

Problem Set V

Content: Confidence Intervals and Hypothesis Testing

Question 1

Supporters claim that a new windmill can generate an average of at least 800 kilowatts of power per day. Daily power generation for the windmill is assumed to be normally distributed with a standard deviation of 120 kilowatts. A random sample of 100 days is taken to test this claim against the alternative hypothesis that the true mean is less than 800 kilowatts. The claim will be accepted if the sample mean is 776 kilowatts or more and rejected otherwise.

- What is the probability α of a Type I error using the decision rule if the population mean is in fact 800 kilowatts per day?
- What is the probability β of a Type II error and power of the test using this decision rule if the population mean is in fact 740 kilowatts per day?
- Suppose that the same decision rule is used, but with a sample of 200 days rather than 100 days.
 - Would the value of α be larger than, smaller than, or the same as that found in (a)?
 - Would the value of β be larger than, smaller than, or the same as that found in (b)?

Solution:

We have the data: $\sigma = 120, n = 100$ $H_0 : \mu \geq 800 \rightarrow \text{reject } H_0 \text{ if } \bar{X} < 776$

$$H_1 : \mu < 800$$

Type I error = Rejecting null hypothesis when it is true.

Type II error = Fail to reject null hypothesis when it is false.

a. $P(\bar{X} < 776 \mid \mu = 800) = \alpha$

$$P\left(\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} < \frac{776 - 800}{\frac{120}{\sqrt{100}}}\right) = \alpha \rightarrow P(Z < -2) = \alpha = 0.0228$$

- b. $P(\bar{X} \geq 776 \mid \mu = 740) = P\left(\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \geq \frac{776 - 740}{\frac{120}{\sqrt{100}}}\right) = \beta \rightarrow P(Z \geq 3) = \beta = 0.0013$
- c. Now we take $n = 200$
- α would be smaller, because as we increase the size of the sample, standard error of the sample decreases.
 - β would be smaller too.

Question 2

Twelve randomly selected mature citrus trees of one variety have a mean height of 13.8 feet with a standard deviation of 1.2 feet, and 15 randomly selected mature citrus trees of another variety have a mean height of 12.9 feet with a standard deviation of 1.5 feet. Assuming that the random samples were selected from normal populations with equal variances;

- Construct a 95% confidence interval for the difference between the true average heights of the two kinds of citrus trees.
- Construct a 98% confidence interval for the ratio of the variances of the two populations sampled.

Solution:

- We use s_p since both of these samples are small and their population variances are equal. And also, we don't know population variances (that is, unknown σ s + small samples) so we use t distribution.

$$s_p^2 = [11 \cdot (1.2)^2 + 14 \cdot (1.5)^2] / 25 = 1.8936 = 1.376 \quad s_p = 1.376$$

$$(\bar{X}_1 - \bar{X}_2) - t_{\alpha/2, n_1 + n_2 - 2} \cdot s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} < \mu_1 - \mu_2 < (\bar{X}_1 - \bar{X}_2) + t_{\alpha/2, n_1 + n_2 - 2} \cdot s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

2.060 is the t-table value of $0.025(\alpha/2)$ and degrees of freedom $n_1 + n_2 - 2 = 12 + 15 - 2 = 25$

$$(13.8 - 12.9) \pm 2.060(1.376) \sqrt{(1/12 + 1/15)} = 0.9 \pm 1.098 \text{ therefore;}$$

$$-0.198 < \mu_1 - \mu_2 < 1.998 \text{ feet}$$

- If s_1^2 and s_2^2 are the values of the variances of "independent" random samples of twelve and n_2 from normal populations, then $(1 - \alpha)100\%$ confidence interval for σ_1^2/σ_2^2 is:

$$(s_1^2/s_2^2) (1/f_{\alpha/2, n_1-1, n_2-1}) < \sigma_1^2/\sigma_2^2 < (s_1^2/s_2^2) (f_{\alpha/2, n_2-1, n_1-1})$$

remember that

$$\rightarrow (1/f_{\alpha/2, n2-1, n1-1}) = f_{1-\alpha/2, n1-1, n2-1} \quad (\text{dofs also change place})$$

so the answer is:

$$\begin{aligned} & ((1.2)^2/(1.5)^2) (1/f_{0.01, 11, 14}) < \sigma_1^2/\sigma_2^2 < ((1.2)^2/(1.5)^2) f_{0.01, 14, 11} \\ & = (0.64)/(3.87) < \sigma_1^2/\sigma_2^2 < (0.64) \cdot (4.3) \rightarrow 0.165 < \sigma_1^2/\sigma_2^2 < 2.752 \end{aligned}$$

Note that the interval obtained here includes the possibility that the ratio is 1, meaning, there is no real evidence against the assumption of equal population variances.

However, we already know that they are given equal in the question. So the truth of our answer is proved.

Question 3

When a process producing ball bearings is operating correctly, the weights of the ball bearings have a normal distribution with mean 5 ounces and standard deviation 0.1 ounce. An adjustment has been made to the process, and the plant manager suspects that this has raised the mean weight of ball bearings produced, leaving the standard deviation unchanged. A random sample of sixteen ball bearings is taken, and their mean weight is found to be 5.038 ounces.

- Test the null hypothesis that the true mean weight of ball bearings is 5 ounces against the alternative that the true mean is bigger than 5 ounces. Use $\alpha = 5\%$.
- Find the probability that the null hypothesis will be accepted if the true mean weight is 5.05 ounces.
- What is the power of the test?

Solution:

- $H_0 : \mu = 5 \quad n = 16, \sigma = 0.1, Z_\alpha = Z_{0.05} = 1.645$
 $H_1 : \mu > 5$
 $Z^c = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{5.038 - 5}{0.1/\sqrt{16}} = 1.52 < 1.645 \rightarrow \text{so fail to reject!}$
- The null hypothesis is rejected if

$$\frac{\bar{x} - 5}{0.1/4} > 1.645 \text{ or } \bar{x} \leq 5.041$$

In effect, the question is to find the probability that the sample mean does not exceed 5.041 for a random sample of sixteen observations from a normal population with mean $\mu_1 = 5.05$ and standard deviation 0.1.

This is;

$$P\left(Z \leq \frac{5.041 - \mu_1}{\sigma/\sqrt{n}}\right) = P\left(Z \leq \frac{5.041 - 5.05}{0.1/\sqrt{16}}\right) = P(Z \leq -0.36) = 0.3594 \text{ the probability of Type II error} = \beta$$

- c. power of the test $= 1 - \beta = 1 - 0.3594 = 0.6406$.

Question 4

Suppose $X_1, \dots, X_{16}$ is a random sample of size $n = 16$ from a normal distribution, $X_i \sim N(\mu, 1)$, and we wish to test $H_0 : \mu = 20$ at significance level $\alpha = 0.05$, based on the sample mean $\bar{X}$.

- Find the probability of a Type II error and power of the test for the alternative $H_1 : \mu = 21$.
- Find the probability of a Type II error and power of the test for the alternative $H_1 : \mu = 19$.

Solution:

- a. The rejection region for a 0.05 level test was given by

$$P(\bar{X} > k \mid \mu = \mu_0) = \alpha$$

$$P\left(\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} > \frac{k - \mu_0}{\sigma/\sqrt{n}}\right) = \alpha$$

$$P\left(z > \frac{k - 20}{1/\sqrt{16}}\right) = 0.05 \quad \frac{k - 20}{1/\sqrt{16}} = 1.645 \quad k = 20.411$$

$$\beta = P(\text{do not reject } H_0 \mid \text{under } H_1) = P(\bar{X} \leq k \mid \mu = \mu_1)$$

$$P\left(\frac{\bar{X} - \mu_1}{\sigma/\sqrt{n}} \leq \frac{k - \mu_1}{\sigma/\sqrt{n}}\right) = P\left(z \leq \frac{20.411 - 21}{1/\sqrt{16}}\right) = P(z \leq -2.356) = 0.0091$$

- b. $P(\bar{X} < c \mid \mu = \mu_0) = \alpha$

$$P\left(\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} < \frac{c - \mu_0}{\sigma/\sqrt{n}}\right) = \alpha$$

$$P\left(z < \frac{c-20}{1/\sqrt{16}}\right) = 0.05 \quad \frac{c-20}{1/\sqrt{16}} = -1.645 \quad c = 19.589$$

$$\beta = P(\text{do not reject } H_0 \mid \text{under } H_1) = P(\bar{X} \geq c \mid \mu = \mu_1)$$

$$P\left(\frac{\bar{X}-\mu_1}{\sigma/\sqrt{n}} \geq \frac{c-\mu_1}{\sigma/\sqrt{n}}\right) = P\left(z \geq \frac{19.589-19}{1/\sqrt{16}}\right) = P(z \geq 2.356) = 0.0091$$

Question 5

The mean mass of 50 male students who showed above average participation in college athletics was 68.2 kg with a standard deviation of 2.5 kg, while 50 male students who showed no interest in such participation had a mean mass of 67.5 kg with a standard deviation of 2.8 kg.

- Test the hypothesis that male students who participate in college athletics are more massive than other male students.
- By how much should the sample size of each of the two groups be increased in order that the observed difference of 0.7 kg in the mean masses be significant at the level of significance i) 0.05 and ii 0.01

Solution:

- We must decide between the hypotheses: $H_0 : \mu_1 = \mu_2$, there is no difference between the mean masses. $H_1 : \mu_1 > \mu_2$, mean mass of first group is greater than that of second group.

Under null hypothesis;

$$\mu_{\bar{X}_1 - \bar{X}_2} = 0 \text{ and } \sigma_{\bar{X}_1 - \bar{X}_2} = \sqrt{\frac{\sigma_1^2}{N_1} + \frac{\sigma_2^2}{N_2}} = \sqrt{\frac{(2.5)^2}{50} + \frac{(2.8)^2}{50}} = 0.53$$

where we have used the sample standard deviations as estimates of σ_1 and σ_2 .

$$\text{Then, } Z = (\bar{X}_1 - \bar{X}_2) / \sigma_{\bar{X}_1 - \bar{X}_2} = (68.2 - 67.5) / 0.53 = 1.32$$

On the basis of a one-tailed test at a level of significance of 0.05, we would reject the hypothesis H_0 if the Z score were greater than 1.645. thus we cannot reject the hypothesis at this level of significance.

- Suppose the sample size of each group is N and that the standard deviations for the two groups remain the same.

Then under the hypothesis H_0 , we have

$$\mu_{\bar{X}_1 - \bar{X}_2} = 0 \text{ and } \sigma_{\bar{X}_1 - \bar{X}_2} = \sqrt{\frac{\sigma_1^2}{N_1} + \frac{\sigma_2^2}{N_2}} = \sqrt{\frac{(2.5)^2}{N} + \frac{(2.8)^2}{N}} = 3.75/\sqrt{N}$$

For an observed difference in mean masses of 0.7 kg,

$$Z = (\bar{X}_1 - \bar{X}_2) / \sigma_{\bar{X}_1 - \bar{X}_2} = \frac{0.7}{3.75/\sqrt{N}} = \frac{0.7\sqrt{N}}{3.75}$$

- The observed difference will be significant at a 0.05 level if $\frac{0.7\sqrt{N}}{3.75} = 1.645$ at least, so that N must be 78 at least. Thus we must increase the sample size in each group by at least $(78 - 50) = 28$.

Another method: $\frac{0.7\sqrt{N}}{3.75} \geq 1.645, N \geq 78$.

- ii. the observed difference will be significant at a 0.01 level if $\frac{0.7\sqrt{N}}{3.75} \geq 2.33$ and so; $N \geq 157$ Hence we must increase the sample size in each group by at least $(157 - 50) = 107$.

Question 6

One suggested method for solving the electric-power shortage in a region involves constructing floating nuclear power plants a few miles offshore in the ocean. Concern about the the possibility of a ship collision with the floating (but anchored) plant has rised the need for an estimate of the density of ship traffic in the area. The number of ships passing within 10 miles of the proposed power-plant location per day, recorded for $n = 60$ days during July and August, possessed a sample mean and variance of $\bar{y} = 7.2$ and $s^2 = 8.8$.

- Find a 95% confidence interval for the mean number of ships passing within 10 miles of the proposed power-plant location during a 1 -day time period.
- The density of ship traffic was expected to decrease during the winter months. A sample of $n = 90$ daily recordings of ship sightings for December, January and February yielded a mean and variance of $\bar{y} = 4.7$ and $s^2 = 4.9$. Find a 90% confidence interval for the difference in mean density of ship traffic between the summer and winter months.

Solution:

- $\bar{y} \pm 1.96 \left(\frac{s}{\sqrt{n}} \right) = 7.2 \pm 1.96 \sqrt{\frac{8.8}{60}} = (6.45, 7.95)$
- $(\bar{y}_1 - \bar{y}_2) \pm 1.645 \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} = (7.2 - 4.7) \pm 1.645 \sqrt{\frac{8.8}{60} + \frac{4.9}{90}} = (1.762, 3.238)$

Question 7

A single observation of a random variable having an exponential distribution is used to test the null hypothesis is that the mean of the distribution is $\theta = 2$ against that it is $\theta = 5$. The null hypothesis is accepted if and only if the observed value of the random variable is less than 3 . Find the probability of:

- Type-I error.
- Type-II error.
- Power of the test.

(Recall: Y has an exponential distribution if $f(y) = \left(\frac{1}{\theta}\right) e^{-\frac{y}{\theta}}$, where $\theta > 0$ and $0 < y < \infty$)

Solution:

- $\alpha = P(\text{reject } H_0/H_0 \text{ is true})$

$$= P(x > 3/\theta = 2)$$

$$= \int_3^\infty \frac{1}{2} e^{-x/2} dx = -e^{-x/2} \Big|_3^\infty = e^{-1.5} = 0.223$$

$$\text{b. } \beta = P(\text{ do not reject } H_0 / H_0 \text{ is false}) \quad \begin{array}{l} H_0 : \theta = 2 \\ H_1 : \theta = 5 \end{array} \quad f_x = \frac{1}{\theta} e^{-x/\theta} = P(x < 3/\theta = 5)$$

$$= \int_0^3 \frac{1}{5} e^{-x/5} dx = 0.451$$