

Marginal & Conditional Prob Distr.

If Y_1 & Y_2 are jointly discrete random variables with prob. fun. $P(Y_1, Y_2)$ then marginal prob. of Y_1 & Y_2 are

$$P_1(Y_1) = \sum_{Y_2} P(Y_1, Y_2)$$

$$P_2(Y_2) = \sum_{Y_1} P(Y_1, Y_2)$$

If Y_1 & Y_2 are continuous random variables & $f(Y_1, Y_2)$ is value of their joint pdf of (Y_1, Y_2) . then functions given by

$$f_1(Y_1) = \int_{-\infty}^{\infty} f(Y_1, Y_2) dY_2$$

$$f_2(Y_2) = \int_{-\infty}^{\infty} f(Y_1, Y_2) dY_1$$

are called marginal density functions of Y_1 & Y_2

Ex: Given joint pdf

$$f(Y_1, Y_2) = \begin{cases} \left(\frac{2}{3} Y_1 + 2Y_2\right) & 0 < Y_1 < 1, 0 < Y_2 < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the marginal densities of Y_1 & Y_2

$$f_1(Y_1) = \int_{-\infty}^{\infty} f(Y_1, Y_2) dY_2 = \int_0^1 \frac{2}{3} (Y_1 + 2Y_2) \cdot dY_2$$

$$= \frac{2}{3} (Y_1 + 1) \quad f_1(Y_1) = \frac{2}{3} (Y_1 + 1) \quad \text{for } 0 < Y_1 < 1$$

$$f_2(Y_2) = \int_{-\infty}^{\infty} f(Y_1, Y_2) dY_1 = \int_0^1 \frac{2}{3} (Y_1 + 2Y_2) \cdot dY_1 = \frac{1}{3} (1 + 4Y_2)$$

$$f_2(Y_2) = \begin{cases} \frac{1}{3} (1 + 4Y_2) & 0 < Y_2 < 1 \\ 0 & \text{elsewhere} \end{cases}$$

for the multivariate case, we can speak not only marginal distributions of individual random variables but also of joint marginal distribution of random variables

If joint prob. density of continuous random variables $y_1, y_2, y_3, \dots, y_n$ has values $f(y_1, y_2, \dots, y_n)$ marginal density of y_2 alone is

$$f_2(y_2) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} f(y_1, y_2, \dots, y_n) dy_1 \dots dy_{n-1}$$

for $-\infty < y_1 < \infty, \dots, -\infty < y_n < \infty$

Joint marginal Density of y_1 & y_n is

$$g(y_1, y_n) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} f(y_1, \dots, y_n) dy_2 \dots dy_{n-1}$$

Ex:

If tri-variate pdf of y_1, y_2 & y_3 is

$$f(y_1, y_2, y_3) = \begin{cases} (y_1 + y_2) e^{-y_3} & 0 < y_1 < 1, 0 < y_2 < 1, y_3 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

find joint marginal density of y_1 & y_3 and marginal density of y_1 alone

$$\begin{aligned} g(y_1, y_3) &= \int_0^1 (y_1 + y_2) e^{-y_3} dy_2 \\ &= (y_1 + \frac{1}{2}) e^{-y_3} \end{aligned}$$

$$g(y_1, y_3) = \begin{cases} (y_1 + \frac{1}{2}) e^{-y_3} & 0 < y_1 < 1, y_3 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$f_1(y_1) = \int_0^{\infty} \int_0^{\infty} (y_1 + y_2) e^{-y_3} dy_2 dy_3$$

$$= \int_0^{\infty} g(y_1, y_3) dy_3 = \int_0^{\infty} (y_1 + \frac{1}{2}) e^{-y_3} dy_3$$

$$= y_1 + \frac{1}{2}$$

$$f_1(y_1) = \begin{cases} y_1 + \frac{1}{2} & , 0 < y_1 < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Conditional Density

If Y_1 & Y_2 are jointly discrete random variables with joint prob. func. $P(Y_1, Y_2)$ and marginal prob. funcs. $p_1(Y_1)$ & $p_2(Y_2)$, then the conditional discrete prob. function of Y_1 given Y_2 is

$$P(Y_1 = y_1 | Y_2 = y_2) \\ = \frac{P(Y_1 = y_1, Y_2 = y_2)}{P(Y_2 = y_2)} = \frac{P(Y_1, Y_2)}{p_2(Y_2)}$$

if $f(Y_1, Y_2)$ is value of joint density of continuous random variable Y_1 & Y_2 at (y_1, y_2) & $f_2(Y_2)$ is value of marginal density of Y_2 at y_2 , func. given by

$$f(y_1 | y_2) = \frac{f(y_1, y_2)}{f_2(y_2)} \quad \text{for } -\infty < y_1 < \infty$$

is called conditional density of Y_1 given $Y_2 = y_2$, where $f_2(y_2) \neq 0$

If $f_1(y_1)$ is value of marginal density of Y_1 at y_1

$$f(y_2 | y_1) = \frac{f(y_1, y_2)}{f_1(y_1)} \quad -\infty < y_2 < \infty$$

is called conditional density of Y_2 given $Y_1 = y_1$, $f_1(y_1) \neq 0$

Ex: Given joint pdf

$$f(y_1, y_2) = \begin{cases} \frac{2}{3} (y_1 + 2y_2) & \text{if } 0 < y_1 < 1 \text{ and } 0 < y_2 < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find conditional density of Y_1 given $Y_2 = y_2$ & use it to calculate $P(Y_1 \leq \frac{1}{2} | Y_2 = \frac{1}{2})$

$$f_2(y_2) = \frac{1}{3} (1 + 4y_2)$$

$$f(y_1 | y_2) = \frac{f(y_1, y_2)}{f_2(y_2)}$$

$$= \frac{\frac{2}{3} (y_1 + 2y_2)}{\frac{1}{3} (1 + 4y_2)} = \frac{2y_1 + 4y_2}{1 + 4y_2}$$

$$f(y_1, y_2) = \begin{cases} \frac{2y_1 + 4y_2}{1 + 4y_2} & 0 \leq y_1 < 1 \\ & 0 \leq y_2 < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$f(y_1 | \frac{1}{2}) = \frac{2y_1 + 4(\frac{1}{2})}{1 + 4(\frac{1}{2})} = \frac{2y_1 + 2}{3}$$

$$P(y_1 \leq \frac{1}{2} | y_2 = \frac{1}{2}) = \int_0^{\frac{1}{2}} \frac{2y_1 + 2}{3} \cdot dy_1 = \frac{5}{12}$$

Ex: Given joint pdf

$$f(y_1, y_2) = \begin{cases} 4y_1 y_2 & , 0 \leq y_1 < 1 \quad 0 \leq y_2 < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find marginal densities of y_1 & y_2 & conditional density of y_1 given $y_2 = y_2$

$$f_1(y_1) = \int_{-\infty}^{\infty} f(y_1, y_2) dy_2 = \int_0^1 4y_1 y_2 dy_2$$

$$= 2y_1 y_2^2 \Big|_0^1 = 2y_1$$

$$f_1(y_1) = \begin{cases} 2y_1 & , 0 \leq y_1 < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$f_2(y_2) = \int_{-\infty}^{\infty} f(y_1, y_2) dy_1 = \int_0^1 4y_1 y_2 \cdot dy_1$$

$$2y_1^2 y_2 \Big|_0^1 = 2y_2$$

$$f_2(y_2) = \begin{cases} 2y_2 & , 0 \leq y_2 < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$f(y_1 | y_2) = \frac{f(y_1, y_2)}{f_2(y_2)} = \frac{4y_1 y_2}{2y_2} = 2y_1$$

$$f(y_1, y_2) = \begin{cases} 2y_1 & , \quad 0 < y_1 < 1 \\ C & \text{elsewhere} \end{cases}$$

Multi-variate Case

When we are dealing with more than two random variable, we can consider various different kinds of conditional densities

for example if we have 4 random variables $f(y_1, y_2, y_3, y_4)$ is value of joint distribution of cont. random variables y_1, y_2, y_3 & y_4 of $f(y_1, y_2, y_3, y_4)$ we can write

$$f(y_3 | y_1, y_2, y_4) = \frac{f(y_1, y_2, y_3, y_4)}{g(y_1, y_2, y_4)}$$

$$g(\dots) \neq C$$

for value of conditional density of y_3 of y_3 given $y_1 = y_1, y_2 = y_2, y_4 = y_4$

$g(y_1, y_2, y_4) \Rightarrow$ value of joint marginal density of y_1, y_2 & y_4

We can also write

$$g(y_2, y_1 | y_1, y_3) = \frac{f(y_1, y_2, y_3, y_4)}{m(y_1, y_3)} \quad \rightarrow \text{marginal density}$$

$$r(y_2, y_3, y_4 | y_1) = \frac{f(y_1, y_2, y_3, y_4)}{K(y_1)} \quad K(y_1) \neq 0$$

When we are dealing with two or more than two r.v.'s $\Rightarrow$ question of independence is important

If $(y_1, y_2, \dots, y_n)$ is value of joint pdf of n continuous random variables $y_1, y_2, \dots, y_n$ at $(y_1, \dots, y_n)$

$f_i(y_i)$ is value of marginal density of y_i at y_i

$i = 1, 2, \dots, n$ then n random variables are independent iff

$$f(y_1, y_2, y_3, \dots, y_n) = f_1(y_1) \cdot f_2(y_2) \cdot f_3(y_3) \dots$$

for all $(y_1, y_2, \dots, y_n)$ within their range

Independent Random Variables

Let Y_1 have a distribution function $F_1(y_1)$ Y_2 have a distr. func. $F_2(y_2)$ Y_1 & Y_2 have a joint distr. func. $F(y_1, y_2)$. Then Y_1 & Y_2 are said to be independent iff

$$F(y_1, y_2) = F_1(y_1) \cdot F_2(y_2)$$

for every pair of real numbers (y_1, y_2)

If Y_1 & Y_2 are discrete random variables with joint prob. func. $P(y_1, y_2)$ and marginal prob. func. $P_1(y_1)$ & $P_2(y_2)$ then Y_1 & Y_2 are independent iff

$$P(y_1, y_2) = P_1(y_1) \cdot P_2(y_2)$$

If Y_1 & Y_2 are continuous rvs with joint density function $f(y_1, y_2)$ & marginal density func.

$f_1(y_1)$ & $f_2(y_2)$ then Y_1 & Y_2 are independent iff

$$f(y_1, y_2) = f_1(y_1) \cdot f_2(y_2)$$

for all pairs of real numbers (y_1, y_2)

If $f(y_1, y_2, \dots, y_n)$ is value of joint prob. density/distr. of n continuous or discrete random variables $Y_1, Y_2, \dots, Y_n$ at their values $y_1, y_2, \dots, y_n$ and $f_i(y_i)$ is value of marginal density distr. of Y_i at y_i for $i = 1, 2, \dots, n$. then n random variables are independent iff

$$f(y_1, y_2, \dots, y_n) = f_1(y_1) \cdot f_2(y_2) \cdot \dots \cdot f_n(y_n)$$

for all $(y_1, y_2, \dots, y_n)$ within their range

Ex: Given independent rv's Y_1, Y_2, Y_3 with pdfs

$$f_1(y_1) = \begin{cases} e^{-y_1} & \text{for } y_1 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$f_2(y_2) = \begin{cases} 2e^{-2y_2} & \text{for } y_2 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$f_3(y_3) = \begin{cases} 3e^{-3y_3} & \text{for } y_3 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

find their joint pdf & use it to evaluate prob. that $P(Y_1 + Y_2 \leq 1, Y_3 > 1)$

$$f(y_1, y_2, y_3) = f_1(y_1) \cdot f_2(y_2) \cdot f_3(y_3)$$

$$= e^{-y_1} \cdot 2e^{-2y_2} \cdot 3e^{-3y_3}$$

$$= 6e^{-y_1 - 2y_2 - 3y_3}$$

$$f(y_1, y_2, y_3) = \begin{cases} 6e^{-y_1 - 2y_2 - 3y_3} & , y_1 > 0, y_2 > 0, y_3 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$P(Y_1 + Y_2 \leq 1, Y_3 > 1) = \int_0^{\infty} \int_0^{1-y_1} \int_1^{\infty} 6e^{-y_1 - 2y_2 - 3y_3} \cdot dy_1 \cdot dy_2 \cdot dy_3$$

$$= (1 - 2e^{-1} + e^{-2})e^{-3} = 0.020$$

Let Y_1 & Y_2 have a joint density $f(y_1, y_2)$ that is positive if $a \leq y_1 \leq b$ & $c \leq y_2 \leq d$ for constants a, b, c, d and $f(y_1, y_2) = 0$ otherwise. Then Y_1 & Y_2 are independent variables iff

$$f(y_1, y_2) = g(y_1) \cdot h(y_2)$$

where $g(y_1)$ is nonnegative func. of y_1 alone & $h(y_2)$ is a non-negative func. of y_2 alone

Expected Values of a Function of Random Variables

Let $g(Y_1, Y_2, \dots, Y_k)$ be a function of discrete random variables which have prob. functions $p(Y_1, Y_2, \dots, Y_k)$. The expected value of $g(Y_1, Y_2, \dots, Y_k)$ is

$$E[g(Y_1, Y_2, \dots, Y_k)] = \sum_{y_k} \dots \sum_{y_2} \sum_{y_1} g(y_1, y_2, \dots, y_k) \cdot p(y_1, y_2, \dots, y_k)$$

If $Y_1, Y_2, \dots, Y_k$ are continuous r.v.'s with joint density function $f(y_1, y_2, \dots, y_k)$ then

$$E[g(Y_1, Y_2, \dots, Y_k)] = \int_{y_k} \dots \int_{y_2} \int_{y_1} g(y_1, y_2, \dots, y_k) \cdot f(y_1, y_2, \dots, y_k) \cdot dy_1 \cdot dy_2 \cdot \dots \cdot dy_k$$

Consider two random variables Y_1 & Y_2 with density func. $f(y_1, y_2)$ we wish to find exp. value of $g(Y_1, Y_2) = Y_1$

$$E(Y_1) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y_1 \cdot f(y_1, y_2) \cdot dy_2 \cdot dy_1$$

$$= \int_{-\infty}^{\infty} y_1 \cdot \underbrace{\left[\int_{-\infty}^{\infty} f(y_1, y_2) dy_2 \right]}_{\text{marginal density func of } Y_1} dy_1$$

$$= E(Y_1) = \int_{-\infty}^{\infty} y_1 \cdot f_1(y_1) dy_1$$

Ex: Let Y_1 & Y_2 have a joint pdf

$$f(y_1, y_2) = \begin{cases} 3y_1 & , \quad 0 \leq y_1 \leq 1 \quad 0 \leq y_2 \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

find $E(Y_1, Y_2)$ & $E(Y_1)$

$$E(Y_1, Y_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y_1 y_2 f(y_1, y_2) dy_1 dy_2$$

$$= \int_0^1 \int_0^1 y_1 y_2 (3y_1) dy_1 dy_2 = \int_0^1 y_2 \left(\frac{3y_1^2}{2} \right) \Big|_0^1 dy_2$$

$$= \int_0^1 y_2 dy_2 = \frac{y_2^2}{2} \Big|_0^1 = \frac{1}{2}$$

$$E(Y_1) = \int_0^1 \int_0^1 y_1 \cdot 3y_1 \cdot dy_1 \cdot dy_2 = \int_0^1 \frac{3y_1^3}{3} \Big|_0^1 dy_2$$

$$= \int_0^1 1 dy_2 = y_2 \Big|_0^1 = 1$$

Let c be a constant

$$E(c) = c$$

If c is a constant

$$E(c \cdot Y) = c \cdot E(Y)$$

If $c_1, c_2, c_3, \dots, c_n$ constants

$$E\left(\sum_{i=1}^n c_i g_i(Y)\right) = \sum_{i=1}^n c_i \cdot E[g_i(Y)]$$

Let $g(y_1, y_2)$ be a func of random variables y_1 & y_2
and $g_1(y_1, y_2), g_2(y_1, y_2) \dots g_k(y_1, y_2)$ be functions
of y_1 & y_2

Then,

$$E[g_1(y_1, y_2) + g_2(y_1, y_2) + \dots + g_k(y_1, y_2)] \\ = E[g_1(y_1, y_2)] + E[g_2(y_1, y_2)] + \dots + E[g_k(y_1, y_2)]$$

Ex: The joint pdf of y_1 & y_2 is

$$f(y_1, y_2) = \begin{cases} 4y_1 y_2 & , \quad 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find $E(y_1 - y_2)$

$$f_1(y_1) = \int_{-\infty}^{\infty} f(y_1, y_2) dy_2 = \int_0^1 4y_1 y_2 dy_2$$

$$= 4y_1 \left(\frac{y_2^2}{2} \right) \Big|_0^1 = 2y_1 \quad \text{for } 0 \leq y_1 \leq 1$$

$$f_2(y_2) = \int_{-\infty}^{\infty} f(y_1, y_2) dy_1 = \int_0^1 4y_1 y_2 dy_1 = 4y_2 \left(\frac{y_1^2}{2} \right) \Big|_0^1$$

$$= 2y_2 \quad \text{for } 0 \leq y_2 \leq 1$$

$$E(y_1) = \int_0^1 2y_1 y_1 dy_1 = \int_0^1 2y_1^2 dy_1 = \frac{2}{3}$$

$$E(y_2) = \int_0^1 2y_2 y_2 dy_2 = \int_0^1 2y_2^2 dy_2 = \frac{2}{3}$$

$$E(y_1 - y_2) = E(y_1) - E(y_2) = \frac{2}{3} - \frac{2}{3} = 0$$

Let y_1 & y_2 be independent random variables & $g(y_1) h(y_2)$ be
functions of only y_1 & y_2 . The

$$E[g(y_1) h(y_2)] = E[g(y_1)] \cdot E[h(y_2)]$$

$$E(y_1, y_2) = \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9} = E(y_1) \cdot E(y_2)$$