

Spring 2024-
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ECON 205

Problem Set 9 - Answers

1)

Let Y be a random variable with a density function given by

$$f(y) = \begin{cases} (3/2)y^2, & -1 \leq y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

- a Find the density function of $U_1 = 3Y$.
- b Find the density function of $U_2 = 3 - Y$.
- c Find the density function of $U_3 = Y^2$.

SOLUTION

The distribution function of Y is $F_Y(y) = \int_{-1}^y (3/2)t^2 dt = (1/2)(y^3 + 1)$, $-1 \leq y \leq 1$.

- a. $F_{U_1}(u) = P(U_1 \leq u) = P(3Y \leq u) = P(Y \leq u/3) = F_Y(u/3) = \frac{1}{2}(u^3/18 + 1)$. Thus,
 $f_{U_1}(u) = F'_{U_1}(u) = u^2/18$, $-3 \leq u \leq 3$.
- b. $F_{U_2}(u) = P(U_2 \leq u) = P(3 - Y \leq u) = P(Y \geq 3 - u) = 1 - F_Y(3 - u) = \frac{1}{2}[1 - (3 - u)^3]$.
Thus, $f_{U_2}(u) = F'_{U_2}(u) = \frac{3}{2}(3 - u)^2$, $2 \leq u \leq 4$.
- c. $F_{U_3}(u) = P(U_3 \leq u) = P(Y^2 \leq u) = P(-\sqrt{u} \leq Y \leq \sqrt{u}) = F_Y(\sqrt{u}) - F_Y(-\sqrt{u}) = u^{3/2}$.
Thus, $f_{U_3}(u) = F'_{U_3}(u) = \frac{3}{2}\sqrt{u}$, $0 \leq u \leq 1$.

2)

A parachutist wants to land at a target T , but she finds that she is equally likely to land at any point on a straight line (A, B) , of which T is the midpoint. Find the probability density function of the distance between her landing point and the target. [Hint: Denote A by -1 , B by $+1$, and T by 0 . Then the parachutist's landing point has a coordinate X , which is uniformly distributed between -1 and $+1$. The distance between X and T is $|X|$.]

SOLUTION

If we let $(A, B) = (-1, 1)$ and $T = 0$, the density function for X , the landing point is
 $f(x) = 1/2$, $-1 < x < 1$.

We must find the distribution of $U = |X|$. Therefore,

$$F_U(u) = P(U \leq u) = P(|X| \leq u) = P(-u \leq X \leq u) = [u - (-u)]/2 = u.$$

So, $f_U(u) = F'_U(u) = 1$, $0 \leq u \leq 1$. Therefore, U has a uniform distribution on $(0, 1)$.

3)

A fluctuating electric current I may be considered a uniformly distributed random variable over the interval (9, 11). If this current flows through a 2-ohm resistor, find the probability density function of the power $P = 2I^2$.

SOLUTION

Since P is an increasing function of I for I in (9,11), the method of transformations can be used.

The density function for I is $f_I(i) = 1/2, 9 \leq i \leq 11$. For $P = 2I^2, I = \sqrt{P/2}$ and $\frac{di}{dp} = (1/2)^{3/2} p^{-1/2}$. Then, $f_p(p) = \frac{1}{4\sqrt{2p}}, 162 \leq p \leq 242$.

4) The amount of flour used per day by a bakery is a random variable Y that has an exponential distribution with mean equal to 4 tons. The cost of the flour is $U=3Y+1$. Find the probability density function of U .

SOLUTION

The distribution function of Y is $F_Y(y) = 1 - e^{-\frac{y}{4}}, 0 \leq y$.

$$F_U(u) = P(U \leq u) = P(3Y + 1 \leq u) = P\left(Y \leq \frac{u-1}{3}\right) = F_Y\left(\frac{u-1}{3}\right) = 1 - e^{-\frac{(u-1)}{12}} \text{ for } \frac{u-1}{3} \geq 0.$$

$$f_U(u) = F'_U(u) = \frac{1}{12} e^{-\frac{(u-1)}{12}}, u \geq 1$$

$$f_U(u) = \begin{cases} \frac{1}{12} e^{-\frac{(u-1)}{12}}, & \text{for } u \geq 1 \\ 0, & \text{elsewhere} \end{cases}$$

5) For the given joint density function, find the probability density function of $U=Y_1-Y_2$.

$$f(y_1, y_2) = \begin{cases} e^{-y_1}, & 0 \leq y_2 \leq y_1 \leq \infty \\ 0, & \text{elsewhere} \end{cases}$$

SOLUTION

$$F_U(u) = P(U \leq u) = P(Y_1 - Y_2 \leq u) = P(Y_1 \leq u + Y_2)$$

$$\text{For } u \geq 0, P(Y_1 \leq u + Y_2) = \int_0^\infty \int_{y_2}^{u+y_2} e^{-y_1} dy_1 dy_2 = 1 - e^{-u}$$

$$\text{For } u < 0, P(Y_1 \leq u + Y_2) = 0$$

$$f_U(u) = F'_U(u) = e^{-u}, u \geq 0$$

$$f_U(u) = \begin{cases} e^{-u}, & u \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

6)

Let Y be a random variable with probability density function given by

$$f(y) = \begin{cases} 2(1-y), & 0 \leq y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

a Find the density function of $U_1 = 2Y - 1$.

b Find the density function of $U=Y^2$

c Find the density function of $U=1-2Y$

Note: Use the method of transformation.

SOLUTION

a. Note that $U=2Y-1$ is a (strictly) monotonic function of Y for all y such that $f_y(y)>0$, the transformation method can be used.

If $U=2Y-1$, then $Y=\frac{U+1}{2}$. Thus, $\frac{dy}{du} = \frac{1}{2}$ and $f_U(u) = \frac{1}{2} 2 \left(1 - \frac{u+1}{2}\right) = \frac{1-u}{2}$, $-1 \leq u \leq 1$.

$$f_U(u) = \begin{cases} \frac{1-u}{2} & \text{for } -1 \leq u \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

b. Note that $U=Y^2$ is a (strictly) monotonic function of Y for all y such that $f_y(y)>0$, the transformation method can be used.

If $U=Y^2$, then $Y=\sqrt{U}$. Thus, $\frac{dy}{du} = \frac{1}{2\sqrt{u}}$ and $f_U(u) = \frac{1}{2\sqrt{u}} 2(1 - \sqrt{u}) = \frac{1-\sqrt{u}}{\sqrt{u}}$, $0 < u \leq 1$.

$$f_U(u) = \begin{cases} \frac{1}{\sqrt{u}} - 1 & \text{for } 0 < u \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

c. Note that $U=1-2Y$ is a (strictly) monotonic function of Y for all y such that $f_y(y)>0$, the transformation method can be used.

If $U=1-2Y$, then $Y=\frac{1-U}{2}$. Thus, $\left|\frac{dy}{du}\right| = \frac{1}{2}$ and $f_U(u) = \frac{1}{2} 2 \left(1 - \frac{1-u}{2}\right) = \frac{1+u}{2}$, $-1 \leq u \leq 1$.

$$f_U(u) = \begin{cases} \frac{1+u}{2} & \text{for } -1 \leq u \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

7)

Let the probability density function of Y be given by

$$f(x) = \begin{cases} 6x(1-x), & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

- a) Find the probability density of $Y=X^3$.
- b) Find the probability density of $Y=X^4$.

SOLUTION

a)

Let $F_Y(y)$ denote the value of the cumulative distribution function of Y at y and write

$$F_Y(y) = P(Y \leq y) = P(X^3 \leq y) = P(X \leq y^{1/3}) \quad (1)$$

For $0 < y^{1/3} < 1$, that is for $0 < y < 1$:

$$F_Y(y) = \int_0^{y^{1/3}} 6x(1-x)dx = \int_0^{y^{1/3}} (6x - 6x^2)dx \quad (2)$$

$$F_Y(y) = (3x^2 - 2x^3) \Big|_{x=0}^{x=y^{1/3}} = 3y^{2/3} - 2y$$

So,

$$F_Y(y) = \begin{cases} 0, & y \leq 0 \\ 3y^{2/3} - 2y, & 0 < y < 1 \\ 1, & y \geq 1 \end{cases} \quad (3)$$

Now we differentiate $F_Y(y)$ to obtain the density function $f_Y(y)$.

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{d}{dy} (3y^{2/3} - 2y)$$
$$f_Y(y) = \begin{cases} 2y^{-1/3} - 2 = 2(y^{-1/3} - 1), & 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

b)

For $0 < y < 1$: $F_Y(y) = P(Y \leq y) = P(X^4 \leq y) = P\left(-y^{\frac{1}{4}} \leq X \leq y^{\frac{1}{4}}\right) = \int_0^{y^{\frac{1}{4}}} (6x - 6x^2) dx = 3x^2 - 2x^3 \Big|_{x=0}^{y^{\frac{1}{4}}} = 3y^{\frac{1}{2}} - 2y^{\frac{3}{4}}$

$$F_Y(y) = \begin{cases} 0, & y \leq 0 \\ 3y^{\frac{1}{2}} - 2y^{\frac{3}{4}}, & 0 < y < 1 \\ 1, & y \geq 1 \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{3}{2}y^{-1/2} - \frac{3}{2}y^{-\frac{1}{4}}, & 0 < y < 1 \\ 0, & \text{elsewhere} \end{cases}$$

8)

X is a continuous variable with the following probability density function:

$$f_X(x) = 3x^2, \text{ for } 0 < x < 1$$

- i) Find the probability density of $Y = X^2$
- ii) Now assume that probability density function of X is given by

$$f_X(x) = 3(1-x)^2, \text{ for } 0 < x < 1$$

Find the probability density function of $Y = (1 - X)^3$

SOLUTION

- i) Since $Y = X^2$ is a (strictly) monotonic function of X for all x such that $f_X(x) > 0$, the transformation method can be used.

Note that the function:

$$Y = X^2$$

defined over the interval $0 < x < 1$ is an invertible function. The inverse function is:

$$x = v(y) = \sqrt{y} = y^{1/2} \text{ for } 0 < y < 1 \quad (1)$$

Now, taking derivative of $v(y)$, we get:

$$v'(y) = \frac{1}{2}y^{-1/2} \quad (2)$$

Therefore, the transformation method:

$$f_Y(y) = f_X(v(y)) * |v'(y)| \quad (3)$$

tells us that the probability function of Y is:

$$f_Y(y) = 3[y^{1/2}]^2 * \frac{1}{2}y^{-1/2} \quad (4)$$

And, simplifying we get that the probability density function of Y is:

$$f_Y(y) = \begin{cases} \frac{3}{2}y^{1/2}, & 0 < y < 1 \\ 0, & \text{otherwise} \end{cases} \quad (5)$$

- ii) Since $Y = (1-X)^3$ is a (strictly) monotonic function of X for all x such that $f_X(x) > 0$, the transformation method can be used.

Note that the function:

$$Y = (1 - X)^3$$

defined over the interval $0 < x < 1$ is an invertible function. The inverse function is:

$$x = v(y) = 1 - y^{1/3} \text{ for } 0 < y < 1 \quad (6)$$

Now, taking derivative of $v(y)$, we get:

$$v'(y) = -\frac{1}{3}y^{-2/3} \quad (7)$$

Therefore, the transformation method:

$$f_Y(y) = f_X(v(y)) * |v'(y)| \quad (8)$$

tells us that the probability function of Y is:

$$f_Y(y) = 3[1 - (1 - y^{1/3})]^2 * \left| -\frac{1}{3}y^{-2/3} \right| = 3y^{2/3} * \frac{1}{3}y^{-2/3} \quad (9)$$

And, simplifying we get that the probability density function of Y is:

$$f_Y(y) = \begin{cases} 1, & 0 < y < 1 \\ 0, & \text{otherwise} \end{cases} \quad (10)$$

- 9) Suppose X_1 and X_2 are independent exponential random variables with parameter $\lambda = 1$ so that

$$f_{x_1}(x_1) = e^{-x_1}, \quad 0 < x_1 < \infty$$

$$f_{x_2}(x_2) = e^{-x_2}, \quad 0 < x_2 < \infty$$

The joint pdf is given by

$$f(x_1, x_2) = f_{x_1}(x_1)f_{x_2}(x_2) = e^{-x_1-x_2}, \quad 0 < x_1 < \infty, \quad 0 < x_2 < \infty$$

Consider the transformation $Y_1 = X_1 - X_2, Y_2 = X_1 + X_2$. Find the joint distribution of Y_1 and Y_2 .

SOLUTION

First, note that the given transformation is a one-to-one transformation with the following inverse: We write $x_1 = v_1(y_1, y_2)$ and $x_2 = v_2(y_1, y_2)$

$$x_1 = v_1(y_1, y_2) = \frac{y_1 + y_2}{2}, \quad (1)$$

$$x_2 = v_2(y_1, y_2) = \frac{y_2 - y_1}{2} \quad (2)$$

The Jacobian, J is

$$J = \begin{vmatrix} \frac{\partial \left(\frac{y_1 + y_2}{2}\right)}{\partial y_1} & \frac{\partial \left(\frac{y_1 + y_2}{2}\right)}{\partial y_2} \\ \frac{\partial \left(\frac{y_2 - y_1}{2}\right)}{\partial y_1} & \frac{\partial \left(\frac{y_2 - y_1}{2}\right)}{\partial y_2} \end{vmatrix} \quad (3)$$

$$\det(J) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2} \quad (4)$$

Therefore,

$$g(y_1, y_2) = e^{-v_1(y_1, y_2) - v_2(y_1, y_2)} \cdot \left| \frac{1}{2} \right| \quad (5)$$

$$g(y_1, y_2) = e^{-\frac{y_1 + y_2}{2} - \frac{y_2 - y_1}{2}} \cdot \left| \frac{1}{2} \right| = \frac{1}{2} e^{-y_2} \quad (6)$$

Now, we determine the support of (Y_1, Y_2) . Since $0 < x_1 < \infty$, $0 < x_2 < \infty$, we have $0 < \frac{y_1 + y_2}{2} < \infty$, $0 < \frac{y_2 - y_1}{2} < \infty$ or $0 < y_1 + y_2 < \infty$, $0 < y_2 - y_1 < \infty$. This may be written as $-y_2 < y_1 < y_2$, $0 < y_2 < \infty$.

Hence the joint pdf of Y_1 and Y_2 is :

$$g(y_1, y_2) = \begin{cases} \frac{1}{2} e^{-y_2} & \text{for } -y_2 < y_1 < y_2 \text{ and } 0 < y_2 < \infty \\ 0, & \text{elsewhere} \end{cases}$$

10) Suppose X has the density

$$f(x) = \begin{cases} \frac{\theta}{x^{\theta+1}} & \text{for } x > 1 \\ 0, & \text{elsewhere} \end{cases}$$

Find the density of $Y=\ln X$ where θ is a positive parameter.
 $Y=\ln X, x=e^y$.

SOLUTION

$Y=\ln X$ is a (strictly) monotonic function of X for all y such that $f_X(x)>0$, the transformation method can be used.

$$\left| \frac{dx}{dy} \right| = e^y$$

Note that $x>1$ if and only if $e^y > 1$ if and only if $y>0$.

$$f(y) = \frac{\theta}{(e^y)^{\theta+1}} e^y = \theta e^{-y\theta} \text{ if } y > 0$$
$$f(y) = \begin{cases} \theta e^{-y\theta} & \text{if } y > 0 \\ 0, & \text{elsewhere} \end{cases}$$