

Problem Set II

Content: Order Statistics and Estimation

Question 1

Suppose that the length of time Y it takes a worker to complete a certain task has the probability density function given by

$$f(y) = \begin{cases} e^{-(y-\theta)}, & y > \theta \\ 0, & \text{elsewhere} \end{cases}$$

where θ is a positive constant that represents the minimum time until task completion. Let $Y_1, Y_2, \dots, Y_n$ denote a random sample of completion times from this distribution. Find

- The density function for $Y_{(1)} = \min(Y_1, Y_2, \dots, Y_n)$
- $E(Y_{(1)})$

Question 2

Let $Y_1, Y_2, \dots, Y_n$ be independent, uniformly distributed random variables on the interval $[0, \theta]$. Find the

- Probability distribution function of $Y_{(n)} = \max(Y_1, Y_2, \dots, Y_n)$
- Density function of Y_n
- Mean and variance of Y_n

Question 3

Let $Y_1, Y_2, \dots, Y_n$ denote a random sample of size n from a population whose density is given by. Find

$$f(y) = \begin{cases} \alpha y^{\alpha-1/\theta^\alpha}, & 0 \leq y \leq \theta \\ 0, & \text{elsewhere} \end{cases}$$

where $\alpha > 0$ is a known, fixed value but θ is unknown. Consider the estimator $\hat{\theta} = \max(Y_1, Y_2, \dots, Y_n)$.

- Show that $\hat{\theta}$ is a biased estimator for θ .
- Find a multiple of $\hat{\theta}$ that is an unbiased estimator of θ .
- Derive $MSE(\hat{\theta})$.

Question 4

Let $X_1, X_2, \dots, X_n$ denote a random sample from a Uniform $(0, \theta)$ distribution with $\theta > 0$ as the unknown parameter. Let $\bar{X}$ denote the sample mean.

- Is $\bar{X}$ an unbiased estimator for θ ?
- Find an unbiased estimator for θ .
- Find the variance of the estimator in the part b.

Question 5

Suppose that $\hat{\theta}$ is an estimator for parameter θ and $E(\hat{\theta}) = a\theta + b$ for some nonzero constants a and b .

- In terms of a , b and θ , what is $B(\hat{\theta})$?
- Find a function $\hat{\theta}^*$, say $\hat{\theta}^*$ that is an unbiased estimator for θ .
- Consider $\hat{\theta}^*$ you found in b. Express $MSE(\hat{\theta}^*)$ as a function of $V(\hat{\theta})$.
- Give an example of a value of a for which $MSE(\hat{\theta}^*) < MSE(\hat{\theta})$ and $MSE(\hat{\theta}^*) > MSE(\hat{\theta})$.

Question 6

Suppose $S^2 = \frac{\sum_{i=1}^n (Y_i - \bar{Y})^2}{n-1}$ $S_1^2 = \frac{\sum_{i=1}^n (Y_i - \bar{Y})^2}{n}$ Then S_1^2 is a biased estimator of σ^2 , but S^2 is an unbiased estimator of the same parameter. If we sample from a normal population,

- Find $V(S^2)$ and show that $V(S^2) > V(S_1^2)$
- Which is a better estimator $V(S^2)$ or $V(S_1^2)$ (Hint: compare mean square errors).
- $V(S^2)$ and $V(S_1^2)$ are two estimators of σ^2 of the form $S^2 = c \sum_{i=1}^n (Y_i - \bar{Y})^2$. What value for c yields the estimator σ^2 with the smallest mean square error among all estimators of the form $S^2 = c \sum_{i=1}^n (Y_i - \bar{Y})^2$.

Question 7

Suppose that $Y_1, Y_2, \dots, Y_n$ are iid exponential observations with mean θ . Derive an unbiased estimator for $T(\theta) = 1/\theta$

Question 8

Let $Y_1, Y_2, \dots, Y_n$ denote a random sample of size n from a population whose density is given by

$$f(y) = \begin{cases} 3\beta^3 y^{-4}, & \beta \leq y \\ 0, & \text{elsewhere} \end{cases}$$

where $\beta > 0$ is unknown. Consider the estimator $\hat{\beta} = \min(Y_1, Y_2, \dots, Y_n)$.

- a. Derive the bias of the estimator $\hat{\beta}$.
- b. Derive $MSE(\hat{\beta})$