

METU, Department of Economics

ECON 201 – Problem Set 3

1- A consumer has the utility function over goods x and y,

$$u(x, y) = x^{1/2} + y^{1/2}$$

Let the price of good x be given by P_x , let the price of good y be given by P_y , and let income be given by M.

- Derive the consumer's demand function for good y.
- What is λ ? What does it mean?
- Is good y normal or inferior?
- If we have $P_x = 2$, $P_y = 1$, and $M = 12$, compute the utility maximizing consumption bundle of goods x and y.

Solution:

$$a. \quad u(x, y) = x^{1/2} + y^{1/2}$$

$$x^*P_x + y^*P_y = M$$

$$L = x^{1/2} + y^{1/2} - \lambda(x^*P_x + y^*P_y - M)$$

$$(1) \quad \partial L / \partial x = 1/2 x^{-1/2} - \lambda P_x = 0$$

$$(2) \quad \partial L / \partial y = 1/2 y^{-1/2} - \lambda P_y = 0$$

$$(3) \quad \partial L / \partial \lambda = x^*P_x + y^*P_y - M = 0$$

$$(1): \lambda = \frac{1}{2P_x} x^{-1/2}$$

$$(2): \lambda = \frac{1}{2P_y} y^{-1/2}$$

$$(1) - (2): \frac{1}{2P_x} x^{-1/2} = \frac{1}{2P_y} y^{-1/2}$$

$$P_x^2 x^{1/2} = P_y^2 y^{1/2}$$

$$x^{1/2} = P_y^2 y^{1/2} / P_x^2$$

$$x = P_y^2 y / P_x^2$$

Insert into (3):

$$P_y^2 y / P_x + y^*P_y = M$$

$$y(P_y^2 / P_x + P_y) = M$$

$$y = \frac{M P_x}{P_y^2 + P_y P_x}$$

b.

$$\lambda = \frac{1}{2P_y} \left(\frac{M \cdot P_x}{P_y^2 + P_y \cdot P_x} \right)^{-1/2} > 0 \text{ for } P_x, P_y > 0$$

If λ was 0, the Lagrangian would be

$$L = x^{1/2} + y^{1/2}$$

which would simply be an unconstrained optimization problem (the case when prices are 0 if you think about it). In reality, since prices are higher than 0, the budget constraints constrains the optimization problem where $\lambda > 0$

c. $y = \frac{M \cdot P_x}{P_y^2 + P_y \cdot P_x}$
 $\frac{\partial y}{\partial M} = \frac{P_x}{P_y^2 + P_y \cdot P_x} > 0$
 y is a normal good.

d. $y = \frac{M \cdot P_x}{P_y^2 + P_y \cdot P_x} = \frac{12 \cdot 2}{1 + 2} = 8$

$$\begin{aligned} x \cdot P_x + y \cdot P_y &= M \\ x \cdot 2 + 8 &= 12 \\ x &= 2 \end{aligned}$$

2- Emre consumes only Tea (T) and Cigarettes (C) each week with his 100TL income. Tea costs 1TL, while cigarettes cost only 0.50TL. Emre has Cobb-Douglas preferences given by:

$U(T, C) = T(C - c)$ where c is the quantity of cigarettes that his neighbor Ekin consumes. Assume throughout this question that Emre always consumes more cigarettes than Ekin.

a. How does Ekin's cigarette consumption influence Emre's utility function?

Specifically, compute and determine the sign of $\partial U / \partial c$. Intuitively, what does this tell you about Emre's happiness and Ekin's consumption of cigarettes?

b. Holding c fixed, compute Emre's optimal consumption bundle of Tea and Cigarettes as functions of Ekin's consumption of cigarettes, c .

c. How does Ekin's consumption of cigarettes affect Emre's optimal bundle? Specifically calculate and determine the sign of $\partial T^* / \partial c$ and $\partial C^* / \partial c$ where (T^*, C^*) is Emre's optimal consumption bundle.

Solution:

a.

$$U(T, C) = T(C - c)$$

$$\partial U / \partial c = -T < 0$$

Emre does not want Ekin to consume cigarettes.

b. Using Lagrangian method

$$L = TC - Tc - \lambda(T*1 + C*0.5 - 100)$$

$$\partial L / \partial T = C - c - \lambda = 0$$

$$\partial L / \partial C = T - 0.5*\lambda = 0$$

$$\partial L / \partial \lambda = T + 0.5*C - 100 = 0$$

$$\lambda = C - c$$

$$T = 0.5*(C-c)$$

$$C - 0.5c - 100 = 0$$

$$C = 100 + 0.5c$$

$$T = 0.5*(100 + 0.5c - c) = 50 - c/4$$

c.

$$\partial T^* / \partial c = -1/4$$

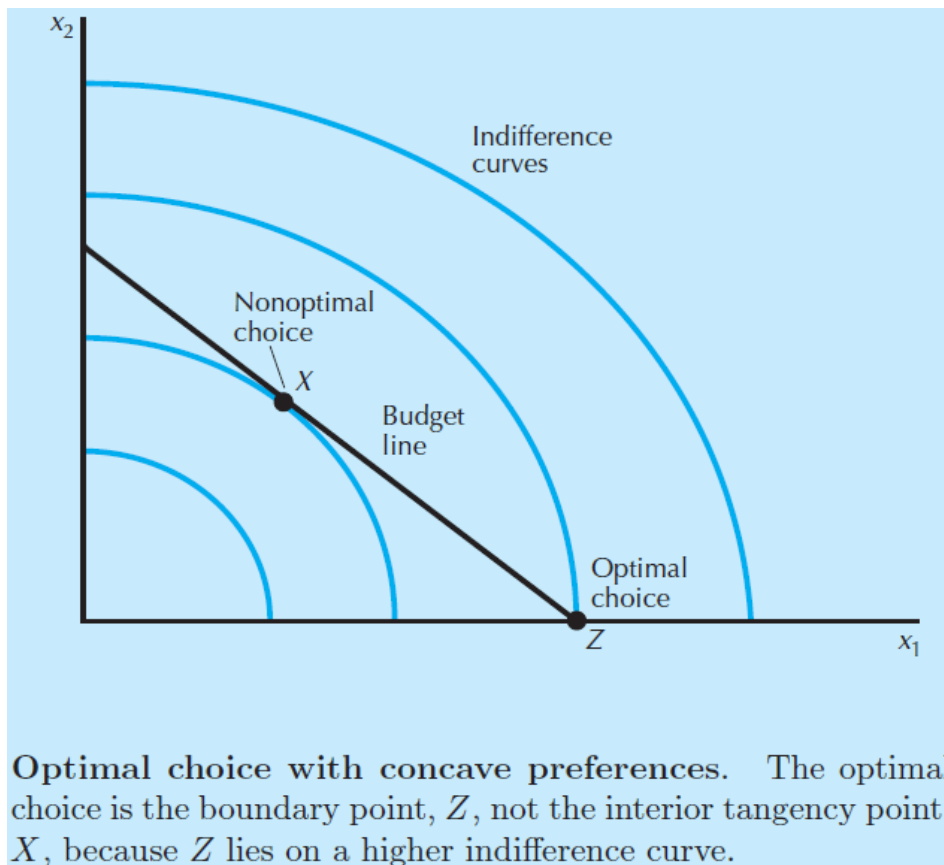
$$\partial C^* / \partial c = 1/2$$

As $c \uparrow$, Emre will consume more cigarettes and less tea.

- 3- If preferences are concave what does this imply in terms of the MRS? In such a case, will the consumer ever consume both of the goods together?

Solution:

When the preferences are concave it means, the person does not like to consume the two goods together. The optimal choice for these preferences is always going to be a boundary choice where one of the goods is not consumed.



- 4- Suppose that there are two goods (a and b) in an economy and a consumer with the following utility function:

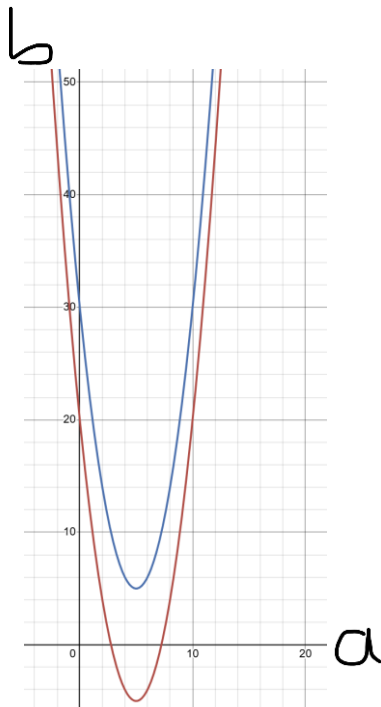
$$u(a, b) = 10a - a^2 + b$$

Assume that price of a is P_a and price of P_b is 1.

- Sketch two indifference curves of this consumer.
- How many a would the household demand at $p_a = 0$?
- At what price would the household be indifferent between $a - 1$ and a ?

Solution:

-



b) $p_a = 0$
 $b = M$

$$u(a, b) = 10a - a^2 + b$$

$$= 10a - a^2 + M$$

$$\partial u / \partial a = 10 - 2a = 0$$

$$a = 5$$

c)

$$u(a, b) = 10a - a^2 + b$$

$$a \cdot P_a + b = M$$

$$b = M - a \cdot P_a$$

$$u(a, b) = 10a - a^2 + M - a \cdot P_a$$

$$10a - a^2 + M - a \cdot P_a = 10(a - 1) - (a - 1)^2 + M - (a - 1) \cdot P_a$$

$$10a - 10 - a^2 + 2a - 1 + M - a \cdot P_a - P_a = 10a - a^2 + M - a \cdot P_a$$

$$-10 + 2a - 1 - P_a = 0$$

$$P_a = 2a - 11$$

5- A consumer has a utility function: $u(x_1, x_2) = x_1 x_2^{1-\alpha}$

a) Derive the Marshallian demands of both goods for prices P_1 , P_2 and income M .

b) Derive the income expansion path.

c) Derive the Engel curve for both goods.

Solution:

a) $\text{MAX } L(x_1, x_2, \lambda) = x_1^\alpha x_2^{1-\alpha} + \lambda(M - P_1 x_1 - P_2 x_2)$

F.O.C.s

$$\frac{\partial L}{\partial x_1} = 0 \rightarrow \alpha x_1^{\alpha-1} x_2^{1-\alpha} - \lambda P_1 = 0 \quad \left. \begin{array}{l} \frac{\alpha}{1-\alpha} \cdot \frac{x_2}{x_1} = \frac{P_1}{P_2} \quad (i) \end{array} \right\} \Rightarrow$$

$$\frac{\partial L}{\partial x_2} = 0 \rightarrow (1-\alpha) x_1^\alpha x_2^{-\alpha} - \lambda P_2 = 0$$

$$\frac{\partial L}{\partial \lambda} = 0 \rightarrow M - P_1 x_1 - P_2 x_2 = 0 \quad \rightarrow M = P_1 x_1 + P_2 x_2 \quad (ii)$$

$$\Rightarrow M = \frac{\alpha}{1-\alpha} P_2 x_2 + P_2 x_2 = P_2 x_2 \frac{1}{1-\alpha} \Rightarrow x_2^* = \frac{(1-\alpha) M}{P_2} //$$

$$x_1^* = \frac{\alpha M}{P_1} //$$

b) Income exp. path: $f(x_1^*) = f(x_2^*)$

from (i) $x_1^* = \frac{P_2}{P_1} \frac{\alpha}{1-\alpha} x_2^*$

c) Engel curve: $x_i^* = f(M, P)$

$$x_1^* = \frac{\alpha}{P_1} M, \quad x_2^* = \frac{1-\alpha}{P_2} M$$

6- A consumer derives utility from consuming x_1 and x_2 . His utility function is

$$u(x_1, x_2) = 8\sqrt{x_1 x_2}$$

Suppose he knows the maximum utility, which is exactly $u = \bar{u}$ given P_1 and P_2 .

- Formulate the constrained expenditure minimization problem.
- Determine the optimum quantities of x_1 and x_2 in order to achieve $\bar{u}$ given prices P_1 and P_2 (i.e. find compensated/Hicksian demand function).
- What is his minimum expenditure while receiving utility level of $\bar{u}$?

Solution:

$$a) \min_{x_1, x_2} P_1 x_1 + P_2 x_2 \quad s.t. \quad 8\sqrt{x_1 x_2} = \bar{u}$$

$$b) L(x_1, x_2, \lambda) = P_1 x_1 + P_2 x_2 + \lambda (\bar{u} - 8\sqrt{x_1 x_2})$$

$$\frac{\partial L}{\partial x_1} = 0 \rightarrow P_1 - 4\lambda \left(\frac{x_2}{x_1}\right)^{1/2} = 0 \quad \left. \begin{array}{l} \frac{P_1}{P_2} = \frac{x_2}{x_1} \quad (i) \end{array} \right\}$$

$$\frac{\partial L}{\partial x_2} = 0 \rightarrow P_2 - 4\lambda \left(\frac{x_1}{x_2}\right)^{1/2} = 0$$

$$\frac{\partial L}{\partial \lambda} = 0 \rightarrow \bar{u} = 8\sqrt{x_1 x_2} \rightarrow \text{using (i)} \quad \bar{u} = 8\sqrt{\frac{P_1}{P_2} x_1^2}$$

$$\text{Hicksian (compensated) demand functions} \leftarrow \begin{cases} \Rightarrow x_1^c = \frac{\bar{u}}{8} \sqrt{\frac{P_2}{P_1}} \\ \Rightarrow x_2^c = \frac{\bar{u}}{8} \sqrt{\frac{P_1}{P_2}} \end{cases}$$

$$c) E(P_1, P_2, \bar{u}) = P_1 x_1 + P_2 x_2 = P_1 \frac{\bar{u}}{8} \sqrt{\frac{P_2}{P_1}} + P_2 \frac{\bar{u}}{8} \sqrt{\frac{P_1}{P_2}} = \frac{\bar{u}}{4} \sqrt{P_1 P_2}$$

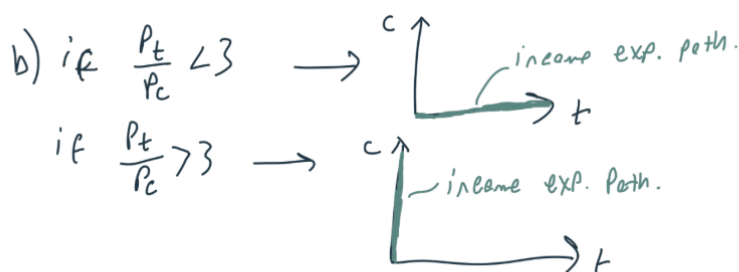
7- Suppose that the consumer now faces a choice between drinking tea and coffee, from which she obtains the utility $u(c, t) = \ln(3t + c)$.

- What are the Marshallian demand functions of tea and coffee, given their prices as P_t and P_c , respectively.
- What is the income expansion path?
- Derive the Engel curves of both goods.

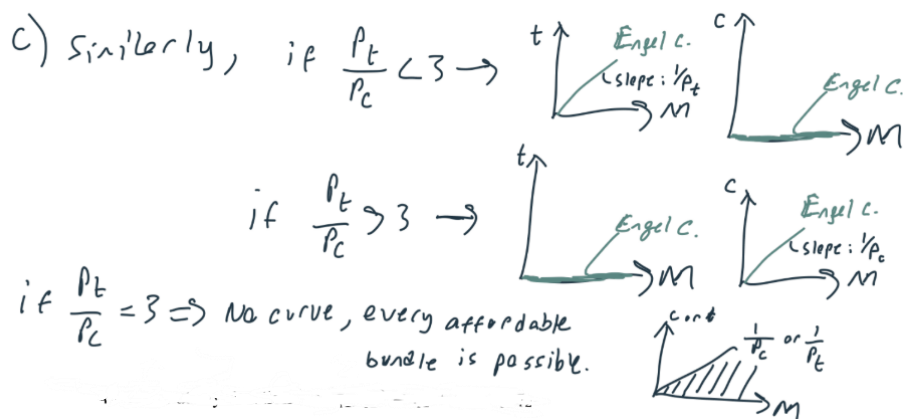
Solution:

a. $U(c, t) = \ln(3t + c)$ is equivalent to utility function $u(c, t) = 3t + c$. So, she considers them as perfect substitutes.

$$(t^*, c^*) = \begin{cases} \left(\frac{M}{P_t}, 0\right) & \text{if } \frac{P_t}{P_c} < MRS = \frac{MU_t}{MU_c} = 3 \\ \{ (t, c) \mid P_t t + P_c c = M \} & \text{if } \frac{P_t}{P_c} = 3 \\ \left(0, \frac{M}{P_c}\right) & \text{if } \frac{P_t}{P_c} > 3 \end{cases}$$



if $\frac{P_t}{P_c} = 3 \Rightarrow$ There is no path, it becomes the whole region.



- 8- Use the Slutsky equation to show that a Giffen good must be an inferior good, but an inferior good need not be a Giffen good. Show the latter case graphically.

Solution:

$$\text{Slutsky decomposition: } \frac{\partial X}{\partial P_X} = \underbrace{\frac{\partial X^c}{\partial P_X}}_{\text{S.E.}} - \underbrace{X \frac{\partial X}{\partial I}}_{\text{I.E.}}$$

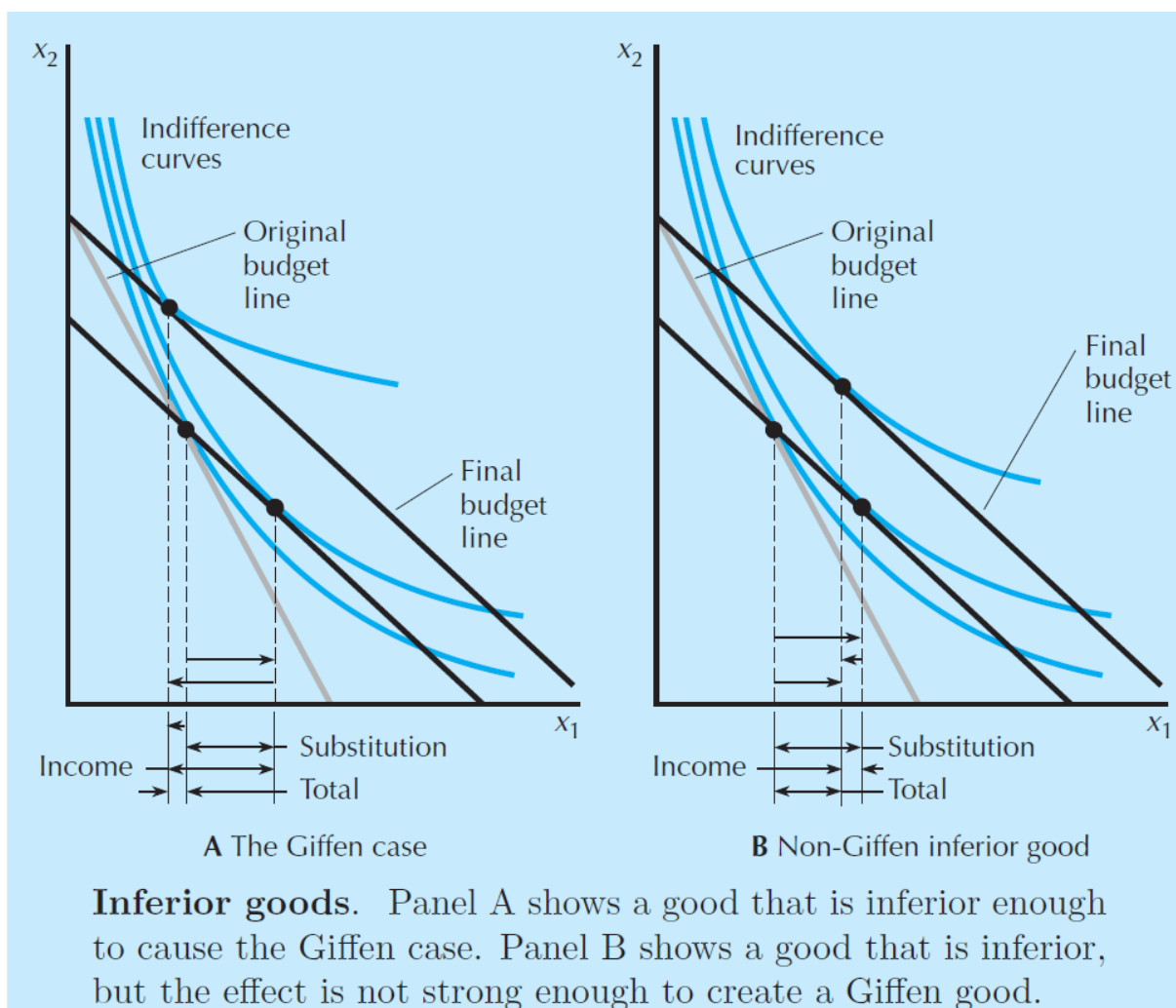
$\overset{\text{inferior}}{\text{inferior}}$

Giffen means $\frac{\partial X}{\partial P_X} > 0$

Since $\frac{\partial X^c}{\partial P_X} < 0$ (always)

$-X \frac{\partial X}{\partial I}$ has to be positive.

So $\Rightarrow \frac{\partial X}{\partial P_X} < 0 \Leftrightarrow X$ is inferior
opposite is not true



- 9- Suppose Rafael spends his monthly stipend only on ice cream (M) and chocolate (C) and his utility function for these goods is $U(M,C) = MC$. Rafael has a monthly income of 60€ and the prices are $P_M = 5€$ & $P_C = 6€$. What will be the income and substitution effects of an increase of 1.5 € in the price of chocolate?

Solution: $P_C = 6 \rightarrow P_C = 7$ $\mathcal{L} = MC + \lambda(I - P_C C - P_M M)$

$$\left. \begin{array}{l} \frac{\partial \mathcal{L}}{\partial M} = C - \lambda P_M = 0 \\ \frac{\partial \mathcal{L}}{\partial C} = M - \lambda P_C = 0 \end{array} \right\} \rightarrow \frac{P_M}{P_C} = \frac{C}{M}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \Rightarrow I = P_C C + P_M M \Rightarrow \left(C_0^* = \frac{I}{2P_C}, M_0^* = \frac{I}{2P_M} \right)$$

$$\Rightarrow M_0^* = 6 \quad C_0^* = 5 \quad \text{and} \quad M_1^* = 6, C_1^* = 4$$

T.E. for C^* is $\Delta C^* = -1$

To find the S.E. we need to hold the purchasing power constant. In other words, compensate for the income loss.

$$\min_{c, m} P_c C + P_m M \quad \text{s.t.} \quad PM = \bar{U} = 6.5 = 30$$

$$\mathcal{L} = P_c C + P_m M + \mu(30 - MC)$$

$$\frac{\partial \mathcal{L}}{\partial C} = P_c - \mu M = 0 \quad \left. \vphantom{\frac{\partial \mathcal{L}}{\partial C}} \right\} \rightarrow \frac{M}{C} = \frac{P_c}{P_m} \Rightarrow M = \frac{C P_c}{P_m}$$

$$\frac{\partial \mathcal{L}}{\partial M} = P_m - \mu C = 0$$

$$\frac{\partial \mathcal{L}}{\partial \mu} = 0 \Rightarrow 30 - MC = 0 \Rightarrow 30 = \frac{C^2 P_c}{P_m} \Rightarrow C^c = \sqrt{30 \frac{P_m}{P_c}} \quad M^c = \sqrt{30 \frac{P_c}{P_m}}$$

$$\begin{aligned} P_c = 6, P_m = 5 &\Rightarrow M^c = 6, C^c = 5 \\ P_c = 7.5, P_m = 5 &\Rightarrow M^c = 3\sqrt{5}, C^c = 2\sqrt{5} \end{aligned} \quad \left. \vphantom{\begin{aligned} P_c = 6, P_m = 5 \\ P_c = 7.5, P_m = 5 \end{aligned}} \right\} \text{S.E. is } \Delta C^c = 2\sqrt{5} - 5 \cong -0.5$$

$$IE = T.E. - S.E. = -1 + 0.5 = -0.5$$