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$$u(x, y) = xy + 20y \quad \text{s.t.} \quad 4x + y \leq 40$$

1st way: (we only keep in mind the nonnegativity constraints, i.e. $x \geq 0, y \geq 0$, but do not write in Lagrange) $\mathcal{L} = xy + 20y + \lambda(40 - 4x - y)$

Hence, the necessary conditions for a point to be a maximum are:

$$\begin{aligned} \mathcal{L}_x &= y - 4\lambda \leq 0 & x &\geq 0 & x(y - 4\lambda) &= 0 \\ \mathcal{L}_y &= x + 20 - \lambda \leq 0 & y &\geq 0 & y(x + 20 - \lambda) &= 0 \\ 40 - 4x - y &\geq 0 & \lambda &\geq 0 & \lambda(40 - 4x - y) &= 0 \end{aligned}$$

Case 1: $x > 0$ and $y > 0$. This implies $y - 4\lambda = 0$ and $x + 20 - \lambda = 0$

$$\frac{y}{x+20} = 4 \Rightarrow y = 4x + 80 \quad \begin{aligned} &\rightarrow y = 4\lambda \text{ and since } y > 0 \text{ also } \lambda > 0 \\ &\text{which means that } 40 - 4x - y = 0 \\ &40 - 4x - 4x - 80 = 0 \\ &x = -5 \text{ X contradiction} \end{aligned}$$

Case 2: $x > 0$ and $y = 0$. This implies $y - 4\lambda = 0$ and $x + 20 - \lambda \leq 0$

$$\text{since } y = 0 \Rightarrow \lambda = y/4 = 0$$

$$\hookrightarrow x \leq -20 \text{ X contradiction}$$

Case 3: $x = 0$ and $y > 0$. This implies $y - 4\lambda \leq 0$ and $x + 20 - \lambda = 0$.

$$\lambda = 20$$

$$\text{Since } \lambda > 0, 40 - 4x - y = 0 \Rightarrow y = 40$$

$$\text{So, } (x, y) = (0, 40)$$

Case 4: $x = 0$ and $y = 0$. This implies $y - 4\lambda \leq 0$ and $x + 20 - \lambda \leq 0$

$$\text{so } \lambda \geq 20 > 0$$

$$\text{which implies that } 40 - 4x - y = 0$$

$$\text{But since } x = 0 \text{ and } y = 0 \Rightarrow 40 \neq 0 \text{ contradiction!}$$

• We have solved the system and found the only solution, $(x, y) = (0, 40)$ and $\lambda = 20$.
 $u(x, y) = 800$

If the following conditions are satisfied

- ① $u(x, y)$ is differentiable & concave
- ② each constraint function is differentiable and convex
- ③ a point (x, y) satisfies the Kuhn-Tucker conditions, then that point is a maximum.

Kuhn-Tucker
sufficiency
conditions

• All linear functions are both convex and concave, so BC is certainly convex, and clearly is differentiable. For u , we need to check Hessian, if it is concave or not

$$H = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \quad \begin{aligned} |H_1| &= 0 & |H_2| &= -1 \\ &\leq 0 & &\leq 0 \end{aligned}$$

not bordered Hessian ∇ we are trying to decide if utility function is concave or not.

indefinite & does not satisfy K-T conditions.

2nd way: (we also integrate non-negativity constraints)

$$\mathcal{L}(x, y, \lambda) = xy + 20y + \lambda_1 x + \lambda_2 y + \lambda_3 (40 - 4x - y)$$

$$\partial \mathcal{L} / \partial x : y + \lambda_1 - 4\lambda_3 = 0 \quad (1)$$

$$\partial \mathcal{L} / \partial y : x + 20 + \lambda_2 - \lambda_3 = 0 \quad (2)$$

$$\bullet \lambda_1 \geq 0, \quad x \geq 0, \quad \lambda_1 x = 0$$

$$\bullet \lambda_2 \geq 0, \quad y \geq 0, \quad \lambda_2 y = 0$$

$$\bullet \lambda_3 \geq 0, \quad 40 - 4x - y \geq 0, \quad \lambda_3 (40 - 4x - y) = 0$$

• either the constraint is binding, that is, $p_1 x_1 + p_2 x_2 - m = 0$, or the constraint is not binding in which case the multiplier λ must be 0.

• we have $2^3 = 8$ cases in total (wrt 3 constraints) depending on $\lambda_j = 0$ or not, $j = 1, 2, 3$.

(i) $\lambda_1 > 0, \lambda_2 > 0$ and $\lambda_3 > 0 \Rightarrow x = 0, y = 0$ and $\begin{matrix} 40 - 4x - y = 0 \\ 0 \quad 0 \end{matrix}$
 $40 = 0$ (impossible ∇)

(ii) $\lambda_1 = 0, \lambda_2 > 0$ and $\lambda_3 > 0 \Rightarrow x \geq 0, y = 0$ and $4x + y = 40$
 From (1) $y = 4\lambda_3$ and since $\lambda_3 > 0 \Rightarrow y > 0$ (contradiction ∇)

(iii) $\lambda_1 > 0, \lambda_2 = 0$ and $\lambda_3 > 0 \Rightarrow x = 0, y \geq 0$ and $4x + y = 40$
 From (2) $\begin{matrix} x + 20 - \lambda_3 = 0 \\ 0 \end{matrix} \Rightarrow \lambda_3 = 20$ and from (1) $\lambda_1 = 40 > 0$ consistent ∇
 so, $(x, y) = (0, 40)$ and $(\lambda_1, \lambda_2, \lambda_3) = (40, 0, 20)$

(iv) $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 > 0 \Rightarrow x \geq 0, y \geq 0$ and $4x + y = 40$
 From (1) & (2): $\frac{y}{x+20} = 4 \Rightarrow \begin{matrix} y = 4x + 80 \\ 8x = -40 \end{matrix} \xrightarrow{\uparrow} x = -5$ (contradiction ∇)

(v) $\lambda_1 > 0, \lambda_2 > 0$ and $\lambda_3 = 0 \Rightarrow x = 0, y = 0$ and $40 - 4x - y \geq 0$
 From (1) $\lambda_1 = 0$ (contradiction ∇)

(vi) $\lambda_1 = \lambda_3 = 0, \lambda_2 > 0 \Rightarrow x \geq 0, y = 0$ and $40 - 4x - y \geq 0$
 From (2) $x + 20 + \lambda_2 = 0 \Rightarrow x = -20 - \lambda_2 < 0$ (contradiction ∇)

(vii) $\lambda_1 > 0, \lambda_2 = \lambda_3 = 0 \Rightarrow x = 0, y \geq 0$ and $40 - 4x - y \geq 0$
 From (2) $x = -20 < 0$ (contradiction ∇)

(viii) $\lambda_1 = \lambda_2 = \lambda_3 = 0 \Rightarrow x \geq 0, y \geq 0$ and $40 - 4x - y \geq 0$
 From (2) $x = -20 < 0$ (contradiction ∇)

• Sufficient conditions are stated in the 1st way.

④ (a) $\max \ln(x_1+1) + \ln(x_2+1) \quad \text{s.t.} \quad p_1 x_1 + p_2 x_2 \leq m, \quad x_1 \geq 0, x_2 \geq 0.$

$$\mathcal{L} = (x_1+1) + \ln(x_2+1) + \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 (m - p_1 x_1 - p_2 x_2)$$

Kuhn-Tucker FOCs:

(1) $\partial \mathcal{L} / \partial x_1: \frac{1}{x_1+1} + \lambda_1 - \lambda_3 p_1 = 0$

(2) $\partial \mathcal{L} / \partial x_2: \frac{1}{x_2+1} + \lambda_2 - \lambda_3 p_2 = 0$

$$\lambda_1 \geq 0, \quad x_1 \geq 0, \quad \lambda_1 x_1 = 0$$

$$\lambda_2 \geq 0, \quad x_2 \geq 0, \quad \lambda_2 x_2 = 0$$

$$\lambda_3 \geq 0, \quad m - p_1 x_1 - p_2 x_2 \geq 0, \quad \lambda_3 (m - p_1 x_1 - p_2 x_2) = 0$$

Case 1: $x_1 > 0, x_2 > 0 \Rightarrow \lambda_1 = \lambda_2 = 0$

Solving (1) & (2) $\frac{x_2+1}{x_1+1} = \frac{p_1}{p_2}$

$$p_2(x_2+1) = p_1(x_1+1)$$

$$p_1 x_1 = p_2 x_2 + p_2 - p_1$$

$$\left. \begin{aligned} p_2 x_2 + p_2 - p_1 + p_2 x_2 &= m \\ x_2^* &= \frac{m + p_1 - p_2}{2p_2} & x_1^* &= \frac{m + p_2 - p_1}{2p_1} \\ \text{and } \lambda_3 &= \frac{2}{m + p_1 + p_2} \end{aligned} \right\}$$

For x_1^* and x_2^* to be strictly positive, it has to be that $m + p_2 > p_1$ and $m + p_1 > p_2$.
Thus, Case 1 applies with x_1^* and x_2^* as listed above if $m + p_2 > p_1$ and $m + p_1 > p_2$.

Case 2: $x_1 > 0, x_2 = 0$

From (1) $\lambda_3 = \frac{1}{m + p_1} > 0$; so the BC is binding.

From the BC $x_1 = \frac{m}{p_1}$.

From (2) $1 + \lambda_2 - \frac{1}{m + p_1} \cdot p_2 = 0 \Rightarrow \lambda_2 = \frac{p_2}{m + p_1} - 1 = \frac{p_2 - m - p_1}{m + p_1}$

Since $x_2 = 0$, λ_2 should be positive. So, case 2 applies (with $x_1^* = m/p_1$ and $x_2^* = 0$ if $p_2 \geq m + p_1$)

Case 3: $x_1 = 0, x_2 > 0$

This case is very similar to Case 2.

$$\lambda_1 = \lambda_3 p_1 \quad \text{and} \quad x_2 = \frac{1}{\lambda_3 p_2} - 1$$

$$\lambda_3 = 0 \text{ violates so it should be that } \lambda_3 > 0 \Rightarrow m - p_1 x_1 - p_2 x_2 = 0$$

$$m - p_2 x_2 = 0 \Rightarrow x_2 = m/p_2$$

$$\frac{m}{p_2} + 1 = \frac{1}{\lambda_3 p_2} \Rightarrow \lambda_3 = \frac{1}{m + p_2} \quad \text{and} \quad \lambda_1 = \lambda_3 p_1 - 1 = \frac{p_1}{m + p_2} - 1$$

Thus, Case 3 applies (with $x_1^* = 0, x_2^* = m/p_2$) if $p_1 \geq m + p_2$

Case 4: $x_1^* = x_2^* = 0$ cannot hold since it means $m = 0$.

(b) $\max x_1^2 + (x_2 - 5)^2 \quad \text{s.t.} \quad x_1 \geq 0, x_2 \geq 0 \text{ and } 2x_1 + x_2 \leq 4$

$$\mathcal{L}(x_1, x_2, \lambda) = x_1^2 + (x_2 - 5)^2 + \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 (4 - 2x_1 - x_2)$$

Kuhn-Tucker Conditions:

$$\partial \mathcal{L} / \partial x_1 = 2x_1 + \lambda_1 - 2\lambda_3 = 0$$

$$\partial \mathcal{L} / \partial x_2 = 2(x_2 - 5) + \lambda_2 - \lambda_3 = 0$$

$$x_1 \geq 0, x_2 \geq 0, 4 - 2x_1 - x_2 \geq 0, \lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0$$

$$\lambda_1 x_1 = 0, \lambda_2 x_2 = 0, \lambda_3 (4 - 2x_1 - x_2) = 0$$

• Totally eight possibilities depending whether $\lambda_j = 0$ or not, $j = 1, 2, 3$.

(i) $\lambda_1 = 0, \lambda_2 = 0$ and $\lambda_3 > 0 \Rightarrow x_1 \geq 0, x_2 = 0$ and $2x_1 + x_2 = 4$. So $(x_1, x_2) = (2, 0)$.

From $4 - 2\lambda_3 = 0$ and $-10 + \lambda_2 - \lambda_3 = 0$ $(\lambda_1, \lambda_2, \lambda_3) = (0, 12, 2)$

(ii) $\lambda_1 > 0, \lambda_2 > 0, \lambda_3 > 0 \Rightarrow x_1 = 0, x_2 = 0$, and $2x_1 + x_2 = 4$ (impossible $\forall$)

(iii) $\lambda_1 > 0, \lambda_2 = 0$ and $\lambda_3 > 0 \Rightarrow x_1 = 0, x_2 \geq 0$ and $2x_1 + x_2 = 4$. So $(x_1, x_2) = (0, 4)$.

From $\lambda_1 - 2\lambda_3 = 0$ and $-2 - \lambda_3 = 0$, we have $(\lambda_1, \lambda_2, \lambda_3) = (-4, 0, -2)$. Impossible $\forall$

(iv) $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 > 0 \Rightarrow x_1 \geq 0, x_2 \geq 0$ and $2x_1 + x_2 = 4$. So, from $2x_1 - 2\lambda_3 = 0$,

$2(x_2 - 5) - \lambda_3 = 0$ and $2x_1 + x_2 = 4$, we have $(x_1, x_2) = (-2/5, 24/5)$. Impossible $\forall$

(v) $\lambda_1 > 0, \lambda_2 > 0$ and $\lambda_3 = 0 \Rightarrow x_1 = 0, x_2 = 0$ and $2x_1 + x_2 \leq 4$. So, $(x_1, x_2) = (0, 0)$.

From $\lambda_1 = 0$ and $-10 + \lambda_2 = 0$ we have $(\lambda_1, \lambda_2, \lambda_3) = (0, 10, 0)$. Impossible $\forall$

(vi) $\lambda_1 = \lambda_3 = 0, \lambda_2 > 0 \Rightarrow x_1 \geq 0, x_2 = 0$, and $2x_1 + x_2 \leq 4$. From $2x_1 = 0$ and $-10 + \lambda_2 = 0$,

we have $(x_1, x_2) = (0, 0)$ and $(\lambda_1, \lambda_2, \lambda_3) = (0, 10, 0)$.

(vii) $\lambda_1 > 0, \lambda_2 = \lambda_3 = 0 \Rightarrow x_1 = 0, x_2 \geq 0$ and $2x_1 + x_2 \leq 4$. So from $\lambda_1 = 0$ and $2(x_2 - 5) = 0$,

we have $(x_1, x_2) = (0, 5)$ and $(\lambda_1, \lambda_2, \lambda_3) = (0, 0, 0)$. Impossible $\forall$

(viii) $\lambda_1 = \lambda_2 = \lambda_3 = 0 \Rightarrow x_1 \geq 0, x_2 \geq 0$ and $2x_1 + x_2 \leq 4$. So from $2x_1 = 0$ and $2(x_2 - 5) = 0$

we have $(x_1, x_2) = (0, 5) \Rightarrow$ impossible $\forall$

• Candidate maximizers are $(2, 0)$ and $(0, 0)$. The objective function values at these two candidates are 29 and 25, so $(2, 0)$ is the maximizer and the associated Lagrange multipliers are $(0, 12, 2)$. — check by replacing the points in utility function $\forall$

• Kuhn-Tucker sufficiency conditions are stated in Q3.

All linear functions are both convex and concave, so bc is certainly convex, and clearly is differentiable. For $u(x_1, x_2)$ we need to check Hessian to decide if it is concave or not.

$$H = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0 \quad M_1 = 2 > 0, M_1' = 4 > 0 \quad \text{convex b}$$

does not satisfy
sufficiency conditions

(c) $u(x_1, x_2) = x_1 x_2$ s.t. $x_1 + x_2 \leq 100$, $x_1 \leq 40$, $x_1 \geq 0$ and $x_2 \geq 0$.

$$\mathcal{L}(x_1, x_2, \lambda) = x_1 x_2 + \lambda_1 (100 - x_1 - x_2) + \lambda_2 (40 - x_1) + \lambda_3 x_1 + \lambda_4 x_1$$

$$K-T1 \begin{cases} \partial \mathcal{L} / \partial x_1 = x_2 - \lambda_1 - \lambda_2 + \lambda_3 = 0 \\ \partial \mathcal{L} / \partial x_2 = x_1 - \lambda_1 + \lambda_4 = 0 \end{cases}$$

$$K-T2 \begin{cases} \lambda_1 \geq 0 & x_1 + x_2 \leq 100 & \lambda_1 (100 - x_1 - x_2) = 0 \\ \lambda_2 \geq 0 & x_1 \leq 40 & \lambda_2 (40 - x_1) = 0 \\ \lambda_3 \geq 0 & x_1 \geq 0 & \lambda_3 x_1 = 0 \\ \lambda_4 \geq 0 & x_2 \geq 0 & \lambda_4 x_2 = 0 \end{cases}$$

- There are 16 possibilities to consider! By simple inspection we can discard some of them. Here it does not make any sense to select x_1 or x_2 equal to zero, as then $u(x_1, x_2) = 0$. Therefore, by (K-T2) we know that λ_3 and λ_4 should be both zero at a global maximum. In addition, since u is strictly increasing on $\mathbb{R}^2_+$ the first constraint will be binding, i.e. $x_1 + x_2 = 100$.

By these two arguments the 16 possibilities reduce to only 2.

Let us study the two final possibilities. Let $\lambda_2 = 0$. From (K-T1) we get $x_1 = x_2 = \lambda_1$. Since $x_1 > 0$ and $x_2 > 0$, then the candidate is $x_1 = x_2 = \lambda_1 = 50$. This contradicts the fact that $x_1 \leq 40$.

The other possibility is $x_1 = 40$. Since $x_1 + x_2 = 100$, then $x_2 = 60$. From (K-T1) $\lambda_1 = 40$ and $\lambda_2 = 20$. This second case satisfies all the Kuhn-Tucker conditions.

$$H = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 < 0 \Rightarrow u(x_1, x_2) \text{ is indefinite}$$

$$M_1 = 0 \quad M_1' = 0$$

Also BC is linear $\Rightarrow$ it is convex & " BC is differentiable but since $u(x_1, x_2)$ is indefinite does not satisfy K-T sufficient conditions.