 0^+} \frac{f(x)}{x} \neq \lim_{x\to 0^-} \frac{f(x)}{x}$ $\therefore\ the\ function\ f\ is\ NOT\ differentiable\ at\ x=0$

4- **Example**: $f(x) = \begin{cases} x^2, & x\neq 0 \\ 0, & x = 0 \end{cases}$ $\quad Find\ f'(0)\ if\ it\ exists$

Answer using **(Formal) Definition of the Derivative**	Answer using **Definition of the Derivative**
$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ $f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h}$ $\lim_{h\to 0} \frac{f(h)}{h} = \lim_{h\to 0} \frac{h^2}{h} = h = 0$ $\therefore\ the\ function\ f\ is\ differentiable\ at\ x=0$	$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ $f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{f(x)}{x}$ $\lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} \frac{x^2}{x} = x = 0$, $\therefore\ the\ function\ f\ is\ differentiable\ at\ x=0$

5- Example: $f(x) = \begin{cases} -x, & x < 0 \\ x^2, & x \geq 0 \end{cases}$ $\quad$ *Find* $f'(x)$ *for* $x \neq 0$

Answer: $f'(x) = \begin{cases} -1, & x < 0 \\ 2x, & x > 0 \end{cases}$ *To solve this question, you need to be aware of Derivatives Rules (section 3)*

Note: if $f(g(x))$ is defined on interval containing a, and if $f(x)$ is continuous at L and $\lim\limits_{x \to a} f(g(x))$ then

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(L)$$

Ex: $\lim\limits_{x \to 0} ln(cos(x)) \overset{*}{=} \ln\left(\lim\limits_{x \to 0} \cos(x)\right) = \ln(1) = 0$ $\quad$ $*$ *since* $cos(x)$ *is continous at* $x = 0$

Section 2: Problems session

Question1: $f(x) = \begin{cases} \frac{\sin(2x^2)+x^3}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

a) Show that the function f is continuous at $x = 0$

b) Find $f'(0)$ if it exists.

Question2: let $f: R \to R$ be a function defined by

$$f(x) = \begin{cases} mx - b^2, & x < 119 \\ x^2 & if \;\; x \geq 119 \end{cases}$$

Using the definition of derivative, find all values of $\boldsymbol{m}$ and ***b*** that make f differentiable at 119.

Question3: let $f: R \to R$ be a function defined by

$$f(x) = \begin{cases} x + x^2 & if \;\; x < 1 \\ x^3 + 1 & if \;\; x \geq 1 \end{cases}$$

A. Find the derivative $f'(x)$ for $x \neq 1$.

B. Find $f'(1)$ if it exists.

C. Is the **derivative** function $y = f'(x)$ continuous at $x = 1$? Explain

Question4: let $f: R \to R$ be a function defined by

Consider $f(x) = \begin{cases} \frac{1-\cos x}{x} & if \quad -\frac{\pi}{2} \le x < 0 \\ ax + b & if \quad 0 \le x \le 1 \end{cases}$

Determine the values of a, b such that $f'(x)$ exists for all x with $-\frac{\pi}{2} < x < 1$

Question5: $find\ f'(0)\ if\ f(x) = \begin{cases} \frac{x^3 \sin\left(\frac{1}{x}\right)}{\sin x} & if \quad x \neq 0 \\ 0 & if \quad x = 0 \end{cases}$

Show your work.

Question6: Consider $f(x)$ is a continuous function for all x. Find $\boldsymbol{a}$, where

$$f(x) = \begin{cases} 2x + (x-1)\sin\dfrac{1}{x-1} & if \quad x < 1 \\ ax + 2 & if \quad x \ge 1 \end{cases}$$

Question7: Find the number $a \neq 0$ such that the function

$$f(x) = \begin{cases} \dfrac{\sin(ax)}{x} + \cos x & for \quad x < 0 \\ x^2 + 3e^{2x} & for \quad x \geq 0 \end{cases}$$

Is continuous **everywhere**

Question8:

Let $f(x) = \begin{cases} x^2 + \arcsin x & for \quad x \geq 0 \\ x + e^{x^2} & for \quad x < 0 \end{cases}$

$\leftarrow Review\ Section\ 3$

Find $f'(0)$ if it exists, or explain why it does NOT exist

Question9:

Let $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & for \quad x < 0 \\ x^2 & for \quad x \geq 0 \end{cases}$

a) Show that the function $f(x)$ is continuous at $x = 0$

b) Use the definition of derivative to determine whether $f`(0)$ exists.

Question10:

$$let\ f(x) = \begin{cases} e^x(x^2+a) & if\ x<0 \\ 2 & if\ x=0 \\ bx^2+2 & if\ x>0 \end{cases}$$

$If\ possible, determine\ a, b\ so\ that\ f`(0)\ exists.\ \ If\ not\ possible\ explain\ why?$

Section 3: Derivatives

Basic Properties/Formulas

$I.\ \dfrac{d}{dx}(cf(x)) = c\,f'(x),\ \ c\ is\ any\ contant.$ $\quad II.\ \big(f(x) \pm g(x)\big)' = f'(x) \pm g'(x)$

$III.\ \ \dfrac{d}{dx}f(x) = f'(x) = y' = \dfrac{dy}{dx}$

Common Derivatives

Polynomials

1) $\dfrac{d}{dx}(c) = 0$ 2) $\dfrac{d}{dx}(x) = 1$ 3) $\dfrac{d}{dx}(cx) = c$ 4) $\dfrac{d}{dx}(x^n) = nx^{n-1}$ 5) $\dfrac{d}{dx}(cx^n) = ncx^{n-1}$

Trig Functions

1) $\dfrac{d}{dx}(\sin x) = \cos x$ 2) $\dfrac{d}{dx}(\cos x) = -\sin x$ 3) $\dfrac{d}{dx}(\tan x) = \sec^2 x$ 4) $\dfrac{d}{dx}(\sec x) = \sec x \tan x$

Inverse Trig Functions

1) $\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}$ 2) $\dfrac{d}{dx}(\cos^{-1} x) = -\dfrac{1}{\sqrt{1-x^2}}$ 3) $\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1+x^2}$

4) $\dfrac{d}{dx}(arcsin\,x) = \dfrac{1}{\sqrt{1-x^2}}$ 5) $\dfrac{d}{dx}(\arccos x) = -\dfrac{1}{\sqrt{1-x^2}}$ 6) $\dfrac{d}{dx}(\arctan x) = \dfrac{1}{1+x^2}$

Exponential/Logarithm Functions [*where* ***a*** *is constant*]

1) $\frac{d}{dx}(a^x) = a^x \ln(a)$ 2) $\frac{d}{dx}(e^x) = e^x$ [*hint* $\boldsymbol{e} \approx 2.7$] 3) $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$, $x > 0$

4) $\frac{d}{dx}(\log_a(x)) = \frac{1}{x \ln a}$, $x > 0$ 5) $\frac{d}{dx}\left(a^{g(x)}\right) = a^{g(x)}\,(g'(x)\ln a)$ 6) $\frac{d}{dx}(\ln g(x)) = \frac{g'(x)}{g(x)}$

***Some Remarks* about Exponential/Logarithm Functions**

1) $\ln(xy) = \ln x + \ln y$ 2) $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$ 3) $\ln\left(\frac{1}{x}\right) = -\ln x$ 4) $\ln(x^r) = r \ln x$

5) $\log_a(x) = \frac{\ln x}{\ln a}$ 6) $f(x) = e^{\ln f(x)}$ 7) $f(x)^{g(x)} = e^{\ln f(x)^{g(x)}} = e^{g(x)\ln f(x)}$

Some facts about Exponential/Logarithm Functions: $\ln(e) = 1$; $\ln(1) = 0$; $\ln(0) = -\infty$; $\ln(\infty) = \infty$

Hyperbolic Trig Functions: $\sinh x = \frac{e^x - e^{-x}}{2}$ $\cosh x = \frac{e^x + e^{-x}}{2}$ $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

The rules of Hyperbolic Trig Functions are the same as the Trig Functions in terms of inverse and derivatives

ex: 1) $\frac{d}{dx}(\sinh x) = \cosh x$ 2) $\frac{d}{dx}(\cosh x) = \sinh x$ $(except)$

Inverse Trig Functions:

Function	Domain	Function	Domain
$\sin x$	$\mathbb{R}$	$\arcsin x = \sin^{-1} x$	$-1 \le x \le 1$
$\cos x$	$\mathbb{R}$	$\arccos x = \cos^{-1} x$	$-1 \le x \le 1$
$\tan x$	$\mathbb{R} - \left\{\frac{\pi}{2} + \pi k\right\}, k \in \mathbb{Z}$	$\arctan x = \tan^{-1} x$	$\mathbb{R}$
$\sin(\arcsin x) = x$	$-1 \le x \le 1$	$\arcsin(\sin x) = x$	$-\frac{\pi}{2} \le x \le \frac{\pi}{2}$
$\cos(\arccos x) = x$	$-1 \le x \le 1$	$\arccos(\cos x) = x$	$0 \le x \le \pi$
$\tan(\arctan x) = x$	$\mathbb{R}$	$\arctan(\tan x) = x$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$

Some Remarks about inverse Functions:

$f^{-1}(x) = y \;\rightarrow f(y) = x$!! $f^{-1}(x) \neq \frac{1}{f(x)}$!! $\cos^{-1} x \neq \frac{1}{\cos x}$

$To\ find\ f^{-1\,\prime}(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(a)}$ $We\ assume\ f^{-1}(x) = a \quad \therefore f(a) = x$

One to One *Function / Inverse of a Function (**Uniqueness**)*

1. *If the function is* ***increasing*** *or* ***decreasing****, then it is invertible* $\therefore$ *this is called* ***one to one*** *function*
2. *If* $f'(x) \ge 0 \;\therefore f(x)$ *is increasing* *OR* $f'(x) \le 0 \;\therefore f(x)$ *is decreasing*

OR $f(x_1) = f(x_2)$ *and* $x_1 = x_2$ $\therefore f(x)$ *is invertible and it is called* ***one to one*** *function*

Example: $f(x) = x^3 + 7x + 2 \quad Find \quad f^{-1\prime}(10)$

$\because f'(x) = 3x^2 + 7 > 0 \quad \therefore f(x)$ *is increasing for all* $x \quad \therefore f(x)$ *is invertible* $\quad \therefore f(x)$ *is* ***one to one*** *function*

$$f^{-1\prime}(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(a)} \quad where\ f^{-1}(x) = a \quad \therefore f(a) = x \quad \because x = 10 \quad \therefore f(a) = 10$$

$$a^3 + 7a + 2 = 10 \quad \therefore a = 1 \quad thus\ f^{-1\prime}(10) = \frac{1}{f'(f^{-1}(10))} = \frac{1}{f'(1)} = \frac{1}{3(1)^2 + 7} = \frac{1}{10}$$

Derivatives Rules:

(I) Product Rule: $(fg)' = f'g + fg'$

ex: $\frac{d}{dx}(x^2\ sin\ x) = (x^2\ sin\ x)' = 2x\sin x + x^2\cos x$

(II) Quotient Rule: $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

ex: $\frac{d}{dx}\left(\frac{x^2}{\sin x}\right) = \frac{(\sin x)(2x) - (x^2)(\cos x)}{\sin^2 x}$

(III) Chain Rule: $\frac{d}{dx}\Big(f\big(g(x)\big)\Big) = f'\big(g(x)\big)\ g'(x)$

ex: $\frac{d}{dx}(\sin x^2) = (\cos x^2) * (2x) = 2x\cos x^2$

(IV) Implicit Differentiation: $If\ \frac{d}{dx}(y) = y'\ then\ \frac{d}{dx}(2y^3) = (6y^2)\ y'$

ex: $\frac{d}{dx}(y^2\ sin\ x) = 2y\ y'\sin x + y^2\cos x$

(V) General Techniques: $(1)\ \frac{d}{dx}\left(\sqrt{f(x)}\right) = \frac{f'(x)}{2\sqrt{f(x)}}$ $\qquad (2)\ \frac{d}{dx}\left(e^{f(x)}\right) = e^{f(x)}(f'(x))$

$ex1$: $\frac{d}{dx}\left(\sqrt{x^2\sin x}\right) = \frac{2x\sin x + x^2\cos x}{2\sqrt{x^2\sin x}}$ $\qquad ex2$: $\frac{d}{dx}\left(e^{x^2\sin x}\right) = e^{x^2\sin x}(2x\sin x + x^2\cos x)$

(VI) Higher-Order Derivatives:

1st derivative	y'	$f'(x)$	$\frac{dy}{dx}$	$\frac{d}{dx}f(x)$
2nd derivative	y''	$f''(x)$	$\frac{d^2y}{dx^2}$	$\frac{d^2}{dx^2}f(x)$
3rd derivative	y'''	$f'''(x)$	$\frac{d^3y}{dx^3}$	$\frac{d^3}{dx^3}f(x)$
4th derivative	$y^{(4)}$	$f^{(4)}(x)$	$\frac{d^4y}{dx^4}$	$\frac{d^4}{dx^4}f(x)$
nth derivative	$y^{(n)}$	$f^{(n)}(x)$	$\frac{d^ny}{dx^n}$	$\frac{d^n}{dx^n}f(x)$

ex: $f(x) = \frac{x^4}{4} + \frac{x^3}{9} + \frac{x^2}{3} + 3x + 2019$

$$f'(x) = x^3 + \frac{x^2}{3} + \frac{2x}{3} + 3 \quad \curvearrowright \quad f''(x) = 3x^2 + \frac{2x}{3} + \frac{2}{3} \quad \curvearrowright \quad f'''(x) = 6x + \frac{2}{3} \quad \curvearrowright \quad f^{(4)}(x) = 6$$

Thus: $f^{(n)}(x) = 0 \quad for\ n \geq 5$

Sample problem: $y = (x)^{\sin x}$ $\quad$ *Find* y' $\quad$ (Exponential/Logarithm Question)

Answer: $y = (x)^{\sin x} = e^{\ln(x)^{\sin x}} = e^{[\sin x \ \ln(x)]}$ $\quad$ $(Equivalent)$

$Then,\ y' = e^{[\sin x \ \ln(x)]}\left(\frac{d}{dx}[\sin x \ \ln(x)]\right)$

$where\ \frac{d}{dx}[\sin x \ \ln(x)] = \cos x \ \ln(x) \ + \sin x \ \left(\frac{1}{x}\right)$

$\therefore y` = e^{[\sin x \ \ln(x)]}\left[\cos x \ \ln(x) \ + \sin x \ \frac{1}{x}\right] = (x)^{\sin x}\left[\cos x \ \ln(x) \ + \sin x \ \left(\frac{1}{x}\right)\right]$

(***OR***) ***Answer***: $let\ y = (x)^{\sin x} \implies \ln y = \sin x \ \ln(x)$ $\ Then, take\ \frac{d}{dx}\ to\ both\ sides$

$\frac{y`}{y} = \cos x \ \ln(x) \ + \sin x \ \left(\frac{1}{x}\right) = y\left[\cos x \ \ln(x) \ + \sin x \ \left(\frac{1}{x}\right)\right]$

$\therefore y` = (x)^{\sin x}\left[\cos x \ \ln(x) \ + \sin x \ \left(\frac{1}{x}\right)\right]$

Section 3: Problems session

Part I: Evaluate the following derivatives. There is **no** need to simplify your answer.

a) $\frac{d}{dx}[\arcsin(e^x + x^e)]$

b) $\frac{d}{dx}[(\sin x)^{\cos x}]$

c) $\frac{d}{dx}[\tan^2(\sec(x^2))]$

d) $\frac{d}{dx}\left[(1 + x)^{\sin x}\right]$

e) $\frac{d}{dx}[x^3\arcsin(x^2 - 1)]$

f) $\frac{d}{dx}f(x)$ at point $(x, f(x)) = (1,1)$ where f is a differentiable function satisfying the equation

$$e^{xf(x)} + \ln f(x) = e \qquad for\ all\ x$$

Part II: For the given function $f(x)$, find the derivative function $f'(x)$. DO NOT SIMPLIFY:

a) $f(x) = \cos^2 \sqrt{x} + \frac{1}{\ln x}$

b) $f(x) = \frac{\arcsin\left(\frac{1}{1+x^2}\right)}{1+\cos x}$

c) $f(x) = \left(1 + \frac{3}{x} + \frac{5}{x^2}\right)^x$

d) $f(x) = \frac{\ln(1+x^2)}{\sqrt{e^{2x}+5^x}}$

e) $f(x) = (\sin x)^{\cos(x^3)}$

Part III: Find the derivative $\frac{dy}{dx}$ for the following functions.

a) $y = \sqrt{\frac{\sin(x^2)}{\tan x}}$

b) $y = (x + x^2)^{\ln x}$

Part IV: This question has <u>unrelated</u> parts about derivatives.

a) Evaluate $\frac{d}{dx}\left[\sin^2(2x) + \frac{e^x}{\ln x}\right]$

b) Evaluate $\frac{d}{dx}\left[\ln\left(\frac{\sin x}{\sqrt[3]{2x+x^2}}\right)\right]$

c) Find $f'(x) = \frac{dy}{dx} \quad if \quad y = f(x) = \frac{x^{\pi}+\cos x}{1+x^3}$

d) Find $f'(x) = \frac{dy}{dx} \quad if \quad y = f(x) = \sec(\sec x)$

e) Find $f'(x) = \frac{dy}{dx} \quad if \quad y = f(x) = 3^x + \log_x 3$

f) Suppose that $h(x) = f\left(x^2 + g(x)\right), where\ f(3) = 6,\ f'(3) = -4, f(-2) = 0,\ f'(-2) = 1,\ g(3) = 1, g'(3) = -1,\ g(-2) = -1\ and\ g'(-2) = 2. Compute\ h'(-2).$

g) Find $f'(1) = \left.\frac{dy}{dx}\right|_{x=1} \quad if \quad y = f(x) = \frac{x^{9x}(x-2)^3}{x^4 e^x}$

Part V: <u>Question 1</u>

Let $f(x) = \arctan(x) + e^x$

a) Show that f is invertible, f^{-1} is a function.

b) Find the **derivative of the inverse** of f at 1, that is, find $(f^{-1})'(1)$.

<u>Question 2:</u>

Let $f(x) = 2x + \cos(x)$.

a) Show that f is invertible.

b) Find $(f^{-1})'(\pi)$ and the derivative of the inverse of f at π, that is, find $(f^{-1})'(\pi)$.

<u>Question 3:</u>

Show that $f(x) = 1 + \arctan x + e^x$ has an inverse function. Find the derivative of the inverse of $f(x)$ at $x = 2$, that is, find $(f^{-1})'(2)$.

$Slope\ of\ tangent\ line\ is\ \left(m = y' = f'(x) = \frac{dy}{dx}\right)$

Let: $\boldsymbol{Curve}$ $equation\ is\ y = f(x)$, $\boldsymbol{Tangent}$ $equation\ of\ line\ (L_1)\ and,$ $\boldsymbol{Normal}$ $equation\ of\ line\ (L_2)$

$\boldsymbol{Tangent}$ $equation\ of\ line\ (L_1)\ that\ touches\ a\ curve\ (y)\ at\ one\ and\ only\ one\ point.\ Let\ this\ point\ is\ (a,b) \in curve\ \&\ Tangent\ Line$:

$\boldsymbol{Tangent}$ $line\ eqn\ is\ \dfrac{y-b}{(x-a)} = (Slope\ of\ \boldsymbol{tangent}) = m_1 \quad OR \quad y = m_1(x-a)+b$

$\boldsymbol{Normal}$ $equation\ has\ the\ same\ form\ as\ the$ $\boldsymbol{Tangent}$ $equation. The\ only\ difference\ is\ their\ slopes.$

$\boldsymbol{Normal}$ $line\ eqn\ is\ \dfrac{y-b}{(x-a)} = (Slope\ of\ \boldsymbol{normal}) = m_2 \quad OR \quad y = m_2(x-a)+b$

$\boldsymbol{Note}$: $Slope\ of\ tangent = \dfrac{-1}{slope\ of\ normal}$

Example: Find an equation of both the *tangent* and *normal* lines to the curve $y = x^2$ at the point $(-1,1)$

$1 = 1^2 \quad Thus,\ (-1,1) \in curve\ y = x^2 \quad OR \quad (-1,1)\ is\ on\ the\ curve\ y = x^2$

$y' = 2x \quad \therefore y'|_{x=-1} = -2 = m\ (Slope\ of\ \boldsymbol{tangent})$

$\boldsymbol{Tangent}$ $line\ eqn\ is\ \dfrac{y-1}{(x-(-1))} = -2 \ \rightarrow\ y = -2x - 1$

$\boldsymbol{Normal}$ $line\ eqn\ is\ \dfrac{y-1}{(x-(-1))} = \dfrac{1}{2} \ \rightarrow\ y = \dfrac{1}{2}x + \dfrac{3}{2} = \dfrac{1}{2}(x+3)$

General Notes:

$Case$: $when\ the\ point\ (a,b) \notin curve$

Example: Find an equation of the *tangent* line(s) to the curve $y = x^2 + x$ at the point $(1,1)$

$\boldsymbol{Ans}$: $1 \neq 1^2 + 1 \quad Thus,\ (1,1) \notin curve\ y = x^2 + x \quad let\ x = a\ \ then\ the\ point\ is\ (a, f(a))\ where\ x \neq 1$

The **slope** of the **tangent line** with the point $(a, f(a))$ must be $y' = 2x + 1 \quad \therefore y'|_{x=a} = 2a + 1 \ \rightarrow (1)$

Because the line passes through the point $(1,1)$, the slope of the tangent line is **also**:

$\frac{f(a)-1}{a-1} = \frac{a^2+a-1}{a-1} \rightarrow (2) \quad \because (1) = (2) \therefore \frac{a^2+a-1}{a-1} = 2a+1 \ \rightarrow\ a^2+a-1 = 2a^2 - a - 1 \ \rightarrow\ a^2 - 2a = 0$

$\rightarrow a(a-2) = 0 \ \therefore a = 0 \ \ or \ \ a = 2 \quad \therefore y'|_{x=0} = 1\ and\ y'|_{x=2} = 5$

$Tangent\ line\ eqn\ are\ \ y = (x-1)+1 \ \ and\ \ y = 5(x-1)+1$

!! $When\ the$ $\boldsymbol{slope}$ $of\ line\ is$ $\boldsymbol{zero}$ $this\ means\ the\ line\ is$ $\boldsymbol{horizontal}$ $i.e. f`(x) = 0\ where\ y = f(x) = c\ \ where\ c\ is\ constant$

$Let\ the\ slope\ of\ the\ line\ (L_1)\ is\ m_1 \quad \& \quad the\ slope\ of\ the\ line\ (L_2)\ is\ m_2$

$If\ \ m_1 = m_2 \quad then\ \ (L_1) // (L_2) \quad where\ // \ is\ parallel \qquad ex: y = 2x+1 \ \ \&\ \ y = 2x+3$

$If\ \ m_1 * m_2 = -1\ \ then\ \ (L_1) \perp (L_2)\ \ where\ \perp\ is\ perpendicular \qquad ex: y = 2x+1\ \ \&\ \ y = -\dfrac{x}{2} + 3$

$If\ \ m_1 * m_2 = -1\ \ then\ \ (L_1)$ intersects with (L_2)

Section 4: Problems session

Question 1

Find an equation of the tangent line to the curve $119\left(\frac{x}{y}\right) + 1891\left(\frac{y}{x}\right)^3 = 2010 \quad at\ the\ point\ (-1,-1)$.

Question 2

Let f and g be differentiable functions such that $f(9) = -2,\ g(3) = 8,\ f'(9) = 7, g'(3) = 2$.

(a) Show that $y' = -11$ at the point $(3,1)$ if $yg(x) + f(x^2) = y^4 + 5$.

(b) Find an equation of the line tangent to the curve $yg(x) + f(x^2) = y^4 + 5$ at the point $(3,1)$.

Question 3

Consider the curve given by the equation $xy^2 + \sin\big((x-1)y\big) = 4$

a) Find all points $(1, y)$ on this curve.

b) Find the equations of the tangent lines to the curves at the points you found in part (a).

c) Are any of the tangent lines you found in part (b) parallel? Explain!

Question 4

Suppose the f is a function for which $f\left(\frac{\pi}{2}\right) = 2$ and $f'\left(\frac{\pi}{2}\right) = 1.$ Let $g(x) = f(x)^{\cos x}$

Find an equation of the line tangent to the graph of $g(x)$ at $x = \frac{\pi}{2}$

Question 5

Verify that the point $P_0(\pi, 0)$ is on the curve C defined implicitly by the equation $\sin(x + y) = xy$. Then find the two lines which are **normal** and **tangent** to C at P_0.

Question 6

Find the equations of the tangent lines to the curve $y = x^3 + x$ which pass through the point $(2,2)$.

Question 7

Find the equations of any lines that pass through the point $(-1,0)$ and are tangent to the curve $y = \frac{x-1}{x+1}$.

Question8

Let $f(x) = |x^3|$. Find $f'(0)$ by using the definition of derivative and find an equation of the tangent line to the graph of $f(x)$ at the point where $x = 0$.

I. $\underline{\textit{Formal definition of the limit}}$ (chapter 1.5)

$Rule$: $(\boldsymbol{a})$ $\boldsymbol{Write\ the\ formal\ definition\ (the\ \epsilon[Epsilon]-\delta[Delta]definition)\ of\ \lim_{x\to a} f(x) = L}$

$For\ every\ \epsilon > 0,\quad there\ exists\ \delta > 0\quad such\ that\quad if\ 0 < |x-a| < \delta,\quad then\ |f(x) - L| < \epsilon$

Ex: $(\boldsymbol{b})$ $\boldsymbol{Show\ that\ \lim_{x\to 3}(2x+1) = 7\ \ by\ using\ the\ formal\ definition\ of\ the\ limit}$.

$For\ every\ \epsilon > 0,\ \ choose\ \delta = \frac{\epsilon}{2}\quad such\ that\quad if\ 0 < |x-3| < \delta = \frac{\epsilon}{2},\quad then\ |2x+1-7| = |2x-6| = 2|x-3| < 2\frac{\epsilon}{2} = \epsilon$

II. $\underline{\textit{Mean Value Theorem (MVT)}}$

$Rule$: $Suppose\ f\ (x)\ is\ a\ function\ that\ satisfies\ both\ of\ the\ following$:

1. $f\ (x)\ is\ continuous\ on\ the\ closed\ interval\ [\ a\ ,b\ ]$
2. $f\ (x)\ is\ differentiable\ on\ the\ open\ interval\ (\ a\ ,b\ )$

$Then\ there\ is\ a\ number\quad c\quad such\ that\ a\ <\ c\ <\ b\qquad and\qquad f`(c) = \dfrac{f(b) - f(a)}{b-a}$

$\boldsymbol{Ex}$: $If\ f(0) = 2\ and\ 3 \le f'(x) \le 9\ \ for\ all\ x,\ \ Show\ that\ 5 \le f(1) \le 11.$

Ans: Since f is differentiable for all $x \in \mathbb{R}$ we have, by MVT, $\frac{f(1)-f(0)}{1-0} = f'(c)\quad for\ some\ c \in (0,1\ )$

$Then\quad 3 \le f'(c)\ \ \le 9 = 3 \le \frac{f(1)-2}{1-0} \le 9 \qquad\qquad 3 \le \frac{f(1)-2}{1-0}\ \le 9 \qquad\qquad 5 \le f(1) \le 11$

III. $\underline{\textit{Rolle's Theorem}}$

$Rule$: $Suppose\ f\ (x) is\ a\ function\ that\ satisfies\ both\ of\ the\ following$:

1. $f\ (x)\ is\ continuous\ on\ the\ closed\ interval\ [\ a\ ,b\ ]$
2. $f\ (x)\ is\ differentiable\ on\ the\ open\ interval\ (\ a\ ,b\ )$
3. $f\ (a) = f(b)$

$Then\ there\ is\ a\ number\quad c\quad such\ that\ a\ <\ c\ <\ b\quad and\quad f'(c) = 0$

$\boldsymbol{Ex}$: $Let\ f\ be\ a\ differentiable\ function\ on\ \mathbb{R}\ such\ that\ f''\ exists\ everywhere\ and\ f''(x) > 0\ for\ all$

$x\ in\ \mathbb{R}\ .\boldsymbol{Show\ that\ f\ cannot\ have\ three\ roots}$.

$\boldsymbol{Ans}$: assume to the f has three roots, say a < b < c are roots of f, that is, $f(a) = f(b) = f(c) = 0$

since f is differentiable, then by Rolles`s theorem, there exist $a < x_1 < b\ \ and\ \ b < x_2 < c\ \ such\ that$

$f'(x_1) = 0\ and\ f'(x_2) = 0.\ \ Now\ we\ have\ two\ roots\ x_1 and\ x_2$

$since\ f'\ is\ diffenetiable\ and\ \ f'(x_1) = f'(x_2)\ By$ Rolles`s theorem, there exists

$x_3\ between\ x_1\ and\ x_2\ such\ that\ f''(x_3) = 0, but\ f''(x) > 0\ for\ all\ x.$ $\boldsymbol{Thus, f\ can\ not\ have\ three\ roots}$.

IV. $\underline{\textit{Intermediate Value Theorem (IVT)}}$

$Rule$: $Suppose\ that\ f\ (x)\ is\ continuous\ on\ [\ a\ ,b\ ]\ and\ let\ M\ be\ any\ number\ between\ f\ (a)\ and\ f\ (b)$

$Then\ there\ exists\ a\ number\quad c\quad such\ that \qquad\qquad 1.\ \ a\ <\ c\ <\ b \qquad\qquad 2.\ f\ (\ c\ )\ =\ M$

$\boldsymbol{Ex}$: $Show\ that\ the\ equation\ x^2 + 6x + 3 = 0\ \ has\ a\ root, but\ no\ more\ than\ one.$

$\boldsymbol{Ans}$: $Let\ f(x) = x^2 + 6x + 3\quad then\ f(x)\ is\ cont.\ and\ diff.\ for\ all\ x$

$\begin{cases} f(0) = 3 > 0 \\ f(-1) = -2 < 0 \end{cases}$ $thus\ by\ IVT\ \ f(x)\ is\ cont.on\ [-1,0]\ and\ there\ is\ a\ point\ c_1 \in (-1,0)$

$such\ that\ f(c_1) = 0 \quad which\ means\ x = c_1\ \ is\ a\ root\ of\ f(x)$

$f'(x) = 2x + 6 \geq 0\ for\ all\ x \in (-1,0)\ so\ f(x)\ is\ a\ strictly\ increasing\ function\ and\ hence\ one\ to\ to\ one$

Therefore, the equation $f(x)$ has at most one solution. $f(x)\ has\ only\ one\ root\ which\ is\ c_1$

V. ***One to One*** *Function / Inverse of a Function (**Uniqueness**)*

1. *If the function is **increasing** or **decreasing**, then it is invertible* $\therefore$ *this is called **one to one** function*
2. $If\ f'(x) \geq 0\ \therefore f(x)\ is\ increasing \qquad OR \quad f'(x) \leq 0\ \therefore f(x)\ is\ decreasing$

   $OR\ \ f(x_1) = f(x_2)\ and\ x_1 = x_2 \qquad \therefore f(x) is\ invertible\ \ and\ it\ is\ called$ ***one to one*** *function*

Section 5: Problems session

$Q1$: $Show\ that\ \lim_{x \to 119}(2x + 1) = 239\ \ by\ using\ the\ formal\ definition\ of\ the\ limit.$

$Q2$: $Use\ the\ definition\ of\ the\ limit\ to\ show\ that\ \lim_{x \to 1}(2x + 119) = 121.$

$Q3$: $Use\ the\ definition\ of\ the\ limit\ to\ prove\ that\ \lim_{x \to 2}(3x - 1) = 5.$

$Q4$: $Use\ the\ definition\ of\ the\ limit\ to\ prove\ that\ \lim_{x \to 0} x^3 = 0.$

$Q5$: $Suppose\ that\ 4 \leq f'(x) \leq 6\ \ for\ all\ x \in R\ and\ f(0) = 119.\ Show\ that\ 121 \leq f\left(\frac{1}{2}\right) \leq 122.$

$Q6$: $If\ f(1) = -2\ and\ 3 \leq f'(x) \leq 5\ \ for\ all\ x,\ \ Show\ that\ 1 \leq f(2) \leq 3.$

$Q7$: *Prove that* $\ln\ (1+x^2) \leq 2x$ *for every real number* $x \in (0,1)$.

$Q8$: *Prove that* $\ln\ x < x-1$ *for all* $x > 1$.

$Q9$: *If* $f: \mathbb{R} \to \mathbb{R}$ *satisfies* $f(x) = e^{-x^2}$ *for all* x *and* $f(0) = 0$ *then prove that* $|f(x)| \leq |x|$ *for all* x.

$Q10$: *Show that the equation* $x^3 + x^2 + 2x + 5 = 0$ *has a root, but no more than one*.

$Q11$: *Prove that the equation* $\ln(x) = \frac{1}{x}$ *for* $x > 0$ *has a unique solution. Explain.*

$Q12$: *Show that the equation* $\arctan x = 2 - x - x^3$ *has exactly ONE soluation.*

$Q13$: *let* f *be a continuous function on* $[0,2]$ *and assume that* f' *and* f'' *exists on* $(0,2)$.

$a)$ *If* $f(0) = 0$ *and* $f(1) = 1$ *then show that there exists a number* $a \in (0,1)$ *such that* $f'(a) = 1$.

$b)$ *In addition conditions in part* (a) *if* $f(2) = 2$ *show that there exists a number* $c \in (0,2)$ *such that* $f''(c) = 0$

$Q14$: *Consider the function* $f(x) = e^x + 2x + x^4$.

(a) *Find the line of* ***tangent*** *to the graph of* $y = f(x)$ *at the point* $P(0,1)$.

(b) *Show that there exists a point* $Q(x, y)$ *on the graph of* $y = f(x)$ *at which the* ***tangent*** *line is* ***parallel*** *to the line* $y = 2x + 1$. *(Do NOT find the point* Q.*)*

(c) *In* ***Part (b)***, *show that* Q *is unique, i.e. there is ONLY ONE such point.*

$Q15$: *show that* $e \ln x \leq x$ *for all* $x > 0$.

Section 6: Related Rates

Procedure for Solving Related Rates Problems:

1) Draw a picture (if one is not provided) and define the variables. Assign ALL numbers to variables. Remember, rates ALWAYS correspond to a derivative.

2) Determine and CLEARLY STATE goal of the problem, which is ALWAYS finding a rate/derivative.

3) Build your related rates equation. Usually, this involves the implicit differentiation of an equation from geometry. Note: If there's only ONE rate/derivative in your related rates equation, you did something wrong!

4) Isolate the goal rate in 2) and make sure you have the numbers for all the other variables and rates/derivatives. If not, you have to find the missing numbers, usually by solving another equation.

5) Plug in numbers and solve, making sure to INCLUDE UNITS. Think about the solution and its plausibility!

Sample problem:

Question: How fast is the surface area of a cube changing when the volume of cube is $64\ cm^3$ and is increasing at $2\ cm^3/sec$?

Solution: surface area $S(x) = 6x^2$ volume $V(x) = x^3$ $\frac{dV}{dt} = +2\ cm^3/sec$ & $V(x) = 64\ cm^3$

$V(x) = x^3 = 64 \ \rightarrow x = 4\ cm$, $\dfrac{dV}{dt} = 3x^2 \dfrac{dx}{dt} \ \rightarrow \ 2 = 3(4)^2 \dfrac{dx}{dt}$, $\dfrac{dx}{dt} = \dfrac{1}{24}\ cm/sec$

Thus, $\frac{dS}{dt} = 12x \frac{dx}{dt} = 12(4)\left(\frac{1}{24}\right) = 2\ cm^2/sec$ The surface area of a cube is increasing at rate $2\ cm^2/sec$

Section 6: Problems session

Q1: The height of a rectangular box decreases at $1\ cm\ per\ hour$, its width decreases at $0.5\ cm\ per\ hour$ and its length decreases at $2\ cm\ per\ hour$. When the shape of the box become a cube, its volume decreases at the rate of $14\ cm^3\ per\ hour$. Find the volume of the box when it **becomes a cube**.

Q2: A point is moving to the right along the first-quadrant portion of the curve $x^2y^3 = 72$. When the point has coordinates (3,2), its horizontal velocity is $2\ unites/second$.

What is the rate of change in the distance of the particle to the origin?

Q3: A point (x, y) is moving on the curve $y = e^{x^2}$ so that its $x - coordinate$ is decreasing at rate of $2\ units/sec$. Find the rate of change of the slope of the tangent line to the curve $y = e^{x^2}$ at the moment when $x = 0$.

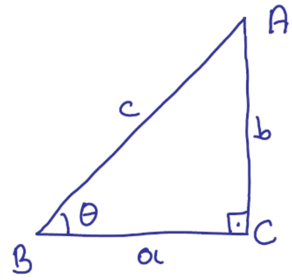
Q4: If the short side of a rectangle is increasing at a rate of $3\ cm\ per\ second$ and the long side is decreasing at a rate of $2\ cm\ per\ second$. Determine how fast the area of the rectangle is changing when the short side is $4\ cm$ and the long side is $6\ cm$.

Q5: A particle moves along the curve $y = 4x - x^3$ and it`s $x - coordinate$ is decreasing at a constant rate of $1/3\ unites\ per\ second$.

a) At what rate does the $y - coordinate$ of the particle change when $x = 2$?

b) At what rate does the slope of the tangent line change when $x = 2$?

Q6: Consider the right triangle ABC at the angle C. It`s side-lengths $a, b, c$ are changing in such a way that it`s area is always $16\ cm^2$. If the side-length a is increasing at a rate of $1/2\ cm/sec$ when a is $8\ cm$, at the rate $(radian/sec)$ is the measure of θ the angle $\hat{B}$ changing at that moment?



Q7: A particle is moving around the ellipse $4x^2 + 16y^2 = 64$. At any time, t, its $x -$ and $y - coordinates$ are given by $x(t) = 4cos\ t$ and $y(t) = 2sin\ t$. At what rate is the particle`s distance to the point $(2,0)$ changing at any time t? At what rate is the distance changing when $t = \frac{\pi}{2}$?

Section 7: Indeterminate Forms

L'Hospital's Rule:

Suppose that we have one of the following cases,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0} \qquad OR \qquad \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$$

where $\boldsymbol{a}$ can be any real number, infinity or negative infinity. In these cases, we have:

$$\lim_{x \to a} \frac{f(x)}{g(x)} \overset{(L`H)}{=\!=\!\Rightarrow} \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Note1: If you see the following cases in a limit, $(0)(\pm\infty)$ $\quad 1^{\infty} \quad 0^{0} \quad \infty^{0} \quad \infty - \infty$

Then you need to change the function of the limit to $\left(\frac{0}{0}\right)$ or $\left(\pm\frac{\infty}{\infty}\right)$ types.

(a) $\quad (0)(\pm\infty) = \dfrac{\pm\infty}{\frac{1}{0}} = \pm\dfrac{\infty}{\infty}$ $\qquad$ (b) $\quad \ln(1^{\infty}) = \infty \ln 1 = \dfrac{\ln 1}{\frac{1}{\infty}} = \dfrac{0}{0}$

(c) $\quad \ln(0^0) = 0 \ln 0 = \dfrac{\ln 0}{\frac{1}{0}} = \dfrac{\infty}{\infty}$ $\qquad$ (d) $\quad \ln(\infty^0) = 0 \ln \infty = \dfrac{\ln \infty}{\frac{1}{0}} = \dfrac{\infty}{\infty}$

when you have the following cases 1^{∞} , $0^0 and\ \infty^0$ in your question, then apply the logaritem rule (ln): (look at following **Example)**

Example: $\lim\limits_{x\to 0^+}(1+4x)^{\frac{2}{x}}$

Ans: $\left[let \lim\limits_{x\to 0^+}(1+4x)^{\frac{2}{x}} = \lim\limits_{x\to 0^+} y\right]$ By taking the logaritem to both sides:

$$\lim_{x\to 0^+} lny = \lim_{x\to 0^+} \frac{2}{x}\, ln(1+4x) = 2\lim_{x\to 0^+} \frac{ln(1+4x)}{x}\left(\frac{0}{0}\right) type = \xrightarrow{L`H} 2\lim_{x\to 0^+}\frac{4}{1+4x} = 2*4 = 8$$

$$\therefore \lim_{x\to 0^+} lny = 8 \qquad \therefore y = e^8 \quad \therefore \lim_{x\to 0^+}(1+4x)^{\frac{2}{x}} = \ e^8$$

<u>Another solution:</u>

$$\lim_{x\to 0^+}(1+4x)^{\frac{2}{x}} \overset{*}{=} e^{\lim\limits_{x\to 0^+}\frac{2}{x} ln(1+4x)} \xrightarrow{solving\ the\ power\ of\ e} \lim_{x\to 0^+}\frac{2}{x}\, ln(1+4x) = 2\lim_{x\to 0^+}\frac{ln(1+4x)}{x}\left(\frac{0}{0}\right)$$

$$= \xrightarrow{L`H} 2\lim_{x\to 0^+}\frac{4}{1+4x} = 2*4 = 8 \xrightarrow{thus} \lim_{x\to 0^+}(1+4x)^{\frac{2}{x}} = e^8$$

* where e^x is continues

<u>Note 2:</u> $(\infty - \infty)$ (in this case you are recommeded to use the following steps to have $\left(\frac{0}{0}\right)$ or $\left(\frac{\pm\infty}{\pm\infty}\right)$ types)

<u>1st: you neet to use the factor method</u>

$\lim\limits_{x\to\pm\infty} a^{f(x)} \pm b^{g(x)} = \lim\limits_{x\to\pm\infty} a^{f(x)}\left[1 \pm \frac{b^{g(x)}}{a^{f(x)}}\right]$, where (a & b can be **Constant or variable**)

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ex: $\lim\limits_{x\to\infty}(\mathrm{e}^x - 2x) = \lim\limits_{x\to\infty} e^x\left(1 - \dfrac{2x}{e^x}\right) = \infty * 1 = \infty$ $\qquad$ $since \lim\limits_{x\to\infty}\dfrac{2x}{e^x}\left(\dfrac{\infty}{\infty}\right) =^{(L`H)} \lim\limits_{x\to\infty}\dfrac{2}{e^x} = \dfrac{2}{\infty} = 0$

<u>2nd: you need to use the limit for both functions separately</u>

$= \left[\lim\limits_{x\to\pm\infty} a^{f(x)}\right].\left[1 \pm \lim\limits_{x\to\pm\infty}\frac{b^{g(x)}}{a^{f(x)}}\right]$ $\qquad$ To solve $\left[\lim\limits_{x\to\pm\infty} a^{f(x)}\right]$ check the hint

Hint: $\lim\limits_{x\to\infty} a^x = \infty,\ \lim\limits_{x\to-\infty} a^x = 0 \quad for\ any\ a > 1$ $\qquad$ $\lim\limits_{x\to\infty} a^x = 0,\ \lim\limits_{x\to-\infty} a^x = \infty \quad for\ any\ 0 < a < 1$

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ex: $\lim\limits_{x\to\infty}\dfrac{3^x - 2^x}{3^{x+1} + 2^x} = \lim\limits_{x\to\infty}\dfrac{1 - \frac{2^x}{3^x}}{3 + \frac{2^x}{3^x}} = \lim\limits_{x\to\infty}\dfrac{1 - \left(\frac{2}{3}\right)^x}{3 + \left(\frac{2}{3}\right)^x} = \dfrac{1}{3}$ $\qquad$ $where \lim\limits_{x\to\infty}\left(\dfrac{2}{3}\right)^x = 0\ and\ \dfrac{2}{3} < 1$

<u>!! Wrong assumption!!</u>

$\ln(a^x - b^x) \neq \ln a^x - \ln b^x$ $\qquad$ since $\ln a^x - \ln b^x = \ln\left(\frac{a^x}{b^x}\right)$

<u>Note 3</u>: $\frac{(\infty-\infty)}{(\infty\pm\infty)}$

<u>You will use the same method as mentioned in “case 2” for both numerator and denominator</u>

Evaluate each of the the following limits if it exists :

1) $\lim\limits_{X\to 0} \frac{e^{x}-1-x}{x^2}$

2) $\lim\limits_{x\to \infty} \frac{(\ln x)^2}{\sqrt{x}}$

3) $\lim\limits_{x\to 1} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$

4) $\lim\limits_{x\to \infty} (e^{x} - x^2)$

5) $\lim\limits_{x\to 0^+} (\sin x)^{\frac{2}{Inx}}$

6) $\lim_{x\to\infty}\left(\cos\left(\frac{2}{x}\right)\right)^{x}$

7) $\lim_{x\to\left(\frac{\pi}{2}\right)^{-}}(\tan x)^{\cos x}$

8) $\lim_{x\to 0^{+}}(1+4x+x^{2})^{\frac{2}{x}}$

9) $\lim_{x\to 0^{+}}(1+\sin x)^{\cot x}$

10) $\lim\limits_{x\to 0} \frac{1-\cos\left(\frac{\sin^2 x}{x}\right)}{\left(\frac{\sin^2 x}{x}\right)^2}$

11) $\lim\limits_{x\to\infty} \frac{5^x-2^x}{5^{x+1}+2^x}$

12) $\lim\limits_{x\to\infty} x\,(2\tan^{-1} x - \pi)$

13) $\lim\limits_{x\to\infty} \frac{119^x - e^x}{119^{x+1} + e^x}$

14) $\lim\limits_{x\to 1^+} \left(\tan\left(\frac{\pi x}{4}\right)\right)^{\left(\frac{1}{1-x}\right)}$

15) $\lim\limits_{x\to 0^+} \frac{x - \sin(x)}{(1-\cos(x))^{3/2}}$

Step 1 — Determine the Domain

•**Defination:** The **domain** of a function $f(x)$ is the set of all input values (x - values) for the function.
•**Rule:** In rational function, which means $\left(\frac{numerator}{denominator}\right)$, the $\boldsymbol{denominator}$ **can not be zero** ($denominator \neq 0$)
•**How to find the domain of the funcation** $\left(y=\frac{A(x)}{B(x)}\right)$: make the $denominator$ fucntion equal to zero to find what value of x that makes $B(x)=0$
•**Example:** $y=\frac{3x}{x^2-5x-14} \xrightarrow{denominator \neq 0} x^2-5x-14 \neq 0 \xrightarrow{\Delta>0\ \therefore\ y\ has\ root} \therefore x \neq -2\ and\ x \neq\ 7\quad Domain\ of\ y = R-\{-2,7\}$
•**Recall:** $\sqrt{f(x)}$, $where\ f(x) \geq 0$ $\qquad \ln(f(x))$, $where\ f(x)>0$

$$x=\frac{-b \pm \sqrt{b^2-4ac}}{2a}, where\ \Delta = b^2-4ac$$

•**Δ < *0*** means the funcation has **no real roots**
•**Δ = *0*** means the funcation has **equal real roots**
Δ > *0* means the funcation has 2 **real roots**

Step 2 — Find both the x and y Intercepts

•:: $\boldsymbol{x-Intercepts}$::
•**Defination:** An $\boldsymbol{x-intercept}$ of a function $f(x)$ is any **point** where the graph crosses **the** $\boldsymbol{x-axis}$.
•**To find the** $\boldsymbol{x-intercepts}$, solve *f*(*x*) = 0. The $\boldsymbol{x-intercept\ point}$ form is: $(x,0)$.
•**Recall:** there is no $\boldsymbol{x-intercept\ point}$ when $\Delta=b^2-4ac<0$

•:: $\boldsymbol{y-Intercepts}$::
•**Defination:** An $\boldsymbol{y-intercept}$ of a function $f(x)$ is any **point** where the graph crosses **the** $\boldsymbol{y-axis}$.
•**To find the** $\boldsymbol{y-intercept}$, $solve\ f(0)\ when\ x=0$. The $\boldsymbol{y-intercept\ point}$ form is: $(0,y)=(0,f(0))$.
•**Recall:** there is no $\boldsymbol{y-intercept\ point}$ when $f(0)\ is\ undefined$

Step 3 — Find Asymptote(s) *(chapter 4.6)*

•::***Vertical asymptote***::
•**Defination:** A **Vertical Asymptote (V . A)** for a function is a **vertical** line $\boldsymbol{x=a}$ showing where the function becomes unbounded.
•**When** we have $\lim_{x\to a^+} f(x)=\pm\infty$, or $\lim_{x\to a^-} f(x)=\pm\infty\ or\ both.$ **Then**, $\boldsymbol{x=a}$ is **Vertical Asymptote (V . A)**
•**Note: At $\boldsymbol{x=a}$, the $\boldsymbol{f(x)\ is\ undefined}$, where "$\boldsymbol{a}$" is constant. Therefore, If the domain is defined and continous for all $\boldsymbol{x}$, then there is no Vertical Asymptote (V . A)**
•::***Horizontal Asymptote***::
•**Defination:** A **Horizontal Asymptote (H . A)** for a function is a **horizontal** line $\boldsymbol{y=b}$ that the graph of the function approaches as x approaches ∞ or $-\infty$.
•**When** we have $\lim_{x\to\infty} f(x)=b\ (right), or\ \lim_{x\to-\infty} f(x)=b\ (left), or\ both.$ **Then**, y = $\boldsymbol{b}$ is **Horizontal Asymptote (H . A)**
•**Note: If** we have $\lim_{x\to\infty} f(x)=\pm\infty$, **then** there is no **Horizontal Asymptote (H . A) on the right.**
Similarly, If we have $\lim_{x\to-\infty} f(x)=\pm\infty$, **then** there is no **Horizontal Asymptote (H . A) on the left.**
•**Recall:The function $\left(\boldsymbol{y=\frac{A(x)}{B(x)}}\right)$ has no Horizontal Asymptote (H . A) when the degree of $\boldsymbol{A(x)\ is\ greater}$ the degree of $\boldsymbol{B(x)}$, but it has oblique asymptote instead.**
•**::Oblique *Asymptote*::**
•**Defination:** An **Oblique Asymptote** for a function is **slanted** line $\boldsymbol{y=ax+b}$ ($where\ a \neq 0$) that the function approaches as x approaches ∞ or $-\infty$.
•**When** we have $\lim_{x\to\infty}\left(f(x)-(\boldsymbol{ax+b})\right)=0\ (right), or\ \lim_{x\to-\infty}\left(f(x)-(\boldsymbol{ax+b})\right)=0\ (left), or\ both.$ **Then**, $\boldsymbol{y=ax+b}$ is **Oblique Asymptote**
•**Recall:The function $\left(\boldsymbol{y=\frac{A(x)}{B(x)}}\right)$ has no Oblique Asymptote when the degree of $\boldsymbol{A(x)\ is\ smaller\ than\ or\ equal\ to}$ the degree of $\boldsymbol{B(x)}$, but it has Horizontal Asymptote (H . A) instead.**

Step 4 — Determine the Intervals of Increase and Decrease and local extreme values *(chapter 4.4)*

- *Find* $f`(x)$
- *When* $f`(x) = 0$, *Find* $x = a_1, a_2, \dots$ *etc*. ***Therefore***, *the* "$x = a_1, a_2, \dots$ *etc*" *are called the* ***Critical Points*** $(\boldsymbol{C.P})$
- ***The singular points*** $(\boldsymbol{S.P})$ *are the points* x *where* $\boldsymbol{f`(x)}$ ***is not defined*** *except the one(s) on domain.*
- ***Draw a table*** *with the* ***values of x*** [*including*: ***critical points***, ***singular points***, ***and undefined points of both*** $\boldsymbol{f(x)}$, ***and*** $\boldsymbol{f`(x)}$]
- ***From the table****,* ***the Intervals of Increase and Decrease can be determined as well as local extreme values (max/min)***
- *If* $\boldsymbol{f`(x) > 0}$ *on an interval, then f is* ***increasing*** *on that interval.*
- *If* $\boldsymbol{f`(x) < 0}$ *on an interval, then f is* ***decreasing*** *on that interval.*
- ***<u>Remark</u>: If*** $\boldsymbol{f`(x)}$ ***is postive (or negative) on both sides of a critical or singular point, the*** $\boldsymbol{f(x)}$ ***has neither a maximum nor a minimum value at that point.***

Step 5 — Determine the Intervals of concave up/down and all Inflection Points *(chapter 4.5)*

- *Find* $f``(x)$
- *When* $f``(x) = 0$, *Find* $x = b_1, b_2, \dots$ *etc*. ***Therefore***, *the* "$x = b_1, b_2, \dots$ *etc*" *are called the* ***Inflection Points.***
- ***Draw a table*** *with the* ***values of x*** [*including*: ***Inflection Points***, ***and undefined points of both*** $\boldsymbol{f(x)}$, ***and*** $\boldsymbol{f``(x)}$]
- ***From the table****,* ***the Intervals of concave up/down can be determined.***
- *If* $\boldsymbol{f``(x) > 0}$ *on an interval, then f is* ***concave up*** *on that interval.*
- *If* $\boldsymbol{f``(x) < 0}$ *on an interval, then f is* ***concave down*** *on that interval.*

Step 6 — Draw the graph *(chapter 4.6)*

- *using the previous step, you can draw the graph.*
- *Determine the symmetry of the graph*:
- ***odd function***: ***symmetric*** *about the* ***origin****, where* $f(-x) = -\boldsymbol{f(x)}$ *for all x in the domain*
- ***even function***: ***symmetric*** *about the* ***y-axis****, where* $f(-x) = \boldsymbol{f(x)}$ *for all x in the domain*
- ***neither odd nor even function***: ***no symmetry***

Step4 $a_1 > a_2 > a_3 > a_4$

x		(C.P) a_1		(Undef) a_2		(C.P) a_3		(C.P) a_4	
$f`(x)$	+	0	-	‖	-	0	+	0	-
$f(x)$	↗		↘	‖	↘		↗		↘
		local Max				Abs. Min		local Max	

Step5 $b_1 > a_2 > b_2 > b_3$

x		(Infl.P) b_1		(Undef) a_2		(Infl.P) b_2		(Infl.P) b_3	
$f``(x)$	-	0	+	‖	-	0	+	0	-
$f(x)$	∩		∪	‖	∩		∪		∩
	Concave Down		Concave Up		Concave Down		Concave Up		Concave Down

Step6 $a_1 > b_1 > a_2 > b_2 > a_3 > b_3 > a_4$

x		(C.P) a_1		(Infl.P) b_1		(Undef) a_2		(Infl.P) b_2		(C.P) a_3		(Infl.P) b_3		(C.P) a_4	
$f`(x)$	+	0	-		-	‖	-		-	0	+		+	0	-
$f``(x)$	-		-	0	+	‖	-	0	+		+	0	-		-
$f(x)$	↗		↘		↘	‖	↘		↘		↗		↗		↘
$f(x)$	∩		∩		∪	‖	∩		∪		∪		∩		∩
$f(x)$	╭		╮		╰	‖	╮		╰		╯		╭		╮

i. Existence of extreme values on closed intervals *(chapter 4.4)*

If the domain of the function f is a ***closed***, finite interval or a union of finitely many such intervals, and if f is ***continuous*** on that domain, then f must have an absolute maximum value and an absolute minimum value.

ii. Existence of extreme values on open intervals *(chapter 4.4)*

If f is continuous on the open interval (a, b), and if

$\lim_{x\to a^+} f(x) = L \text{ and } \lim_{x\to b^-} f(x) = M$, then the following conclusions hold:

(i) *if* $f(u) > L$ *and* $f(u) > M$ *for some* u *in* (a, b), *then* f *has an absolute maximum value on* (a, b)

(i)*if* $f(v) < L$ *and* $f(v) < M$ *for some* v *in* (a, b), *then* f *has an absolute minimum value on* (a, b)

iii. The Second Derivative Test

(a) *if* $f'(x_0) = 0$ *and* $f''(x_0) < 0$, *then* f *has a local maximum value at* x_0.

(b)*if* $f'(x_0) = 0$ *and* $f''(x_0) > 0$, *then* f *has a local minimum value at* x_0.

(a)*if* $f'(x_0) = 0$ *and* $f''(x_0) = 0$, *then no conclusion can be drawn.*

Example for oblique asymptote *(chapter 4.6)*

Consider the function $f(x) = \dfrac{x^2+1}{x} = x + \dfrac{1}{x}$.*The straight line* $y = x$ *is a two − sided oblique asymptote of the graph of* $f(x)$ *because* $\lim_{x\to\pm\infty} (f(x) - x) = \lim_{x\to\pm\infty} \dfrac{1}{x} = 0$

Example for oblique and horizontal asymptote

The graph of $y = \frac{x\,e^x}{1+e^x}$ *has a horizontal asymptote* $y = 0$ *at the left and an oblique asymptote* $y = x$ *at the right*:

$$\lim_{x\to-\infty} \frac{x\,e^x}{1+e^x} = \frac{0}{1} = 0 \quad \therefore y = 0 \text{ is } H.A \text{ of } y \text{ from the left.}$$

$$\lim_{x\to\infty} \left(\frac{x\,e^x}{1+e^x} - x\right) = \lim_{x\to\infty} \left(\frac{x\,e^x - x(1+e^x)}{1+e^x}\right) = \lim_{x\to\infty} \left(\frac{x(\,e^x - 1 - e^x)}{1+e^x}\right) = \lim_{x\to\infty} \left(\frac{-x}{1+e^x}\right) = 0$$

$\therefore y = x$ *is an oblique asymptote of* y *at the right*

Section 8: Problems session

Q1: *let* $f(x) = \dfrac{(3x+1)^2}{(x-1)^2}$.*You are given that* $f'(x) = -\dfrac{8(3x+1)}{(x-1)^3}$ *and* $f''(x) = \dfrac{48(x+1)}{(x-1)^4}$

- ***Find the domain of*** $f(x)$

The domain of $f(x)$ *is* $\mathbb{R} - \{1\}$ *hint*: $(x-1)^2 \neq 0 \;\therefore x - 1 \neq 0 \;\therefore x \neq 1$

- ***Find both the x and y intercepts of*** $f(x)$

x intercept: $\dfrac{(3x+1)^2}{(x-1)^2} = 0 \rightarrow (3x+1)^2 = 0 \rightarrow 3x + 1 = 0 \rightarrow x = -\dfrac{1}{3}$ *x intercept point is* $\left(-\dfrac{1}{3}, 0\right)$

y intercepts: $f(0) = \dfrac{(3(0)+1)^2}{((0)-1)^2} = 1 \rightarrow f(0) = 1$ *y intercept point is* $(0,1)$

- ***Determine all asymptotes of $f(x)$.***

Vertical asymp. $\lim_{x\to1^+} f(x)=\infty$ & $\lim_{x\to1^-} f(x)=\infty$ $\therefore x=1$ *is Vertical asymp.*

Horizontal asymp. $\lim_{x\to\infty} f(x)=9$ & $\lim_{x\to-\infty} f(x)=9$ $\therefore y=9$ *is Horizontal asymp.*

There is no Oblique asymp.

- ***Find all local maximum and local minimum of $f(x)$; and determine intervals of increase and decrease .***

$f'(x)=-\frac{8(3x+1)}{(x-1)^3}$ *when* $f'(x)=0$ $\therefore -\frac{8(3x+1)}{(x-1)^3}=0 \rightarrow 8(3x+1)=0 \rightarrow$ *at* $x=-\frac{1}{3}$

$f(x)$ has a critical point

There is no singular point.

$f(x)$ is increasing on $\left(-\frac{1}{3},1\right)$

$f(x)$ is decreasing on $\left(-\infty,-\frac{1}{3}\right)$ & $(1,\infty)$

$f(x)$ has no local max *point* & *$f(x)$ has local* min *point at* $-\frac{1}{3}$

The critical point is $\left(-\frac{1}{3}, f\left(-\frac{1}{3}\right)\right)$

x	$-\frac{1}{3}$	1	
$f'(x)$	$-$	$+$	$-$
$f(x)$	↘	↗	↘

- ***Determine all inflection point(s) and the intervals where $f(x)$ concave up and concave down.***

$$f''(x)=\frac{48(x+1)}{(x-1)^4} \text{ when } f''(x)=0 \quad \therefore \frac{48(x+1)}{(x-1)^4}=0 \rightarrow 48(x+1)=0$$

at $x=-1$ *$f(x)$ has an inflection point*

$f(x)$ is concaving up on $(-1,1)$ & $(1,\infty)$

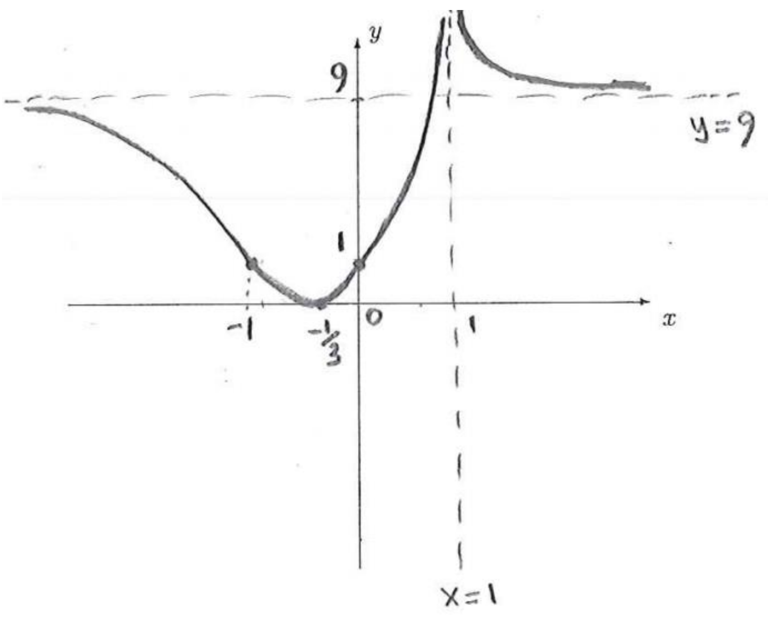
$f(x)$ is concaving down on $(-\infty,-1)$

The inflection point is $(-1, f(-1))$

x	-1	1	
$f''(x)$	$-$	$+$	$+$
$f(x)$	∩	∪	∪

- ***Using the parts a, b, c, d, and e above, sketch the graph $y=f(x)$ in the coordinate axes provided below.***

x	-1	$-\frac{1}{3}$	1	
$f'(x)$	$-$	$-$	$+$	$-$
$f''(x)$	∩	∪	∪	∪
$f(x)$				



$\boldsymbol{Q2}$: $let\ f(x) = \dfrac{x^2 - 2}{(x-1)^2}$ $.You\ are\ given\ that\ f'(x) = \dfrac{4 - 2x}{(x-1)^3}$ $and\ f''(x) = \dfrac{4x - 10}{(x-1)^4}$

- $\boldsymbol{Find\ the\ domain\ of\ f(x)}$

- $\boldsymbol{Find\ both\ the\ x\ and\ y\ intercepts\ of\ f(x)}$

- $\boldsymbol{Determine\ all\ asymptotes\ of\ f(x).}$

- $\boldsymbol{Find\ all\ local\ maximum\ and\ local\ minimum\ of\ f(x);\ and\ determine\ intervals\ of\ increase\ and\ decrease\ .}$

- $\boldsymbol{Determine\ all\ inflection\ point(s)\ and\ the\ intervals\ where\ f(x)\ concave\ up\ and\ concave\ down.}$

- $\boldsymbol{Using\ the\ parts\ above, sketch\ the\ graph\ y = f(x)\ in\ the\ coordinate\ axes\ provided\ below.}$

Q3: *let* $f(x) = (x-3)^2\, e^x$ *.You are given that* $f'(x) = e^x(x-3)(x-1)$ *and* $f''(x) = e^x(x^2-2x-1)$

- ***Find the domain of*** $\boldsymbol{f(x)}$

- ***Find both the x and y intercepts of*** $\boldsymbol{f(x)}$

- ***Determine all asymptotes of*** $\boldsymbol{f(x)}$.

- ***Find all local maximum and local minimum of*** $f(x)$; ***and determine intervals of increase and decrease .***

- ***Determine all inflection point(s) and the intervals where*** $f(x)$ ***concave up and concave down.***

- ***Using the parts above, sketch the graph*** $y = f(x)$ ***in the coordinate axes provided below.***

Q4: $let\ f(x) = \frac{x^3 - 1}{x^3 + 1}$.$You\ are\ given\ that\ f'(x) = \frac{6x^2}{(x^3 + 1)^2}$ $and\ f''(x) = \frac{12x(1 - 2x^3)}{(x^3 + 1)^3}$

- ***Find the domain of $f(x)$***

- ***Find both the x and y intercepts of $f(x)$***

- ***Determine all asymptotes of $f(x)$.***

- ***Find all local maximum and local minimum of $f(x)$; and determine intervals of increase and decrease .***

- ***Determine all inflection point(s) and the intervals where $f(x)$ concave up and concave down.***

- ***Using the parts above, sketch the graph $y = f(x)$ in the coordinate axes provided below.***

Q5: *let* $f(x) = \dfrac{x^3 - 3x^2 + 1}{x^3}$ *. You are given that* $f'(x) = \dfrac{3(x^2 - 1)}{x^4}$ *and* $f''(x) = \dfrac{3(4 - 2x^2)}{x^5}$

- ***Find the domain of*** $\boldsymbol{f(x)}$

- ***Find y intercepts of*** $\boldsymbol{f(x)}$

- ***Determine all asymptotes of*** $\boldsymbol{f(x)}$.

- ***Find all local maximum and local minimum of*** $f(x)$; ***and determine intervals of increase and decrease .***

- ***Determine all inflection point(s) and the intervals where*** $\boldsymbol{f(x)}$ ***concave up and concave down.***

- ***Using the parts above, sketch the graph*** $\boldsymbol{y = f(x)}$ ***in the coordinate axes provided below.***

Q6: *let* $f(x) = \dfrac{x^3}{x^2-1}$ *.You are given that* $f'(x) = \dfrac{x^2(x^2-3)}{(x^2-1)^2}$ *and* $f''(x) = \dfrac{2x(x^2+3)}{(x^2-1)^3}$

- ***Find the domain of*** $\boldsymbol{f(x)}$

- ***Find both the x and y intercepts of*** $\boldsymbol{f(x)}$

- ***Determine all asymptotes of*** $\boldsymbol{f(x)}$.

- ***Find all local maximum and local minimum of*** $\boldsymbol{f(x)}$***; and determine intervals of increase and decrease*** .

- ***Determine all inflection point(s) and the intervals where*** $\boldsymbol{f(x)}$ ***concave up and concave down***.

- ***Using the parts above, sketch the graph*** $\boldsymbol{y = f(x)}$ ***in the coordinate axes provided below***.

$\boldsymbol{Q7}$: $let\ f(x) = \dfrac{e^x}{x-1}$. $You\ are\ given\ that\ f'(x) = \dfrac{e^x(x-2)}{(x-1)^2}\ and\ f''(x) = \dfrac{e^x(x^2-4x+5)}{(x-1)^3}$

- ***Find the domain of $f(x)$***

- ***Find both the x and y intercepts of $f(x)$***

- ***Determine all asymptotes of $f(x)$.***

- ***Find all local maximum and local minimum of $f(x)$; and determine intervals of increase and decrease .***

- ***Determine all inflection point(s) and the intervals where $f(x)$ concave up and concave down.***

- ***Using the parts above, sketch the graph $y = f(x)$ in the coordinate axes provided below.***

Q8: *let* $f(x)$ *be the function defined for all real numbers. suppose that* $f(x)$ *satisfies the following conditions*:

$f(x)$ *is not continous at* $x = \pm 1$,

$f'(x)$ *is not defined at* $x = -2$,

$f'(2) = 0, f''(0) = f''(3) = 0$,

x	-2	-1		0		1		2		3	
$f(x)$	-1	-2		0		4		1		2	
$f'(x)$	$+$	$-$	$-$		$-$		$-$	0	$+$		$+$
$f''(x)$	$+$	$+$	$+$	0	$-$		$+$		$+$	0	$-$

$\lim_{x\to-\infty} f(x) = -\infty$, $\lim_{x\to\infty} f(x) = 3$, $\lim_{x\to-1^+} f(x) = \infty$, $\lim_{x\to-1^-} f(x) = -2$, $\lim_{x\to1^+} f(x) = 4$, $\lim_{x\to1^-} f(x) = -\infty$

Answer the following questions using the given information:

(a) Find all vertical and horizontal asymptotes of $f(x)$.

(b) Give all intervals where $f(x)$ is increasing.

(c) Is $f(x)$ is decreasing on $(-2,2)$? Explain.

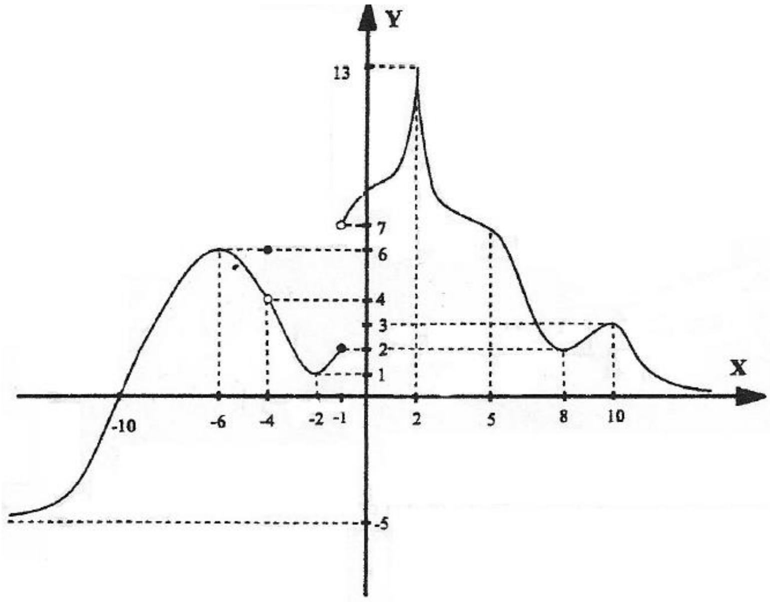
(d) Give all intervals where $f(x)$ is concave down.

(e) Find all local maximum and local minimum points of $f(x)$.

(f) Find all inflection points of $f(x)$.

(g) Sketch a graph of $f(x)$.

$Q9$: *For the function $f(x)$ whose graph is given, find the following*:



$(a)\ \lim_{x\to 2} f(x) =$

$(b)\ \lim_{x\to -4} f(x) =$

$(c)\ \lim_{x\to -\infty} f(x) =$

$(d)\ \lim_{x\to -1} f(x) =$

(e) *Points where* $f'(x) = 0$

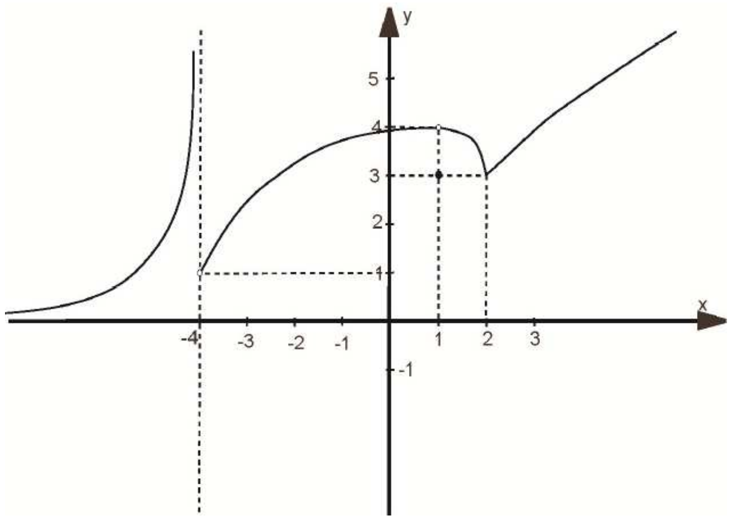
(f) *Intervals where* $f'(x) > 0$

(g) *Intervals where f is continous.*

(h) *Intervals where f is differentiable.*

$(i)\ \lim_{x\to 2^-} f'(x) =$

$Q10$: *The graph of the function $f(x)$ is given below, Answer the following questions*:



(a) *Find the limit* $\lim_{x\to -4^+} f(x)$ *if it exists.*

(b) *Find the limit* $\lim_{x\to 1} f(x)$ *if it exists.*

(c) *Find all intervals which $f(x)$ is continous.*

(d) *Find all intervals which $f(x)$ is differentiable.*

Section 9: Extreme-Value Problem

Example: Suppose a farmer wishes to enclose a rectangular field using $1000\ m$ of fencing in such a way that the area of the field is maximized. Find the dimensions and the Total area of the field when the area of the field is **maximized**.

Solution: Let x and y be the dimensions of the field and let A be the area of the field. Then, $A = xy$

$1000 = 2x + 2y \quad \rightarrow \quad \mathrm{y} = 500 - \mathrm{x}$. where 1000 m is the circumference of the rectangular field

$A = x(500 - x) = 500x - x^2$ The maximum value of A on the interval $x \in [0, 500]$

$\frac{dA}{dx} = 500 - 2x \quad when\ \frac{dA}{dx} = 0 \ \ then\ x = 250\ m \quad and\ y = 250\ m \quad A|_{x=250} = (250)(250) = 62500\ m^2$

$The\ end\ points\!: \ A|_{x=0} = A|_{x=500} = 0 \qquad where\ A|_{x=250} > A|_{x=0} \quad and \quad A|_{x=250} > A|_{x=500}$

So A has a maximum value of $62500\ m$ square when $x = 250\ m$ and $y = 500 - 250 = 250\ m$

Section 9: Problems session

Part 1:

Q1: Find the dimensions of the rectangle of largest area that has its base on the $x - axis$ and its other two vertices are above the $x - axis$ on the parabola $y = 4 - x^2$.

Q2: A cylinder with an open top has total surface area $27\pi\ cm^2$. Find the radius of such a cylinder which has maximum volume. Explain why your answer gives the maximum volume.

Q3: Find the area of the largest triangle with the following properties: one vertex is $(0,0)$, the other vertices are on the ellipse $4x^2 + y^2 = 4$ and one side is parallel to the $y - axis$.

Q4: there are 50 orange trees in a garden. Each tree produces 800 oranges. For each additional tree planted in the garden, the output per tree **drops** by 10 oranges. How many trees should be added to the existing garden in order to maximize the total output of trees?

Q5: Among all triangles formed by the positive $x - axis$, the positive $y - axis$ and a line passing through the point (1,3), find the dimensions of the triangle with the shortest hypotenuse.

Part 2: (Recommed for the final exam)

Q1: Find the absolute mininmum and absolute maximum points and the values of $f(x) = (x^3 - 9x^2 + 15x)^{2/3}$ on $[-1,6]$.

Q2: Find the absolute mininmum and absolute maximum (if they exist) of $f(x) = e^{-\left(\frac{1+x^2}{x}\right)}$ on $(0, \infty)$.

Q3: Consider the function $f(x) = \begin{cases} |x^2 - 1| & x \leq 2 \\ 2 + \cos(x - 2) & x > 2 \end{cases}$

(a) Without finding the absolute extreme values, show that f has absolute maximum and minimum values on the interval $[0,6]$.

(b) Find the absolute maximum and minimum values of f on the interval $[0,6]$.

Section 10: Linear Approximations

The **linearization** of the function f about $\boldsymbol{a}$ is the function $\boldsymbol{L}$ defined by $L(x) = f(a) + f\,`(a)\,(x-a)$

We say that $f(x) \approx L(x) = f(a) + f\,`(a)\,(x-a)$ *provids* ***linear approximations*** *for values of f near to* $\boldsymbol{a}$.

$\boldsymbol{ex}$) **The equation** $\boldsymbol{x^2y^3 - x^3y^2 = 4}$ ***defines a funcation*** $\boldsymbol{y(x)}$. ***Find the lineriztion of*** $\boldsymbol{y(x)}$ ***at*** $(\boldsymbol{1}\,,\boldsymbol{2})$ ***and use it to approximate*** $\boldsymbol{y(1.01)}$

Solution: *using the lineriztion rule*: $L(x) = f(a) + f\,`(a)\,(x-a)$ *hint*: $a = 1$ *and* $x = 1.01$

from $(1,2)$ *since* $y = 2 \xrightarrow{so} y(a) = y(1) = 2$

$\frac{d}{dx}[\,x^2y^3 - x^3y^2\,] = \frac{d}{dx}[\,4\,] \quad \therefore\ 2xy^3 + 3x^2y^2y\,` - 3x^2y^2 - 2x^3yy\,` = 0$ *using the point* $(1\,,2)$

$$2(1)(2)^3 + 3(1)^2(2)^2y\,` - 3(1)^2(2)^2 - 2(1)^3(2)y\,` = 0 \xrightarrow{then} \boldsymbol{y}\,`(\boldsymbol{1}) = y`_{at\ x=1} = \frac{-4}{8} = \frac{-1}{2}$$

$\therefore$ *the* ***lineriztion*** *of y at* $a = 1 \overset{i.e}{\rightarrow} L(x) = y(1) + y\,`(1)\,(x-1) = 2 + \frac{-1}{2}(x-1)$

using the approximation rule, we have $f(1.01) \approx L(1.01) = 2 + \frac{-1}{2}(1.01-1) = 2 - \frac{1}{200} = \frac{399}{200}$

Section 10: Problems session

1) The equation $\cos\left(\frac{\pi y}{x}\right) = \frac{x^2}{y} - 4$ *defines a funcation* $y = f(x)$. *Find the lineriztion of f at* $(3\,,3)$ *and use it to approximate* $f(3.1)$

2) The equation $\sin(xy) + \pi = x + y$ *defines a funcation* $y = f(x)$. *Find the lineriztion of f at* $(\pi, 0)$ *Also, find an approximation for* $f(3)$

3) $Using\ linear\ approximation, estimate\ \sqrt[5]{33}$

Note: review F.T.C (section 14) and integral proprieties (section 13) before solving the following questions

4) $let\ f(x) = \int_1^x \frac{dt}{1+t^6}. Use\ the\ linearization\ of\ f\ at\ a = 1\ to\ approximate\ f\left(\frac{3}{2}\right)$

5) $let\ f(x) = \int_1^{x^2} e^{t^2} dt\ \ be\ given. Using\ the\ linear\ (tangent\ line)\ approximation\ of\ F,$

$and\ estimate\ \int_1^{1.21} e^{t^2} dt$

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\,\Delta x = \int_a^b f(x)\,dx \qquad note\ that \begin{Bmatrix} f(x)\ is\ continous\ on\ [a,b] \\ n\ is\ subinterval\ of\ width\ \Delta x \\ each\ interval\ is\ called\ x_i^* \\ p_n = (x_0, x_1, x_2, \dots\dots, x_n) \end{Bmatrix}$$

$\boldsymbol{Ex}$: $\displaystyle \lim_{n\to\infty}\frac{\pi}{n}\sum_{i=1}^{n} \cos\left(\frac{i\pi}{n}\right)$

$\boldsymbol{Solution}$: $\displaystyle \lim_{n\to\infty}\frac{\pi}{n}\sum_{i=1}^{n} \cos\left(\frac{i\pi}{n}\right) = \lim_{n\to\infty}\Delta x\sum_{i=1}^{n} f(x_i^*) \overset{\blacksquare}{\Rightarrow} \int_0^{\pi} \cos x\,dx = \sin x|_0^{\pi} = 0$

$$\blacksquare \begin{Bmatrix} \Delta x = \frac{\pi}{n}(chosen) = \frac{b-a}{n}(rule),\ \ b-a=\pi \to (1) & \Rightarrow & let\ f(x_i^*) = \cos\left(\frac{i\pi}{n}\right) \to (2) \\ Let\ x_i^* = \frac{i\pi}{n}(chosen) = a + i\,\Delta x\,(rule) = a + i\,\frac{\pi}{n}, & \Rightarrow & \therefore\ a=0\ \therefore b=\pi\ from\ eqn\ (1) \\ P_n = \left\{x_0^* = 0, = x_1^* = \frac{\pi}{n}\dots\dots.., x_n^* = \pi\right\} & \Rightarrow & x_0^* = 0 = a,\ \ x_n^* = \pi = b \\ from\ eqn\ (2), f(x_i^*) = \cos\left(\frac{i\pi}{n}\right) = \cos(x_i^*) & \Rightarrow & let\ f(x) = \cos x\ \ which\ is\ cont.\ on\ [0,\pi] \end{Bmatrix}$$

$\boldsymbol{Ex}$: **For the funcation** $\boldsymbol{f(x) = xe^{(x^2)}}$

write a Riemann Sum obtained by dividing the interval [1, 2] into n equal length subintervals.

$\boldsymbol{Solution}$: $\displaystyle we\ have \int_1^2 xe^{(x^2)}\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\,\Delta x \ \Rightarrow (using\ Riemann\ Sum)$

$$we\ have \Rightarrow \begin{Bmatrix} \Delta x = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n},\ \ P_n = \{x_0^* = 1, \dots\dots.., x_n^* = 2\},\ then\ x_i^* = 1 + \frac{i}{n}\ where\ x_i^* = a + i\,\Delta x \\ where\ f(x) = xe^{(x^2)}\ \ which\ is\ cont.\ on\ [1,2] \end{Bmatrix}$$

$$\int_1^2 xe^{(x^2)}\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(1+\frac{i}{n}\right)\frac{1}{n} \ \Rightarrow (using\ Riemann\ Sum) \Rightarrow \lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{i}{n}\right)e^{\left(1+\frac{i}{n}\right)^2}\frac{1}{n}$$

Section 11: Problems session

$$Q1: \lim_{n\to\infty}\frac{1}{n}\left(\sqrt{1-\frac{1}{n}} + \sqrt{1-\frac{2}{n}} + \cdots + \sqrt{1-\frac{n}{n}}\right) = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sqrt{1-\frac{1}{n}i}$$

$$Q2)\ \lim_{n\to\infty}\frac{1}{n}\left[\left(1+\frac{1}{n}\right)^5 + \left(1+\frac{2}{n}\right)^5 + \cdots + \left(1+\frac{n}{n}\right)^5\right]$$

$Q3)\ \lim_{n\to\infty} \frac{1}{n}\left[\left(\frac{1}{n}\right)^{2/3} + \left(\frac{2}{n}\right)^{2/3} + \cdots + \left(\frac{n-1}{n}\right)^{\frac{2}{3}} + 1\right]$

$Q4)\ \lim_{n\to\infty} \sum_{k=1}^{n} \frac{\pi}{n} \sin\left(\frac{k\pi}{n}\right)$

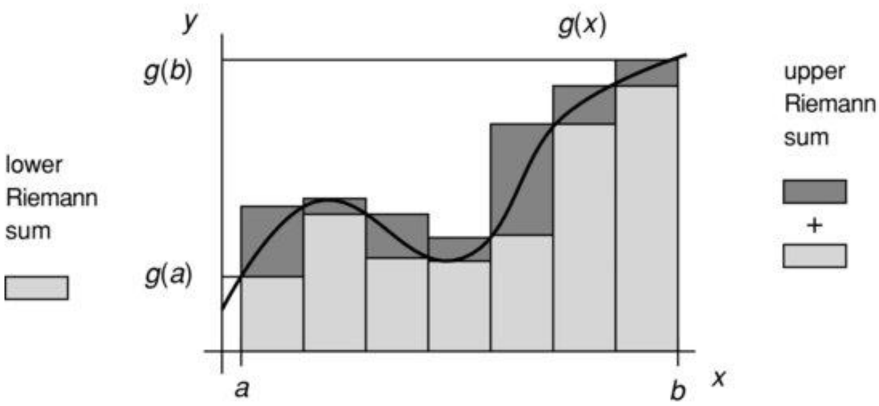
$Q5)\ \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n} \sqrt{1-\frac{i^2}{n^2}}$

$Q6)\ \lim_{n\to\infty} \sum_{i=1}^{n} \frac{4(4j-2)}{n^2} \sqrt{\left(\frac{4j-2}{n}\right)^2 - 1}$

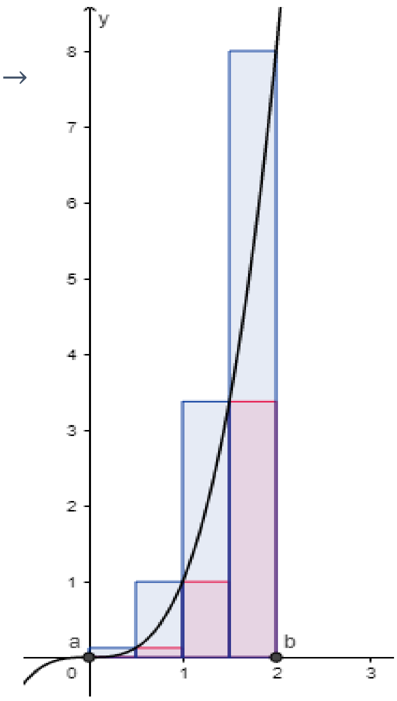
$Q(B)$: ***For the funcation*** $f(x) = (\sin x)^{x+1}$

write a Riemann Sum obtained by dividing the interval $[1,3]$ ***into n equal length subintervals.***

Section 12: The Definite Integral



ex: $f(x) = x^3 \rightarrow$



ex: calculate the upper and lower riemann sum for the funcation $f(x) = x^3$ ***, corresponding to the partition*** P_n ***of*** $[0,2]$ ***into 4 subintervals of equal lenght.*** *[The graph is shown above]*

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$$

$$L(f,p) = \Delta x\,[f(l_1) + f(l_2) + f(l_3) + f(l_4)] = \frac{1}{2}\left[f(0) + f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right)\right] = \frac{1}{2}\left[0 + \left(\frac{1}{2}\right)^3 + 1^3 + \left(\frac{3}{2}\right)^3\right] = \frac{9}{4}$$

$$U(f,p) = \Delta x\,[f(u_1) + f(u_2) + f(u_3) + f(u_4)] = \frac{1}{2}\left[f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right) + f(2)\right] = \frac{1}{2}\left[\left(\frac{1}{2}\right)^3 + 1^3 + \left(\frac{3}{2}\right)^3 + 2^3\right] = \frac{25}{4}$$

$Q1: calculate\ the\ upper\ and\ lower\ riemann\ sum\ for\ the\ funcation\ f(x) = x^2\ , corresponding\ to$
$the\ partition\ \ P_n\ of\ [0,2]\ into\ 4\ subintervals\ of\ equal\ lenght.$

$Q2: calculate\ the\ upper\ and\ lower\ riemann\ sum\ for\ the\ funcation\ f(x) = cosx\ , corresponding\ to$
$the\ partition\ \ P_n\ of\ [0,2\pi]\ into\ 4\ subintervals\ of\ equal\ lenght.$

$Q3: calculate\ the\ upper\ and\ lower\ riemann\ sum\ for\ the\ funcation\ f(x) = |x^2 - 1|\ , corresponding\ to$
$the\ partition\ \ P_n\ of\ [0,2]\ into\ 4\ subintervals\ of\ equal\ lenght.$

The definite integral

Suppose there is exactly one number I such that for every partition P of $[a, b]$ we have $L(f, P) \leq I \leq U(f, P)$.

Then we say that function f is integrable on $[a, b]$, and we call I the definite integral of function f on $[a, b]$. The definite integral is denoted by the symbol $I = \int_a^b f(x)\, dx$.

$ex: By\ using\ ex1,$ Prove that $f(x) = x^3$ $is\ integrable\ on\ the\ interval$ $[0,2]$

Let $I = \int_0^2 x^3\, dx$. such that for every partition P of $[0,2]$ we have $L(f, P) \leq I \leq U(f, P)\ \rightarrow by\ ex1 \rightarrow\ \frac{9}{4} \leq I \leq \frac{25}{4}$

Then the function $f(x)$ is integrable on $[0,2]$, and we call I the definite integral of function f on $[0,2]$.

Q4: $By\ using\ Q1,$ Prove that $f(x) = x^2\ is\ integrable\ on\ the\ interval\ [0,2]$

Q5: $By\ using\ Q2,$ Prove that $\int_0^{2\pi} cosx\ dx \leq \pi$

Q6: $By\ usingQ3,\quad Prove\ 1 \leq I \leq\ 3\qquad where\ \ I = \int_0^2 |x^2 - 1| dx$

Section 13: Properties of the Definite Integral

1) If f is an **odd** function (i.e., $f(-x) = -f(x)$), then $\int_a^{-a} f(x)\ dx = 0.$

$$ex:\ \int_{-\pi}^{\pi} \sin x\ \ dx = 0\quad since\ \ \sin x\ \ is\ odd\ function\quad i.e\quad \sin(-x) = -\sin x$$

2) If f is an **even** function (i.e., $f(-x) = f(x)$), then $\int_a^{-a} f(x)\ dx = 2\int_0^a f(x)\ dx$.

$$ex:\ \int_{-\pi}^{\pi} \cos x\ \ dx = 2\int_0^{\pi} \cos x\ \ dx = -2\sin x|_0^{\pi} = -2(\sin\pi - \sin 0) = 0$$

$since\ \cos x\ \ is\ even\ function\quad i.e\quad \cos(-x) = \cos x$

3) $\int_a^a f(x)\ dx = 0\qquad ex:\ \ \int_2^2 (2+5x)\ dx = 0$

Section 13: Problems session

1) $\int_{-2}^{2} (2+5x)\ dx$

2) $\int_{-\pi}^{\pi} 1 + \sin^3 x \sqrt{9x^2 + 4cos^2\ x}\ \ dx$

3) $\int_{-1}^{1} \frac{\sin x}{1+x^2}\, dx$

4) $\int_{-3}^{3} \sqrt{9-x^2}\, dx$

Section 14: The Fundamental Theorem of Calculus [F.T.C]

$Ex: F(x) = \int_{x}^{3} e^{-t^2}\, dt \quad find\ \frac{d}{dx} F(x)$

$By\ F.T.C,\ \ we\ have\ \ \frac{d}{dx} F(x) = \left[\frac{d}{dx}(3)\right] * e^{-(3)^2} - \left[\frac{d}{dx}(x)\right] * e^{-(x)^2} = [0] * e^{-6} - [1] * e^{-x^2} = -e^{-x^2}$

Section 14: Problems session

1) $G(x) = x^2 \int_{-4}^{5x} e^{-t^2}\, dt \quad find\ \frac{d}{dx} G(x)\ \ at\ x = -\frac{4}{5}$

2) $H(x) = \int_{x^2}^{x^3} e^{-t^2}\, dt \quad find\ \frac{d}{dx} H(x)\ at\ x = 1$

3) $\lim_{x\to 0} \frac{1}{x^3} \int_{0}^{x} tan(t^2)\, dt$

4) $\lim_{x\to\pi/2} \frac{\int_x^{\pi/2} ln\,(\sin t)\ dt}{sinx - 1}$

5) $\lim_{x\to 0^+} \frac{238x + x^{2019}}{\int_0^x (\cos(t^3) + cos^3 t)\ dt}$

Section 15.1: Integral Properties

$\int f(x) \pm g(x)\ dx = \int f(x)\ dx \pm \int g(x)\ dx$

$\int c\ f(x)\ dx = c\int\ f(x)\ dx,\ c\ is\ constant$

$\int\ f(x)\ dx = F(x) + c$ where $F(x)$is an anti-derivative of $f(x)$

$\int_a^b f(x)\ dx = F(x)|_a^b = F(b) - F(a)$

$\int_a^b c\ dx = c(b-a)\ ,\ \ c\ is\ constant$

$\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$

Common Integrals

$\int k\ dx = kx + c$	$\int \sin x\ dx = -\cos x + c$	$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$
$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c, n \neq 1$	$\int \sec^2 x\ dx = \tan x + c$	$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$
$\int x^{-1}\ dx = \int \frac{1}{x}\ dx = \ln\lvert x\rvert + c$	$\int \sec x \tan x\ dx = \sec x + c$	$sin^2\,x + cos^2\,x = 1$
$\int \frac{1}{ax+b} dx = \frac{1}{a}\ln\lvert ax+b\rvert + c$	$\int \csc x \cot x\ dx = -\csc x + c$	$sec^2\,x - tan^2\,x = 1$
$\int e^x\,dx = e^x + c$	$\int \csc^2 x\ dx = -\cot x + c$	$sin(2x) = 2\,sin(x)\,cos(x)$
$\int \ln x\,dx = x\ln x - x + c$	$\int \tan x\ dx = \ln\lvert \sec x\rvert + c$	$cos^2(x) = \frac{1}{2}(1 + cos(2x))$
$\int \cos x\,dx = \sin x + c$	$\int \sec x\ dx = \ln\lvert \sec x + \tan x\rvert + c$	$sin^2(x) = \frac{1}{2}(1 - cos(2x))$

Section 15.2: Substitution

The substitution $u = g(x)$ will convert $\int_a^b f(g(x))g'(x)\ dx = \int_{g(a)}^{g(b)} f(u)\ du\ using\ du = g'(x)\ dx$

$$ex: \int_1^2 5\,x^2\cos(x^3)\,dx \overset{(*)}{\Rightarrow} \int_1^8 \frac{5}{3}\cos(u)\,du = \frac{5}{3}\sin(u)\Big|_1^8 = \frac{5}{3}(\sin(8) - \sin(1))$$

$$\begin{cases} u = x^3 \ \to du = 3x^2dx \to x^2dx = \dfrac{1}{3}du \\ x = 1 \ \to u = 1^3 = 1 \quad and \quad x = 2 \to u = 2^3 = 8 \end{cases}$$

Products and (some) Quotients of Trig Functions

For $\int sin^n(x)\cos^m(x)\ dx$ we have the following:	For $\int tan^n(x)\sec^m(x)\ dx$ we have the following:
1. n odd. Use the substitution $u = cos\ x$ 2. m odd. Use the substitution $u = sin\ x$ (1) & (2) convert rest using $\sin^2 x + \cos^2 x = 1$ 3. n and m both even. Use Trig Formulas: $\sin(2x) = 2\sin(x)\,cos(x)$ $\cos^2(x) = \frac{1}{2}(1 + cos(2x))$ $\sin^2(x) = \frac{1}{2}(1 - cos(2x))$	1. n odd. Use the substitution $u = sec\ x$ 2. m even. Use the substitution $u = tan\ x$ (1) & (2) convert rest using $sec^2 x - tan^2 x = 1$ 3. n odd and m even. Use either 1. or 2. 4. n even and m odd. Each integral will be dealt with differently.

$Ex.\int \tan^3 x\sec^5 x dx$

$$\int \tan^3 x\sec^5 x dx = \int \tan^2 x\sec^4 x\tan x\sec x dx = \int (\sec^2 x - 1)\sec^4 x\tan x\sec x\ dx \qquad (u = \sec x)$$

$$\to (du = \tan x\sec x\ dx)$$

$$= \int (u^2 - 1)u^4 du = \frac{1}{7}u^5 - \frac{1}{5}u^5 + c = \frac{1}{7}\sec^7 x - \frac{1}{5}\sec^5 x + c$$

Ex. $\int \frac{\text{Sin}^5 x}{\cos^3 x}dx = \int \frac{\sin^4 x\sin x}{\cos^3 x}dx = \int \frac{(\sin^2 x)^2\sin x}{\cos^3 x}dx = \int \frac{(1-\cos^2 x)^2\sin x}{\cos^3 x}dx$

$(u = \cos x) \to (du = -\sin x\ dx)$

$$= -\int \frac{(1-u^2)^2}{u^3}du = -\int \frac{1 - 2u^2 + u^4}{u^3}du = -\int \frac{1}{u^3}du + \int \frac{2}{\text{u}}du - \int \text{u}\,du$$

$$= \frac{1}{2}\text{u}^{-2} + 2\ln|\text{u}| - \frac{1}{2}\text{u}^2 + c = \frac{1}{2}\sec^2 x + 2\ln|\cos x| - \frac{1}{2}\cos^2 x + c$$

Section 15.2: Problems session

(1) $\int_0^{\frac{\pi}{3}} \frac{tan\,x\sqrt{sec\,x} + sec\,x\sqrt{tan\,x}}{cos\,x}dx$

(2) $\int_{\frac{\pi}{2}}^{\pi} \cos^5 x \sin^2 x \, dx$

(3) $\int \frac{\sin^3 x}{\sqrt{\cos x}} dx$

(4) $\int_1^2 \sqrt{\frac{1+\sqrt{x}}{x}} dx$

(5) $\int \frac{\ln(\ln x)}{x \ln x} dx$

(6) $\int \tan^5 x \sec x \, dx$

(7) $\int \frac{2x\,dx}{x^2+2x+3}$

(8) $\int \frac{x^2+x+1\,dx}{\sqrt{x}}$

(9) $\int_0^{\frac{1}{2}} \frac{\arcsin x}{\sqrt{1-x^2}} dx$

(10) $\int \frac{1+x}{1+\sqrt{x}} dx$

(11) $\int \frac{x^3}{\sqrt{1+x^2}} dx$

(12) $\int_0^{\frac{\pi}{4}} \frac{\sin x - \cos x}{\sin x + \cos x} dx$

Section 15.3: Integration by Parts

$\int u\, dv = uv - \int v\, du$ and $\int_a^b u\, dv = uv|_a^b - \int_a^b v\, du.$ *Choose* u *and* dv *from integral.*

From $\int u\, dv$ $\begin{cases} u \xrightarrow{(u)'} du \\ dv \xrightarrow{\int dv} u \end{cases}$

The choice of $u \rightarrow$ function to differentiate. The choice of $dv \rightarrow$ function <u>easily</u> to integrate

Ex: $\int x\, e^{-x} dx$ $u = x \qquad dv = e^{-x} dx$ $du = dx \qquad v = -e^{-x}$ $\int x\, e^{-x} dx = -xe^{-x} + \int e^{-x}\, dx$ $= -xe^{-x} - e^{-x} + c$	Ex: $\int_3^5 \ln x\ dx$ $u = \ln x \qquad dv = dx$ $du = \frac{1}{x} dx \qquad v = x$ $\int_3^5 \ln\ x dx = x \ln x|_3^5 - \int_3^5 d\, x = (x \ln (x) - x)|_3^5$ $= 5 \ln (5) - 3 \ln (3) - 2$

How to choose u and dv in Integrate by Parts Questions			
Function	Example	u	dv
Log function	$\int \ln x \, dx$	$\ln x$	dx
Log times algebraic	$\int x^4 \ln x \, dx$	$\ln x$	$x^4 dx$
Log composed with algebraic	$\int \ln x^3 \, dx$	$\ln x^3$	dx
Inverse trig forms	$\int \arcsin x \, dx$	$\arcsin x$	dx
Algebraic times sine	$\int x^2 \sin x \, dx$	x^2	$\sin x \, dx$
Algebraic times cosine	$\int 3x^5 \cos x \, dx$	$3x^5$	$\cos x \, dx$
Algebraic times exponential	$\int \frac{1}{2} x^2 e^{3x} \, dx$	$\frac{1}{2} x^2$	$e^{3x} \, dx$
Algebraic times integral	$\int x \left(\int \frac{1}{\sqrt{1+t^3}} \, dt \right) dx$	$\int \frac{1}{\sqrt{1+t^3}} \, dt$	$x \, dx$
Sine times exponential	$\int e^{\frac{x}{2}} \sin x \, dx$	$\sin x$	$e^{\frac{x}{2}} \, dx$
Cosine times exponential	$\int e^x \cos x \, dx$	$\cos x$	$e^x \, dx$
Order of choosing dv is (easiest: exponential $\Longrightarrow$ sine /cosine $\Longrightarrow$ algebraic) which have a simple form			

Section 15.3: Problems session

(1) $\int e^{6x} \sin(e^{3x}) \, dx$

(2) $\int_1^4 \sqrt{x} \, e^{\sqrt{x}} \, dx$

(3) $\int x \arctan(x^2) \, dx$

(4) $\int x^2 e^x \, dx$

(5) $\int (x-1)^2 \ln(x-1) \; dx$

(6) $\int e^{\sqrt[3]{x}} \; dx$

(7) $\int_1^3 \frac{\arctan \sqrt{x}}{\sqrt{x}} \; dx$

(8) $\int \frac{x^3 e^{x^2}}{(x^2+1)^2} dx$

Section 15.4: Partial Fractions

Partial Fractions: If integrating $\int \frac{f(x)}{g(x)}dx$ where the degree of $f(x)$ is smaller than the degree of $g(x)$. Both $f(x)$ and $g(x)$ are ONLY polynomial functions.

1. Factor denominator as completely as possible
2. Find the partial fraction decomposition (P.F.D.) of the rational expression.
3. Integrate the partial fraction decomposition (P.F.D.).

For each factor in the denominator we get term(s) in the decomposition according to the following table.

Factor in $g(x)$	Term in P. F. D	Factor in $g(x)$	Term in P. F. D
$ax + b$	$\frac{A}{ax+b}$	$(ax + b)^k$	$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_k}{(ax+b)^k}$
$ax^2 + bx + c$	$\frac{Ax+B}{ax^2+bx+c}$	$(ax^2 + bx + c)^k$	$\frac{A_1x+B_1}{ax^2+bx+c}+\cdots+\frac{A_kx+B_k}{(ax^2+bx+c)^k}$

Example: $\int \frac{7x^2+13x}{(x-1)(x^2+4)}dx$

$$\frac{7x^2+13x}{(x-1)(x^2+4)}=\frac{A}{(x-1)}+\frac{Bx+C}{(x^2+4)}=\frac{A(x^2+4)+(Bx+C)(x-1)}{(x-1)(x^2+4)}$$

$$=\begin{cases} A+B=7 \\ C-B=13 \\ 4A-C=0 \end{cases} \qquad A=4,\ \ B=3,\ \ C=16$$

$$\int \frac{7x^2+13x}{(x-1)(x^2+4)}dx=\int \frac{4}{(x-1)}+\frac{3x+16}{(x^2+4)}dx=\int \frac{4}{(x-1)}+\frac{3x}{(x^2+4)}+\frac{16}{(x^2+4)}dx$$

$$=4\ln|x-1|+\frac{3}{2}\ln|x^2+4|+8\tan^{-1}\left(\frac{x}{2}\right)+C$$

Section 15.4: Problems session

(1) $\int \frac{x+1}{x^2(x^2+1)}dx.$

(2) $\int \frac{3x^3-3x^2+x-1}{x^2(2x^2+1)}dx$.

(3) $\int \frac{x^2+11x+10}{x^3+4x^2+5x}dx$. (Assume that ($x \neq 0$))

(4) $\int \frac{8x+1}{x^2(x^2+4)}dx$.

(5) $\int \frac{10}{(x-1)(x^2+9)}dx.$

(6) $\int \frac{x^3+1}{x^3-x^2}dx.$

(7) $\int \frac{1}{x^3(x-1)}dx.$

(8) $\int \frac{1}{(1+x^{1/3})(1+x^{2/3})}dx.$

(9) $\int \frac{dx}{x(1+x^{119})}$.

Section 15.5: Trig Substitutions

Trig Substitutions: If the integral contains the **following roots,** then use the given substitution and formula to convert into an integral involving trig functions. The form type with (◘) symbol has to be at the **denominator** of the integrable function. However, the following roots can be shown at both the **denominator** (commonly) and **numerator** (rarely) of the integrable function.

$\sqrt{\boldsymbol{a^2 - b^2x^2}} \xrightarrow{\text{choose}} \sin\theta = \frac{\sqrt{b^2x^2}}{\sqrt{a^2}} = \frac{bx}{a}$	$\sqrt{\boldsymbol{2^2 - 3^2x^2}} \xrightarrow{\text{choose}} \sin\theta = \frac{\sqrt{3^2x^2}}{\sqrt{2^2}} = \frac{3x}{2}$
$\sqrt{\boldsymbol{b^2x^2 - a^2}} \xrightarrow{\text{choose}} \sec\theta = \frac{\sqrt{b^2x^2}}{\sqrt{a^2}} = \frac{bx}{a}$	$\sqrt{\boldsymbol{3^2x^2 - 2^2}} \xrightarrow{\text{choose}} \sec\theta = \frac{\sqrt{3^2x^2}}{\sqrt{2^2}} = \frac{3x}{2}$
$\sqrt{\boldsymbol{a^2 + b^2x^2}} \xrightarrow{\text{choose}} \tan\theta = \frac{\sqrt{b^2x^2}}{\sqrt{a^2}} = \frac{bx}{a}$	$\sqrt{\boldsymbol{2^2 + 3^2x^2}} \xrightarrow{\text{choose}} \tan\theta = \frac{\sqrt{3^2x^2}}{\sqrt{2^2}} = \frac{3x}{2}$
(◘) $\boldsymbol{a^2 + b^2x^2} \xrightarrow{\text{choose}} \tan\theta = \frac{\sqrt{b^2x^2}}{\sqrt{a^2}} = \frac{bx}{a}$	(◘) $\boldsymbol{2^2 + 3^2x^2} \xrightarrow{\text{choose}} \tan\theta = \frac{\sqrt{3^2x^2}}{\sqrt{2^2}} = \frac{3x}{2}$
Another form:	***Another form*:**
$\sqrt{\boldsymbol{a^2 - b^2(x+c)^2}} \xrightarrow{\text{choose}} \sin\theta = \frac{\sqrt{b^2(x+c)^2}}{\sqrt{a^2}} = \frac{b(x+c)}{a}$	$\sqrt{\boldsymbol{2^2 - 3^2(x+1)^2}} \xrightarrow{\text{choose}} \sin\theta = \frac{\sqrt{3^2(x+1)^2}}{\sqrt{2^2}} = \frac{3(x+1)}{2}$
$\sqrt{\boldsymbol{b^2(x+c)^2 - a^2}} \xrightarrow{\text{choose}} \sec\theta = \frac{\sqrt{b^2(x+c)^2}}{\sqrt{a^2}} = \frac{b(x+c)}{a}$	$\sqrt{\boldsymbol{3^2(x+1)^2 - 2^2}} \xrightarrow{\text{choose}} \sec\theta = \frac{\sqrt{3^2(x+1)^2}}{\sqrt{2^2}} = \frac{3(x+1)}{2}$
$\sqrt{\boldsymbol{a^2 + b^2(x+c)^2}} \xrightarrow{\text{choose}} \tan\theta = \frac{\sqrt{b^2(x+c)^2}}{\sqrt{a^2}} = \frac{b(x+c)}{a}$	$\sqrt{\boldsymbol{2^2 + 3^2(x+1)^2}} \xrightarrow{\text{choose}} \tan\theta = \frac{\sqrt{3^2(x+1)^2}}{\sqrt{2^2}} = \frac{3(x+1)}{2}$
$(*)\ \boldsymbol{a^2 + b^2(x+c)^2} \xrightarrow{\text{choose}} \tan\theta = \frac{\sqrt{b^2(x+c)^2}}{\sqrt{a^2}} = \frac{b(x+c)}{a}$	$(*)\boldsymbol{2^2 + 3^2(x+1)^2} \xrightarrow{\text{choose}} \tan\theta = \frac{\sqrt{3^2(x+1)^2}}{\sqrt{2^2}} = \frac{3(x+1)}{2}$

$Example$: $\int \frac{16x}{\sqrt{4-9x^2}}dx$

$\sin\theta = \frac{3x}{2} \to x = \frac{2}{3}\sin\theta \to dx = \frac{2}{3}\cos\theta\ d\theta$

$$\sqrt{4-9x^2} = \sqrt{4-9\left(\frac{2}{3}\sin\theta\right)^2} = \sqrt{4-4\sin^2\theta} = \sqrt{4\cos^2\theta} = 2\cos\theta$$

$$\int \frac{16x}{\sqrt{4-9x^2}}dx = \int \frac{16\left(\frac{2}{3}\sin\theta\right)}{(2\cos\theta)}\left(\frac{2}{3}\cos\theta\right)d\theta = \frac{32}{9}\int \sin\theta\, d\theta = -\frac{32}{9}\cos\theta + C = -\frac{16}{9}\sqrt{4-9x^2} + C$$

Section 15.5: Problems session

(1) $\int \frac{x^2}{(4x^2-1)^{3/2}}dx$. (Assume that $(x > 1/2)$)

(2) $\int \frac{1}{\sqrt{t^2-6t+13}}dt$

(3) $\int \frac{x^3}{\sqrt{1+x^2}}dx$

(4) $\int \frac{dx}{(4+x^2)^{3/2}}$

(5) $\int \frac{x^3}{\sqrt{1-x^2}}\, dx$

Section 16: Improper Integrals

Integral is called **convergent** if the limit exists and has a finite value and **divergent** if the limit doesn't exist or has infinite value

I. **Direct test:**

If f is continuous on the interval $(a,b]$, we define the improper integral $\int_a^b f(x)\ dx = \lim_{c\to a^+} \int_c^b f(x)\ dx$

If f is continuous on the interval $[a,b)$, we define the improper integral $\int_a^b f(x)\ dx = \lim_{c\to b^-} \int_a^c f(x)\ dx$

>>> **Important**: The following tests are used ONLY for positive functions. <<<

$$P-test: \quad \int_a^{\infty} \frac{1}{x^p}\, dx \ \rightarrow \begin{cases} p>1 & converges \\ p\le 1 & diverges \end{cases} \qquad (\ 0<a<\infty\ ) \qquad \int_0^{a} \frac{1}{x^p}\, dx \ \rightarrow \begin{cases} p<1 & converges \\ p\ge 1 & diverges \end{cases}$$

II. **Comparison test**:

Suppose $f(x)$ and $g(x)$ are two positive functions with $0 \le f(x) \le g(x)$. Then we have the following statements:

$$If \int_a^{\infty} g(x)\ dx\ \ converges\ then\ so\ does \int_a^{\infty} f(x)\ dx.\ \ 0 \le f(x) \le g(x)$$

$$If\ 0 \le g(x) \le f(x) \int_a^{\infty} g(x)\ dx\ \ diverges\ then\ so\ does \int_a^{\infty} f(x)\ dx.$$

Note: the inverse statements are NOT applicable.

$f(x)$ is given in the question, but $g(x)$ is chosen from the question. Generally, $g(x)$ has the form of $\frac{1}{x^p}$

III. **The Limit Comparison Test** (LCT):

Suppose f(x) and g(x) are two positive functions and $I_1 = \int_a^\infty f(x)\ dx \quad and \quad I_2 = \int_a^\infty g(x)\ dx.$

Further suppose that the 3 different conditions are $1.\ \lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = c \qquad 2.\ \lim\limits_{x\to 0^+} \frac{f(x)}{g(x)} = c \qquad 3.\ \lim\limits_{x\to a} \frac{f(x)}{g(x)} = c$

$I_1\ is\ improper \quad 1.\ at\ x = \infty \qquad 2.\ at\ x = 0 \qquad 3.\ at\ x = a \qquad where\ 0 < a < \infty$

1. If $c \neq 0$ and $c \neq \infty$ then both integrals converge of both divergent.

2. If $c = 0$ then $0 \le f(x) \le g(x).\ (*)\ check$ Comparison test

3. If $c = \infty$ then $0 \le g(x) \le f(x).\ (*)\ check$ Comparison test

Example: This problem has 3 unrelated parts about improper integrals:

(a) Evaluate the improper integral $I = \int_0^\infty xe^{-x^2}\ dx$

This integral is improper at $x = \infty$ Using Direct test, we have $\lim\limits_{R\to\infty} \int_0^R xe^{-x^2}\ dx$

$let\ I_R = \int_0^R xe^{-x^2}\ dx = -\frac{1}{2}e^{-x^2}\Big|_0^R = -\frac{1}{2}\left(e^{-R^2} - 1\right) = \frac{1}{2}\left(1 - e^{-R^2}\right)$ Hence, $\frac{1}{2}\lim\limits_{R\to\infty}\left(1 - e^{-R^2}\right) = \frac{1}{2}$

(b) Determine whether the integral $\int_0^\infty \frac{1}{(x+e^x)^{1/3}}\ dx$ is convergent or divergent.

This integral is improper at $x = \infty$ For $x \ge 0$, $x + e^x \ge e^x \quad \to \quad \frac{1}{x+e^x} \le \frac{1}{e^x}$

$$0 \le \frac{1}{(x+e^x)^{\frac{1}{3}}} \le \frac{1}{e^{\frac{x}{3}}} \quad where \int\limits_0^\infty \frac{1}{e^{\frac{x}{3}}}\ dx = \lim_{R\to\infty}\int\limits_0^R \frac{1}{e^{\frac{x}{3}}}\ dx = \lim_{R\to\infty}\int\limits_0^R e^{-\frac{x}{3}}\ dx = -3\lim_{R\to\infty}\left(e^{-\frac{R}{3}} - e^0\right) = 3$$

$\int_0^\infty \frac{1}{(x+e^x)^{\frac{1}{3}}}\ dx\ \ is\ convergent\ by\ comparison\ test.$

(c) Find the values of p for which the integral $\int_0^1 \frac{1}{(\tan x)^p}$ is convergent.

$$\lim_{x\to 0^+} \frac{\frac{1}{(\tan x)^p}}{\frac{1}{(x)^p}} = \lim_{x\to 0^+} \frac{x^p}{(\cos x)^p} * (\cos x)^p = 1 \quad for\ all\ p.\ Recall \int\limits_0^1 \frac{1}{x^p}\ dx \ \to p < 1\ \ converges$$

So, by the Limit Comparison Test, $\int_0^1 \frac{1}{(\tan x)^p}\ is\ converges\ for\ \ p < 1$

Section 16: Problems session

Evaluate the improper integral (1 to 7)

1) $\int_{-\infty}^0 xe^x\ dx$

2) $\int_0^\infty xe^{-x^2}\ dx$

3) $\int_1^\infty \frac{1}{x\,(\ln x)^2}\,dx$

4) $\int_{2019}^\infty \frac{\ln x}{x^2}\,dx$

5) $\int_{120}^\infty \frac{1}{x\,lnx\,\ln(\ln x)}\ dx$

6) $\int_0^3 \frac{x\,dx}{(9-x^2)^{3/2}}$

7) $\int_0^\infty \frac{x\,dx}{1+x^4}$

Determine if the integrals (8 to 17) are convergent or divergent:

8) $\int_0^\infty \frac{e^x}{x^{2/3}}\,dx$

9) $\int_0^{119} \frac{\sin^2 x}{x^{5/2}}\,dx$

10) $\int_0^\infty \frac{dx}{\sqrt{x}+x^3}$

11) $\int_0^\infty \frac{dx}{x^7+x^{1/3}}$

12) $\int_0^\infty \frac{dx}{x^{1/2}+x^2}$

13) $\int_0^{\pi/2} \frac{dx}{x \sin x}$

14) $\int_0^{4} \frac{dx}{x^2+x-6}$

15) $\int_1^{\infty} \frac{1-\cos x}{x^{5/2}+x+119}\,dx$

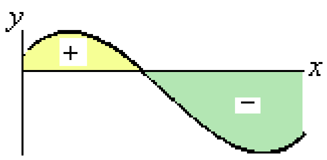
16) $\int_0^3 \frac{(1+e^{-x})\,(x^3+x+1)}{x^5}\,dx$

17) $\int_{\sqrt{3}}^{\infty} x^{-\arctan x}\,dx$

Question: *Let* f *be differentiable function on* $\mathbb{R}$ *such that* $\lim_{x\to+\infty} f(x) = 120$ *and* $\lim_{x\to-\infty} f(x) = 1$. *Find the* ***value*** *of the improper integral* $\int_{-\infty}^{\infty} f\,`(x)\,dx$. *Explain and show your work.*

Section 17: Areas of Plane Regions

Net Area: $\int_a^b f(x)\ dx$ represents the net area between $f(x)$ and the $x-axis$ is with area above $x-axis$ is positive and area below $x-axis$ negative.

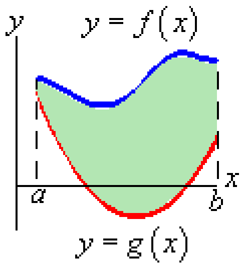


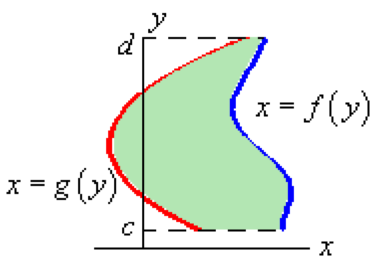
Area Between Curves: The general formulas for the two main cases for each are,

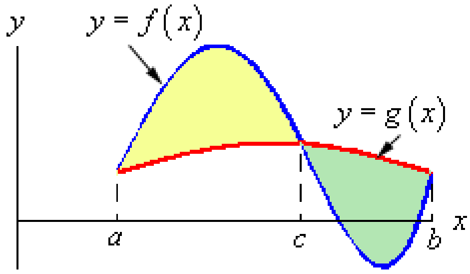
$y = f(x) \Rightarrow A = \int_a^b [upper\ function] - [lower\ function]\ dx$

$x = f(y) \Rightarrow A = \int_c^d [right\ function] - [left\ function]\ dy$

If the curves intersect then the area of each portion must be found individually. Here are some sketches of a couple possible situations and formulas for a couple of possible cases.







$$A = \int_a^b f(x) - g(x)\ dx \qquad A = \int_c^d f(y) - g(y)\ dy \qquad A = \int_a^c f(x) - g(x)\ dx + \int_c^b g(x) - f(x)\ dx$$

Example: find the area of the bounded, plane region R lying between the curves $y = x^2 - 2x$ and $y = 4 - x^2$.

Solution: first, we must find the intersections of the curves, so we solve the equations simultaneously:

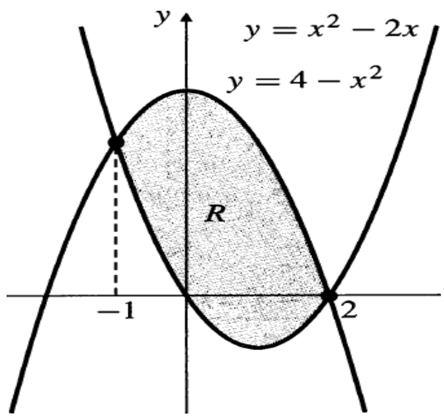
$x^2 - 2x = y = 4 - x^2 \qquad 2x^2 - 2x - 4 = 0 \quad 2(x-2)(x+1) = 0\ \ so\ x = 2\ or\ x = 1$

Since $4 - x^2 \geq x^2 - 2x\ \ for - 1 \leq x \leq 2, the\ area\ A\ of\ R\ is\ given\ by$

$$A = \int_{-1}^{2} \left((4 - x^2) - (x^2 - 2x)\right) dx$$

$$= \int_{-1}^{2} (4 - 2x^2 + 2x)\ dx = \left(4x - \frac{2}{3}x^3 + x^2\right)\Big|_{-1}^{2}$$

$$= 4(2) - \frac{2}{3}(8) + 4 - \left(-4 + \frac{2}{3} + 1\right) = 9\ \ square\ unites.$$



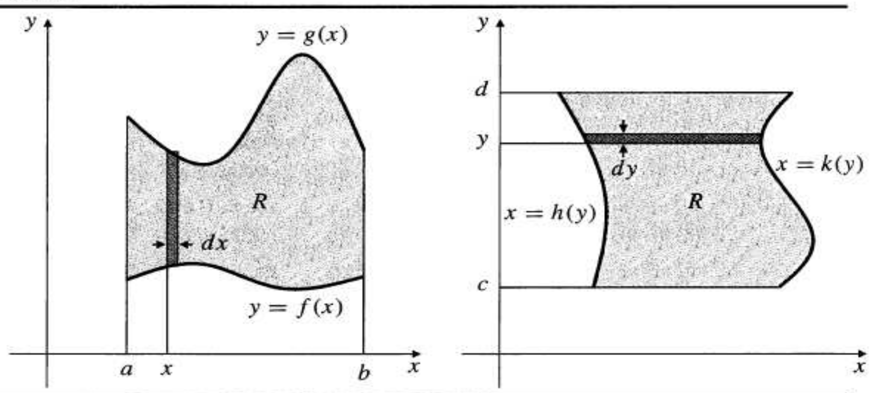
Section 17: Problems session

$Q1: Sketch\ and\ find\ the\ area\ of\ the\ finite\ region\ bounded\ by\ the\ curves\ y = x^3\ an\ y = 4x.$

$Q2$: $Write$ (***DO NOT EVALUATE***) $an\ integral$ ($or\ sum\ of\ the\ ingrals$) $which\ gives\ the\ area\ of\ the\ region\ lying\ below\ the\ line\ \sqrt{7}\,y = x + 1\ and\ inside\ the\ circl\ x^2 + y^2 = 1.$

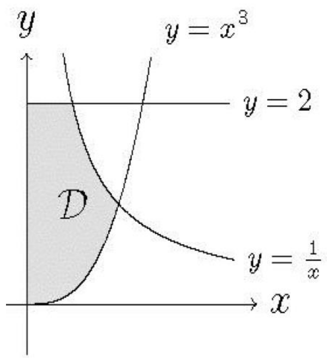
$Q3$: $Sketch\ the\ region\ between\ the\ parabolas\ y = 3 - x^2\ and\ y = x^2 - 2x - 1\ and\ find\ the\ area\ of\ this\ region.$

Table 1. Volumes of solids of revolution



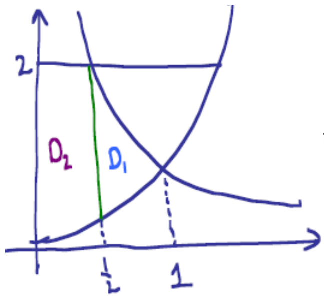
If region R ⟶ is rotated about ↓		
the x-axis	use plane slices $V = \pi \int_a^b ((g(x))^2 - (f(x))^2)\, dx$	use cylindrical shells $V = 2\pi \int_c^d y\,(k(y) - h(y))\, dy$
the y-axis	use cylindrical shells $V = 2\pi \int_a^b x\,(g(x) - f(x))\, dx$	use plane slices $V = \pi \int_c^d ((k(y))^2 - (h(y))^2)\, dy$

Example: let D be the region in the plane described by all four inequalities $x \geq 0,\ y \leq 2,\ y \geq x^3, y \leq \frac{1}{x}$ (see the figure on the right).



a) Let S be the solid of the revolution obtained by rotating the region D about y-axis.

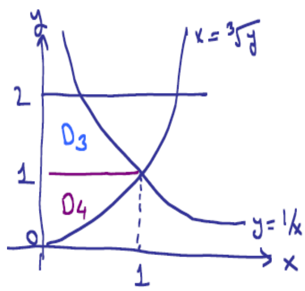
(I) Set up the integral(s) which give the volume of S using cylindrical shells method (DO NOT EVALUATE).



Ans: Divide D into two regions D_1 *and* D_2 using $v = \int_a^b 2\pi rh\, dx$ cylindrical shells method.

Thus, $v = \int_0^{1/2} 2\pi x(2 - x^3)\, dx + \int_{1/2}^{1} 2\pi x\left(\frac{1}{x} - x^3\right)\, dx$

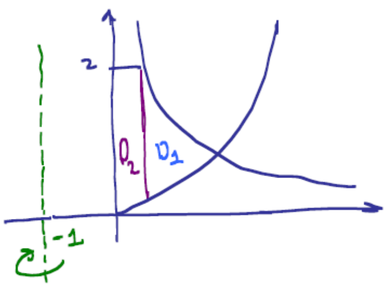
(II) Set up the integral(s) which give the volume of S using washer (slicing, disc) method (DO NOT EVALUATE).



Ans: using $v = \int_a^b \pi(R^2 - r^2)\, dx$ washer method

In D_4: $r = 0,\ R = \frac{1}{y}$ In D_4: $r = 0, R = \sqrt[3]{y}$ $\therefore v = \int_0^1 \pi\left(\sqrt[3]{y}\right)^2 dy + \int_1^2 \pi\left(\frac{1}{y}\right)^2 dy$

b) Let T be the solid of the revolution obtained by rotating the region D about the line $x = -1$. Set up but (DO NOT EVALUATE) the integral(s) which give the volume of T using cylindrical shells method.



Ans: using $v = \int_a^b 2\pi rh\, dx$ cylindrical shells method.

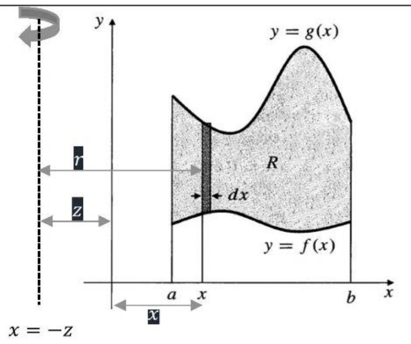
Thus, $v = \int_0^{1/2} 2\pi(x + 1)(2 - x^3)\, dx + \int_{1/2}^{1} 2\pi(x + 1)\left(\frac{1}{x} - x^3\right)\, dx$

$r = x + z \quad as\ x < 0$

$h = g(x) - f(x)$

$At\ z = 0\ (y - axis)$

$\therefore r = x$

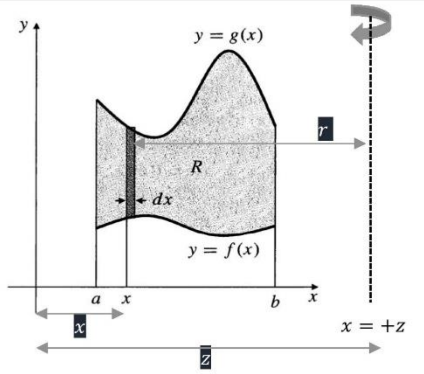


$r = z - x \quad as\ x > 0$

$h = g(x) - f(x)$

$At\ z = 0\ (y - axis)$

$\therefore r = x$

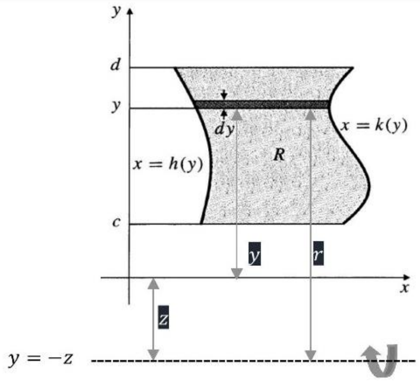


$r = y + z \quad as\ y < 0$

$h = k(y) - h(y)$

$At\ z = 0\ (x - axis)$

$\therefore r = y$

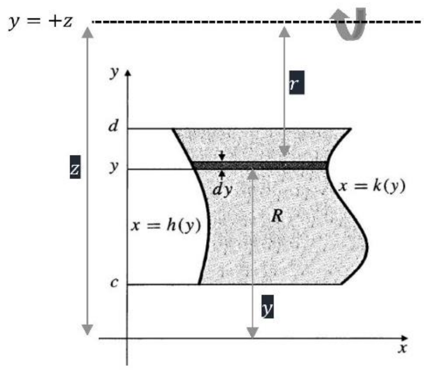


$r = z - y \quad as\ y > 0$

$h = k(y) - h(y)$

$At\ z = 0\ (x - axis)$

$\therefore r = y$



For the cylindrical shell method, the representative rectangle is always parallel to the axis of revolution (rotation)

$$V = 2\pi \int_a^b r\, h\, dx \quad (\text{Vertical Rectangle Element}) \qquad \leftrightarrow \qquad V = 2\pi \int_c^d r\, h\, dy \quad (\text{Horizontal Rectangle Element})$$

r: Distance between the axis of revolution and rectangle element. h: height of rectangle element

In a horizontal rectangle element: y: Distance between the x-axis and rectangle element (fixed). dy is the width of the rectangle element

In a vertical rectangle element: x: Distance between the y-axis and rectangle element (fixed). dx is the width of the rectangle element

$As\ \ x < 0$

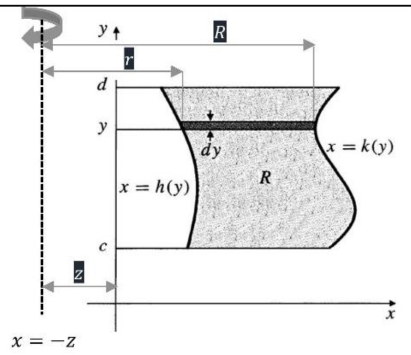
$R = k(y) + z$

$r = h(y) + z$

$At\ \ z = 0\ (y - axis)$

$\therefore R = k(y)$

$\therefore\ r = h(y)$



$As\ \ x > 0$

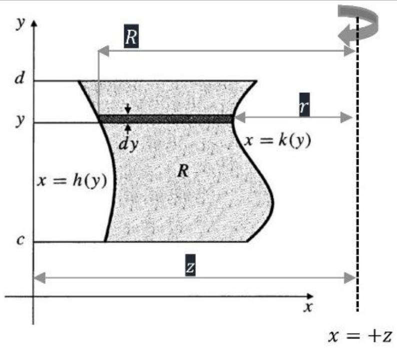
$R = z - h(y)$

$r = z - k(y)$

$At\ \ z = 0\ (y - axis)$

$\therefore R = k(y)$

$\therefore\ r = h(y)$



$As\ \ y < 0$

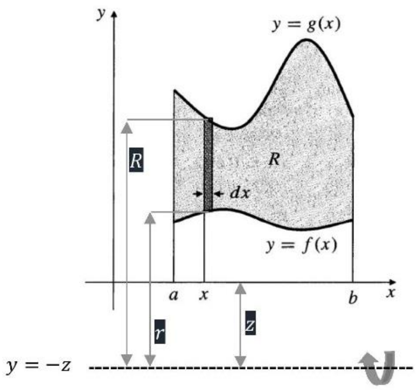
$\therefore R = g(x) + z$

$\therefore r = f(x) + z$

$At\ \ z = 0\ (x - axis)$

$\therefore R = g(x)$

$\therefore\ r = f(x)$



$As\ \ y > 0$

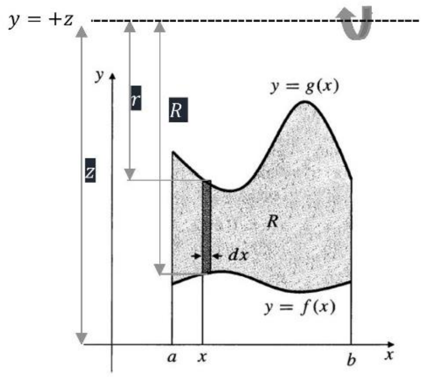
$\therefore R = z - f(x)$

$\therefore r = z - g(x)$

$At\ \ z = 0\ (x - axis)$

$\therefore R = g(x)$

$\therefore\ r = f(x)$



For the disc (slicing, washer) method, the representative rectangle is always perpendicular to the axis of revolution (rotation)

$$V = \pi \int_a^b (R^2 - r^2)\, dx \quad (\text{Vertical Rectangle Element}) \qquad \leftrightarrow \qquad V = \pi \int_c^d (R^2 - r^2)\, dy \quad (\text{Horizontal Rectangle Element})$$

R (Outer Radius): Distance between the axis of revolution and the ending (furthest) edge of the rectangle element.

r (Inner Radius): Distance between the axis of revolution and the starting (nearest) edge of the rectangle element.

In a horizontal rectangle element: dy is the width of the rectangle element

In a vertical rectangle element: dx is the width of the rectangle element

$Q1$: $Let\ R\ the\ region\ bounded\ by\ the\ curves\ y = 10 - (x-1)^2\ \ and\ y = (x-3)^2$

$a)$ $Write\ a\ difinite\ integral\ which\ is\ equal\ to\ the\ volume\ of\ the\ solid\ obtained\ by\ rotating\ the\ region\ R\ about\ x\text{–}axis.\ DO\ NOT\ EVALUATE.$

$b)$ $Write\ a\ difinite\ integral\ which\ is\ equal\ to\ the\ volume\ of\ the\ solid\ obtained\ by\ rotating\ the\ region\ R\ about\ the\ line\ x = 0.\ DO\ NOT\ EVALUATE.$

$Q2$: $Let\ R\ the\ region\ bounded\ from$ above $the\ x\text{–}axis\ and\ from$ below $by\ the\ curv\ y = 2x - x^2.\ R\ is\ rotated\ about\ x = -1\ to\ generate\ a\ solid.\ Write$ ($\boldsymbol{DO\ NOT\ EVALUATE}$) $integrals\ giving\ the\ volume\ V\ of\ the\ resulting\ solid\ of\ the\ revolution\ by\ using\ following\ methods;$

$a)$ $the\ cylindrical\ shell\ method.$

$b)$ $the\ disc\ (slicing, washer)\ method.$

$Q3$: *Let* R be the region bounded by the parabola $y = x^2 + 1$ and the line $y = x + 3$.

Let S the solid obtained by rotated about R about the x–axis.In each part sketch the region R,draw a typical rectangle (area element dA) and a typical volume (volume element dV) and use them to write a difinte integral (or sum of the ingrals,DO NOT EVALUATE) which gives the volume of S using

a) the slicing (disc) method.

b) the cylindrical shell method.

$Q4$: *Let R the finite region bounded by the curves* $y = x^2$ *and* $y = x + 2$

a) Sketch the region R.

b) Set up a difinite integral which gives the ***volume*** *of the solid obtained by rotating the region R about x–axis. DO NOT THE EVALUATE*

c) Set up a difinite integral which gives the ***volume*** *of the solid obtained by rotating the region R about the* ***line*** $x = -1$. *DO NOT THE EVALUATE*

Section 19: Arc Length

$s = \int_{x=a}^{x=b} ds \quad where\ ds = \sqrt{1 + \left(f'(x)\right)^2}dx \qquad \leftrightarrow \qquad s = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}dx$

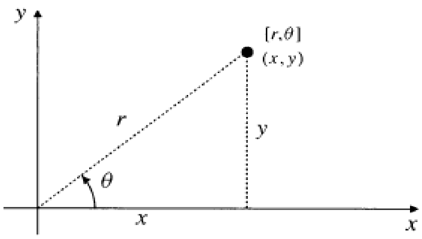
Ex: $Write\ a\ difinite\ integral\ which\ is\ equal\ to\ the\ \boldsymbol{arclengh}\ of\ the\ curve\ y = e^x\ between\ the\ point\ (0,1)\ and\ (1, e).\ DO\ NOT\ EVALUATE.$

$s = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}dx\ \ where \begin{cases} f'(x) = e^x \\ a = 0\ ,\ \ b = 1 \end{cases} \qquad \therefore arclengh = \int_0^1 \sqrt{1 + e^{2x}}dx$

Section 19: Problems session

$Q1$: $Write\ a\ difinite\ integral\ which\ is\ equal\ to\ the\ \boldsymbol{arclengh}\ of\ the\ curve\ y = \ln\left(\frac{x+3}{2-x}\right)\ between\ the\ point\ \left(0, \ln\left(\frac{3}{2}\right)\right)\ an\ (1, \ln(4))\ .\ DO\ NOT\ EVALUATE.$

$Q2$: $Write\ the\ \boldsymbol{arclengh}\ of\ the\ curve$ given $y = f(x) = \int_1^x \sqrt{t^3 - 1}\,dt \quad from\ x = 1\ \ to\ x = 2$



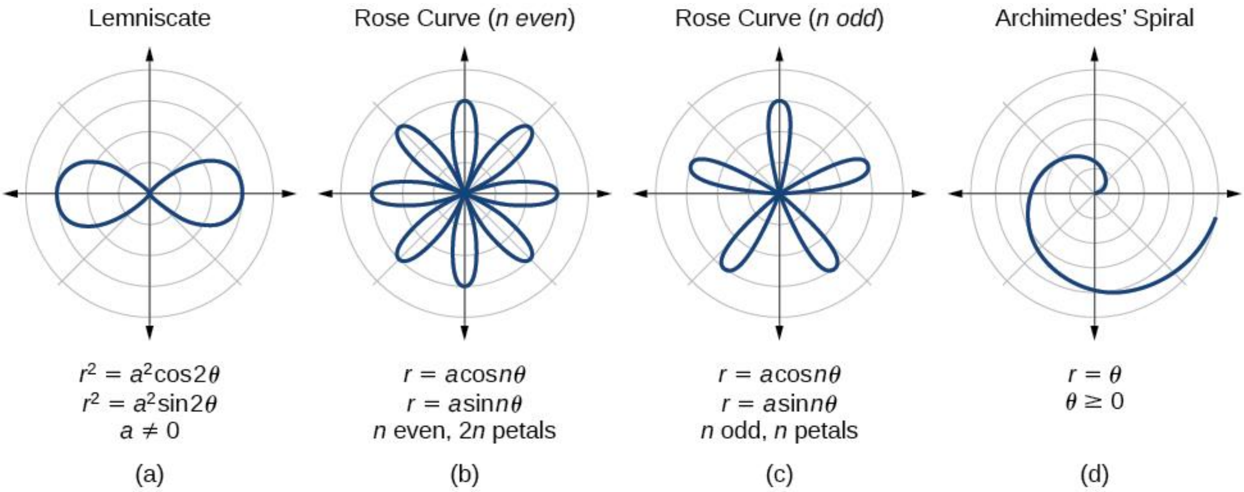
Section 20: Polar Coordinates and Polar Curves

Polar-rectangular conversion

$x = r\cos\theta \qquad\qquad y = r\sin\theta \qquad x^2 + y^2 = r^2 \qquad \tan\theta = \frac{y}{x}$

Ex: $Find\ the\ Cartesian\ equation\ of\ the\ curve\ represented\ by\ the\ polar\ eqation\ r = \sin 2\,\theta$

$r = \sin 2\,\theta\ = 2\sin\theta\cos\theta \quad \leftrightarrow r^3\ = 2\,(r\sin\theta)(r\cos\theta) \quad \rightarrow \sqrt[3]{x^2 + y^2} = 2xy$

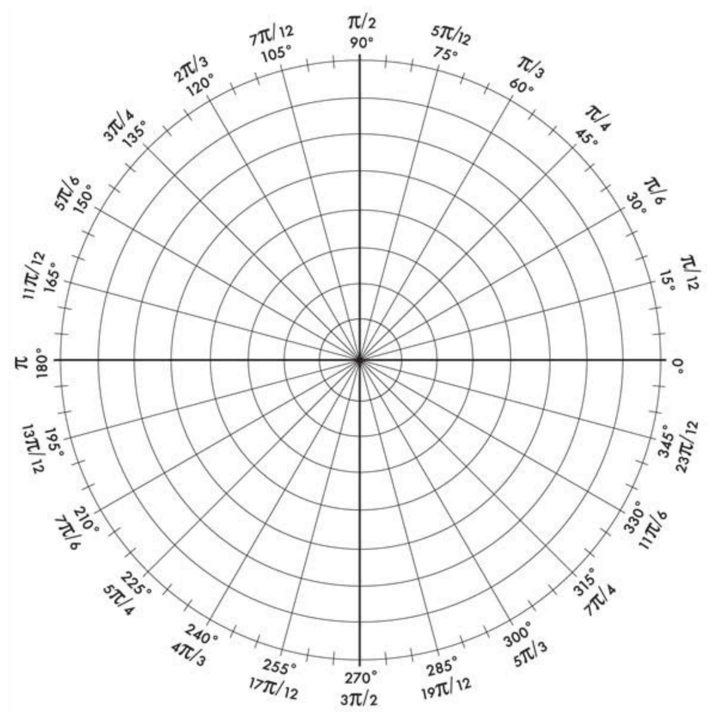


Section 20: Problems session

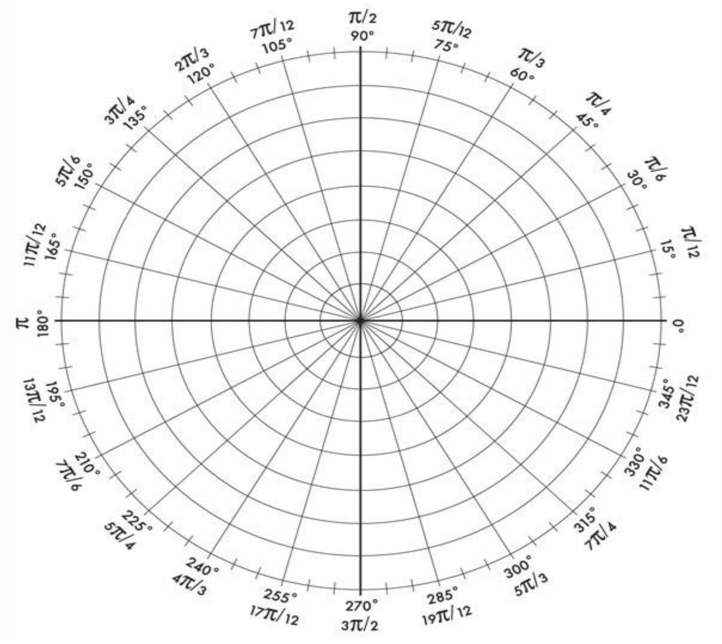
$Q1$: *Find the Cartesian equation of the curve represented by the polar eqation* $r = \sin\theta + \cos\theta$

$Q2$: *Find the Cartesian equation of the curve represented by the polar eqation* $r = \frac{2}{2 - cos\theta}$

$Q3$: *Let C the curve given by the polar equatio* $r = \sin 2\theta$. *Sketch the graph of the part the curve C in the first quadrant* (*this is, where* $x \geq 0,\ y \geq 0$).



$Q4$: *Let C the curve given by the polar equatio* $r = 2(1 - \sin\theta)$. *Sketch the graph of the curve C.*



Answer key:

Section 1:

1. 0
2. $\tan(1)-1$
3. $-\infty$
4. $\frac{7\sqrt{30}}{30}$
5. $\frac{1}{2}$
6. 0
7. 2
8. 0
9. $D.N.E$
10. $D.N.E$
11. $D.N.E$
12. -3
13. -1
14. $D.N.E$
15. 0
16. $D.N.E$
17. 0
18. 0
19. 0
20. 0
21. 0
22. 1
23. 0
24. 0
25. 4
26. 0
27. 0
28. -2
29. 2
30. 1
31. 0
32. 0
33. $\frac{1}{2}$
34. $-\infty$
35. $\frac{1}{2}$
36. $\frac{-1}{2}$
37. 2
38. 4
39. ∞
40. ∞
41. $-\infty$
42. $-\infty$
43. $\frac{1}{2}$
44. $\frac{1}{2}$
45. $\frac{1}{4}$
46. -1
47. 2
48. 2
49. 0
50. ∞
51. $\frac{-2}{\sqrt{3}}$
52. 3
53. -1
54. $324\ln 3$
55. $\frac{\ln 2}{4}$
56. $-4\sin 4$
57. $\frac{e}{6}$

Section 2:

Q1: b) $f'(0)=2$

Q2: $m=2(119)$ $\quad b=\pm 119$

Q3: B. $f'(1)=3$ $\quad$ C. Yes

Q4: $b=0$ $\quad a=\frac{1}{2}$

Q5: $f'(0)=0$

Q6: a $=0$

Q7: a $=2$

Q8: $D.N.E$

Q9: $D.N.E$

Q10: there is NO such a & b points

Section 3:

Part I:

a) $\frac{e^x+ex^{e-1}}{\sqrt{1-(e^x+x^e)^2}}$

b) $(\sin x)^{\cos x}\left[-(\sin x)\ln(\sin x)+\frac{\cos^2 x}{\sin x}\right]$

c) $(2\tan(\sec(x^2)))*(\sec^2(\sec(x^2)))*(\sec(x^2)\tan(x^2))*2x$

d) $(1+x)^{\sin x}\left[\cos x\ \ln(1+x)+\frac{\sin x}{1+x}\right]$

e) $3x^2\arcsin(x^2-1)+x^3\frac{2x}{\sqrt{1-(x^2-1)^2}}$

f) $f'(1)=\frac{-e}{1+e}$

Part II:

a) $2\left(\cos\sqrt{x}\right)\left(-\sin\sqrt{x}\right)*\frac{1}{2\sqrt{x}}+\frac{-1/x}{\ln^2 x}$

b) $\frac{\frac{1}{\sqrt{1-1/\left(1+x^2\right)^2}}\left(\frac{-2x}{\left(1+x^2\right)^2}\right)(1+\cos x)-\ \arcsin\left(\frac{1}{1+x^2}\right)*(-\sin x)}{(1+\cos x)^2}$

c) $\left(1+\frac{3}{x}+\frac{5}{x^2}\right)^x\left[\ln\left(1+\frac{3}{x}+\frac{5}{x^2}\right)+(x)\left(\frac{1}{1+\frac{3}{x}+\frac{5}{x^2}}\right)\left(\frac{-3}{x^2}+\frac{-10}{x^3}\right)\right]$

d) $(\sin x)^{\cos(x^3)}\left[-\sin(x^3)\,(3x^2)\ln(\sin x)+\cos(x^3)\left(\frac{1}{\sin x}\right)(\cos x)\right]$

e) $(\sin x)^{\cos(x^3)}\left[-\sin(x^3)\,(3x^2)\ln(\sin x)+\cos(x^3)\left(\frac{1}{\sin x}\right)(\cos x)\right]$

Part III:

a) $\frac{1}{2\sqrt{\frac{\sin(x^2)}{\tan x}}}\left[\frac{2x\,cos(x^2)-\sin(x^2)\sec^2 x}{\tan^2 x}\right]$

b) $(x+x^2)^{\ln x}\left[\frac{\ln(x+x^2)}{x}+\ln x\frac{(1+2x)}{1+x}\right]$

Part IV:

a) $2\,sin(2x)\,cos(2x)*2+\frac{e^x\ln x-\frac{e^x}{x}}{(\ln x)^2}=2\,sin(4x)+\frac{e^x(x\ln x-1)}{x\,(\ln x)^2}$

b) $\frac{\cos x}{\sin x}-\frac{1}{3}\frac{2+2x}{2x+x^2}$

c) $\frac{(\pi\,x^{\pi-1}-\sin x)(1+x^3)-(x^\pi+\cos x)(3x^2)}{(1+x^3)^2}$

d) $\sec(sec\,x)*\ \tan\,(sec\,x)*\ sec\,x*\ tan\,x$

e) $3^x\ln 3+\ln 3\ \frac{-1}{x\,(\ln x)^2}=\ln 3\left[3^x-\frac{1}{x\ln^2 x}\right]$

f) $h'(-2)=8$

g) $f'(1)=\frac{-1}{e}$

Part V:

Q1: b) $(f^{-1})'(1)=\frac{1}{2}$

Q2: b) $(f^{-1})'(\pi)=1$

Q3: $(f^{-1})'(2)=\frac{1}{2}$

Section 4:

Q1: $y=x$

Q2: b) $y=11(x-3)+1$

Q3: a) $(1,+2)\ and\ (1,-2)$

b) $y=\frac{1}{2}(x-1)-2$ $\quad$ c) NO

Q4: $y=-\ln 2\ \left(x-\frac{\pi}{2}\right)+1$

Q5: $y=\frac{-1}{\pi+1}(x-\pi)$ **Tangent** line

$y=(\pi+1)(x-\pi)$ **Normal** line

Q6: $y=x\ \ and\ y=28(x-3)+30$

Q7: $y=\frac{1}{8}(x+1)$

Q8: $f'(0)=0$

Section 6:

Q1: $V=8\ cm^3$

Q2: $\frac{38}{9\sqrt{13}}\ unit/sec$

Q3: $-4\ unit/\sec^2$

Q4: $10\ cm^2/sec$

Q5: a) $\frac{8}{3}\ unit\ /sec$ $\quad$ b) $4\ \ unit/sec$

Q6: $\frac{-1}{20}\ \ radian\ /sec$

Q7: part 1: $\frac{1}{s}(8\sin t-6\sin 2t)$

Part 2: $2\sqrt{2}\ unit/sec$

Section 7:

1) $\frac{1}{2}$ 2) 0 3) $\frac{1}{2}$
4) 0 5) e^2 6) 1
7) 1 8) e^8 9) e
10) $\frac{1}{2}$ 11) $\frac{1}{5}$ 12) -2
13) $\frac{1}{119}$ 12) $e^{-\frac{\pi}{2}}$ 13) $\frac{\sqrt{2}}{3}$

Section 9:

Part 1:

Q1: $\frac{4}{\sqrt{3}}\ unit$ and $\frac{8}{3}\ unit$
Q2: $3\ cm$
Q3: $1\ unit^2$
Q4: 15 trees
Q5: $x = 1 + 3^{\frac{2}{3}}$ & $y = 3 + 3^{\frac{1}{3}}$ $\quad h = \left(1 + 3^{\frac{3}{2}}\right)^{3/2}$

Part 2:

Q1: $f(-1) = f(5) = 25^{2/3}\ is\ abs.\ max.value$
$f(0) = f\left(\frac{9-\sqrt{21}}{2}\right) = 0\ \ is\ abs.\ min.value$
Q2:
$f(1) = e^{-2}\ is\ abs.\ max.value$
$there\ is\ no\ abs\ min.point\ on\ (0,\infty)$
Q3:
(b) $f(1) = 0\ \ is\ the\ abs\ min.value$
$f(2) = 3\ \ is\ the\ abs\ max.value$

Section 10:

Q1: $L(x) = 2x - 3$ $\quad f(3.1) = 3.2$
Q2: $L(x) = \frac{x-\pi}{\pi+1}$ $\quad f(3) = \frac{3-\pi}{\pi+1}$
Q3: $L(x) = 2 + \frac{1}{80}(x - 32)$ $\quad f(3) = \frac{161}{80}$
Q4: $L(x) = \frac{1}{2}(x-1)$ $\quad f(3) = \frac{1}{4}$
Q5: $L(x) = 2e(x-1)$ $\quad f(3) = \frac{e}{5}$

Section 11:

Q1) $\frac{2}{3}$ Q2) $\frac{21}{2}$ Q3) $\frac{3}{5}$
Q4) 2 Q5) $\frac{\pi}{4}$ Q6) $\frac{1}{3}\left[(17)^{\frac{3}{2}} - 1\right]$
Q(B) $\lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^{n}\left(sin\left(1 + \frac{2}{n}i\right)\right)^{\left(1+\frac{2}{n}i\right)+1}$

Section 12:

Q1: $L(f,p) = \frac{7}{4}$ $\quad U(f,p) = \frac{15}{4}$
Q2: $L(f,p) = -\pi$ $\quad U(f,p) = \pi$
Q3: $L(f,p) = 1$ $\quad U(f,p) = 3$

Section 13:

1) 8 2) 2π
3) 0 4) $\frac{9}{2}\pi$

Section 14:

1) $\frac{16}{5}e^{-\frac{16}{5}}$ 2) $\frac{1}{e}$
3) $\frac{1}{3}$ 4) 0
5) 119

Section 15.2:

(1) $\frac{2^{5/2}}{3} - \frac{2}{3} + 2\left(3^{-1/4}\right)$
(2) $\frac{-8}{105}$
(3) $\frac{2}{5}\cos^{5/2}x - 2\cos^{1/2}x + c$
(4) $\frac{4}{3}\left[\left(1+\sqrt{2}\right)^{3/2} - 2^{3/2}\right]$
(5) $\frac{\ln^2(\ln x)}{2} + c$
(6) $\frac{\sec^5 x}{5} - \frac{2}{3}\sec^3 x + \sec x + c$
(7) $\ln((x+1)^2 + 2) - \sqrt{2}\arctan\left(\frac{x+1}{\sqrt{2}}\right) + c$
(8) $2\left(\frac{x^{5/2}}{5} + \frac{x^{3/2}}{3} + \sqrt{x}\right) + c$
(9) $\frac{\pi^2}{72}$
(10) $2\left(\frac{x^{3/2}}{3} - \frac{x}{2} + 2\sqrt{x} - 2\ln\left|\sqrt{x}+1\right|\right) + c$
(11) $\frac{(1+x^2)^{3/2}}{3} - \sqrt{1+x^2} + c$
(12) $-\frac{\ln 2}{2}$

Section 15.3:

(1) $-\frac{1}{3}e^{3x}\cos(e^{3x}) + \frac{1}{3}\sin(e^{3x}) + c$
(2) $4e^2 - 2e$
(3) $\frac{x^2}{2}\arctan(x^2) - \frac{1}{4}\ln(1+x^4) + c$
(4) $e^x(x^2 - 2x + 2) + c$
(5) $\frac{(x-1)^3}{3}\ln(x-1) - \frac{(x-1)^3}{9} + c$
(6) $3x^{2/3}e^{\sqrt[3]{x}} - 6\sqrt[3]{x}e^{\sqrt[3]{x}} + 2e^{\sqrt[3]{x}} + c$
(7) $\frac{2\sqrt{3}\pi}{3} - \frac{\pi}{2} - \ln 2$
(8) $\frac{-x^2e^{x^2}}{2(x^2+1)} + \frac{1}{2}e^{x^2} + c$

Section 15.4:

(1) $\ln|x| - \frac{1}{x} - \frac{1}{2}\ln(x^2+1) - \arctan x + c$
(2) $\ln|x| + \frac{1}{x} + \frac{1}{4}\ln(x^2+1) - \frac{1}{\sqrt{2}}\arctan\left(\sqrt{2}\,x\right) + c$
(3) $2\ln|x| + 5\arctan(x+2) - \frac{1}{2}\ln((x+2)^2+1) + c$
(4) $2\ln|x| + \frac{1}{4x} - \ln(x^2+4) - \frac{1}{8}\arctan\left(\frac{x}{2}\right) + c$
(5) $\ln\frac{|x-1|}{\sqrt{x^2+9}} - \frac{1}{3}\arctan\left(\frac{x}{3}\right) + c$
(6) $x + \frac{1}{x} + \ln\frac{(x-1)^2}{x} + c$
(7) $-\ln|x| + \frac{1}{x} + \frac{1}{2x^2} - \ln|x-1| + c$
(8) $\frac{3}{2}\ln\left|x^{1/3}+1\right| + \frac{3}{4}\ln\left|x^{2/3}+1\right| - \frac{3}{2}\arctan\left(x^{1/3}\right) + c$
(9) $\frac{1}{119}\ln\left|\frac{x^{119}}{1+x^{119}}\right| + c$

Section 15.5:

(1) $\frac{1}{8}\left[\ln\left|2x + \sqrt{4x^2-1}\right| - \frac{2x}{\sqrt{4x^2-1}}\right] + c$
(2) $\ln\left|\frac{\sqrt{(t-3)^2+4}}{2} + \frac{t-3}{2}\right| + c$
(3) $-\sqrt{1+x^2} + \frac{(1+x^2)^{\frac{3}{2}}}{3} + c$
(4) $\frac{1}{4}\frac{x}{\sqrt{4+x^2}} + c$
(5) $\frac{1}{2}\left[\arcsin x - x\sqrt{1-x^2}\right] + c$

Section 16:

(1) -1 (convergent)
(2) $\frac{1}{2}$ (convergent)
(3) ∞ (divergent)
(4) $\frac{\ln 2019+1}{2019}$ (convergent)
(5) ∞ (divergent)
(6) ∞ (divergent)
(7) $\frac{\pi}{4}$ (convergent)
(8) divergent
(9) convergent
(10) convergent
(11) convergent
(12) convergent
(13) divergent
(14) divergent
(15) convergent
(16) divergent
(17) convergent
(Question) 119

Section 17:

Q1: $8\ unit^2$
Q2: $A = \int_{-1}^{3/4} \frac{1}{\sqrt{7}}(x+1) - (-\sqrt{1-x^2})\, dx + 2\int_{3/4}^{1} \sqrt{1-x^2}\, dx$
Q3: 9

Section 18:

Q1:
a) $V = \pi \int_0^4 [(10-(x-1)^2)^2 - (x-3)^4]\, dx$
b) $V = 2\pi \int_0^4 (4-x)\, [10-(x-1)^2-(x-3)^2]\, dx$

Q2:
a) $V = 2\pi \int_0^2 (1+x)\, [2x - x^2]\, dx$
b) $V = 8\pi \int_0^1 \sqrt{1-y}\, dy$

Q3:
a) $V = \pi \int_{-1}^{2} [(x+3)^2 - (x^2+1)^2]\, dx$
b) $V = 4\pi \int_1^2 y\sqrt{y-1}\, dy + 2\pi \int_2^5 y\left(\sqrt{y-1} - y + 3\right) dy$

Q4:
b) $V = \pi \int_{-1}^{2} [(x+2)^2 - x^4]\, dx$
c) $V = 2\pi \int_{-1}^{2} (1+x)\, [x+2-x^2]\, dx$

Section 19:

Q1: $\int_0^1 \sqrt{1+e^{2x}} dx$
Q2: $\frac{2}{5}\left(\sqrt{32}-1\right)$

Section 20:

Q1: $x^2 + y^2 = y + x$
Q2: $2\sqrt{x^2+y^2} - x = 2$