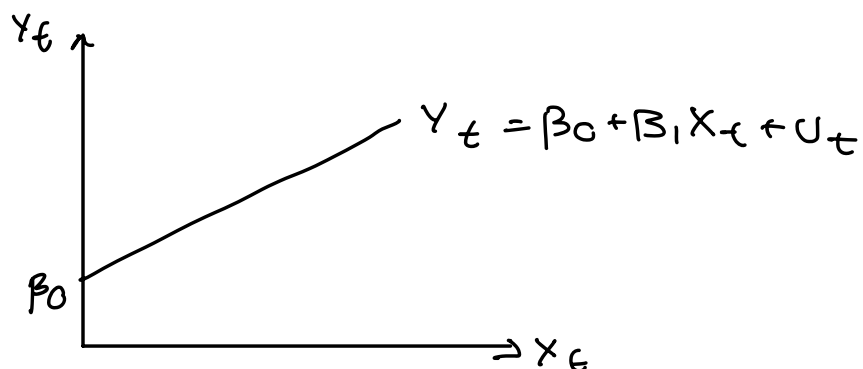


The Model Without an Intercept Term

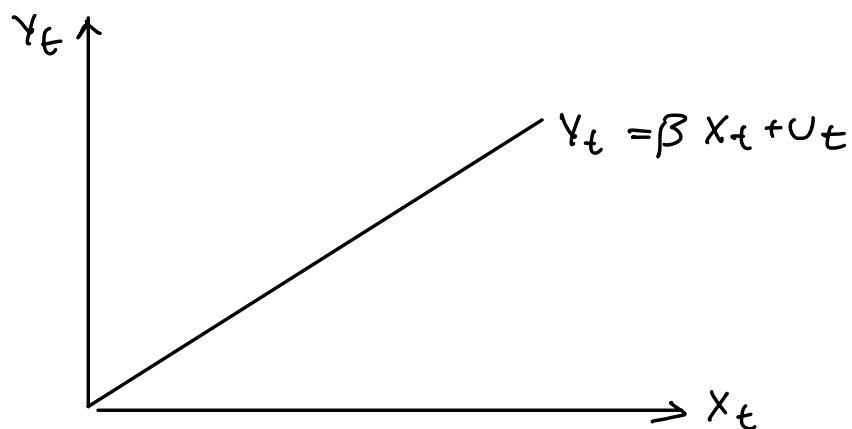
$$Y_t = \beta_0 + \beta_1 X_t + U_t$$

the model assumes that there is an autonomous part of Y_t which is independent of X_t



however, under certain theory priors, we may postulate that the intercept term is zero. The regression line passes through the origin:

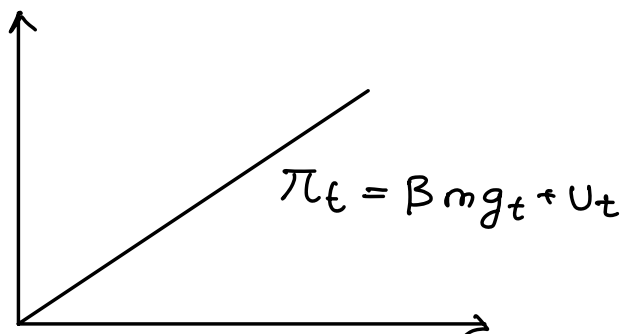
$$Y_t = \beta X_t + U_t$$



Example :

monetarists postulate that inflation (π_t) is solely caused by excess money growth over real income growth (mg_t)

$$\pi_t = \beta mg_t + u_t$$



the model

$$y_t = \beta x_t + u_t$$

We maintain that our standard assumptions are satisfied

$$E(u_t) = 0, \text{Var}(u_t) = \sigma^2, \text{Cov}(u_t, u_s) = 0$$

All x_t are not the same

x_t is exogenous for the evolution of y_t :

$x \rightarrow y$ but not the reverse ~~($y \rightarrow x$)~~

Estimation

$$y_t = \beta x_t + u_t \rightarrow \hat{y}_t = \hat{\beta} x_t \quad \hat{u}_t = y_t - \hat{y}_t$$

minimize SSR w.r.t. $\hat{\beta}$

$$\text{SSR} = \sum \hat{u}_t^2 = \sum (y_t - \hat{y}_t)^2$$

FOC's

$$\frac{\partial SSR}{\partial \hat{\beta}} = \frac{\partial \sum (y_t - \hat{y}_t)^2}{\partial \hat{\beta}} = 0$$
$$= -2 \sum x_t (y_t - \hat{\beta} x_t) = 0$$

$$\Rightarrow \hat{\beta} = \frac{\sum x_t y_t}{\sum x_t^2}$$

Sampling properties

$$\hat{\beta} = \frac{\sum x_t y_t}{\sum x_t^2} = \sum A_t y_t \quad \text{where} \quad A_t = \frac{x_t}{\sum x_t^2}$$

$\hat{\beta}$ is linear

$$\hat{\beta} = \sum A_t y_t = \sum A_t (\beta x_t + u_t)$$

$$\hat{\beta} = \beta \underbrace{\sum A_t x_t}_{=1} + \sum A_t u_t$$

$$\hat{\beta} = \beta + \sum A_t u_t \quad E(\hat{\beta}) = \beta \quad \text{unbiased}$$

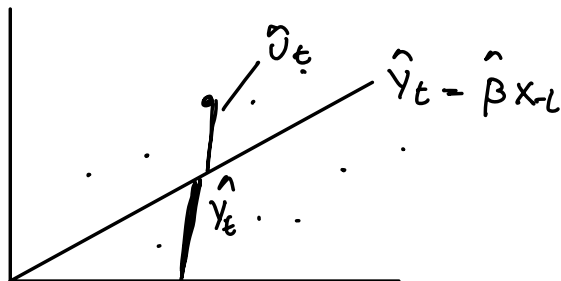
Variance

$$\text{Var}(\hat{\beta}) = E(\hat{\beta} - E(\hat{\beta}))^2 = E(\hat{\beta} - \beta)^2$$
$$= E[\sum A_t u_t]^2 = \sigma^2 \sum A_t^2$$

$$\text{Var}(\hat{\beta}) = \frac{\sigma^2}{\sum x_t^2} \quad \sigma^2 = \frac{\sum \hat{u}_t^2}{T-k}$$

$$\hat{\sigma}_{\beta}^2 = \frac{\hat{\sigma}^2}{\sum x_t^2} = \frac{\frac{\sum \hat{u}_t^2}{T-k}}{\sum x_t^2}$$

the coefficient of determination



$$y_t = \hat{y}_t + \hat{u}_t$$

$$\sum y_t^2 = \sum \hat{y}_t^2 + \sum \hat{u}_t^2 + 2 \underbrace{\sum \hat{y}_t \hat{u}_t}_{=0}$$

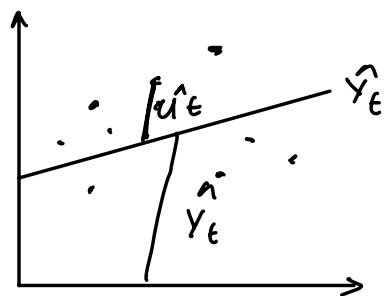
$$\underbrace{\frac{\sum y_t^2}{\sum y_t^2}}_{=1} = \underbrace{\frac{\sum \hat{y}_t^2}{\sum y_t^2}}_{R_r^2} + \frac{\sum \hat{u}_t^2}{\sum y_t^2}$$

$$R_r^2 = 1 - \frac{\sum \hat{u}_t^2}{\sum y_t^2} \quad \text{raw data coefficient of det.}$$

$$0 \leq R_r^2 \leq 1$$

R^2 can be defined also for the model with an intercept term:

$$Y_t = \beta_0 + \beta_1 X_t + u_t$$



$$Y_t = \hat{Y}_t + \hat{u}_t$$

$$\sum Y_t^2 = \sum \hat{Y}_t^2 + \sum \hat{u}_t^2 + 2 \underbrace{\sum \hat{Y}_t \hat{u}_t}_{=0}$$

$$R^2 = \frac{\sum Y_t^2}{\sum Y_t^2} = 1 - \frac{\sum \hat{u}_t^2}{\sum Y_t^2}$$

2- Suppose we have a linear regression model

$$y_t = \beta_0 + \beta_1 x_t + u_t \quad t = 1, \dots, T$$

where

$$E(u_t) = 0$$

$$E(u_t^2) = \sigma$$

$$E(u_t u_s) = 0 \text{ for all } t \neq s$$

$$Y_t = \beta_0 + \beta_1 X_t + u_t \rightarrow \text{population regression}$$

$$\hat{Y}_t = \hat{\beta}_0 + \hat{\beta}_1 X_t \rightarrow \text{estimated regression}$$

$$Y_t - \hat{Y}_t = \hat{u}_t$$

$$\hat{u}_t = y_t - \hat{y}_t = y_t - \hat{\beta}_0 - \hat{\beta}_1 x_t$$

$$SSR = \sum \hat{u}_t^2 = \sum (y_t - \hat{\beta}_0 - \hat{\beta}_1 x_t)^2$$

a) show $\sum \hat{u}_t = 0$

FOC 1 $\frac{\partial SSR}{\partial \hat{\beta}_0} = 0 \rightarrow -2 \sum (y_t - \hat{\beta}_0 - \hat{\beta}_1 x_t) = 0$

$$\hat{u}_t$$

$$-2 \sum \hat{u}_t = 0$$

$$\sum \hat{u}_t = 0$$

b) show that $\text{cov}(x_t, \hat{u}_t) = 0$

$$\text{cov}(x_t, \hat{u}_t) = E(x_t - \underbrace{E(x_t)}_{\bar{x}}) (\hat{u}_t - \underbrace{E(\hat{u}_t)}_{0})$$

where $E(\hat{u}_t) = \frac{\sum \hat{u}_t}{T} = 0$

$$\text{cov}(x_t, \hat{u}_t) = E[(x_t - \bar{x}) \hat{u}_t] =$$

$$\underbrace{E(x_t \hat{u}_t)}_{0} - \bar{x} \underbrace{E(\hat{u}_t)}_{0} = E(x_t \hat{u}_t)$$

FOC 2:

$$\frac{\partial SSR}{\partial \hat{\beta}_1} = 0 \rightarrow -2 \sum x_t (y_t - \hat{\beta}_0 - \hat{\beta}_1 x_t) = 0$$

$$= \hat{u}_t$$

$$-2 \sum x_t \hat{u}_t = 0$$

$$\sum x_t \hat{u}_t = 0$$

$$E(x_t \hat{u}_t) = \frac{\sum x_t \hat{u}_t}{T} = 0 \quad \text{cov}(x_t, \hat{u}_t) = 0$$

c) Show that $\hat{\beta}_1 = \frac{\text{cov}(x_t, y_t)}{\text{var}(x_t)}$

$$\hat{\beta}_1 = \frac{\sum x_t y_t}{\sum x_t^2} = \frac{\sum (x_t - \bar{x})(y_t - \bar{y})}{\sum (x_t - \bar{x})^2}$$

$$\hat{\beta}_1 = \frac{\frac{\sum (x_t - \bar{x})(y_t - \bar{y})}{T}}{\frac{\sum (x_t - \bar{x})^2}{T}} = \frac{E[(x_t - \bar{x})(y_t - \bar{y})]}{E[(x_t - \bar{x})^2]}$$

$$\hat{\beta}_1 = \frac{E[(x_t - E(x_t))(y_t - E(y_t))]}{E[(x_t - E(x_t))^2]} = \frac{\text{cov}(x_t, y_t)}{\text{var}(x_t)}$$

d) Show that $\text{var}(\hat{\beta}_1) = \frac{\sigma^2}{\sum x_t^2}$

$$\hat{\beta}_1 = \frac{\sum x_t y_t}{\sum x_t^2}, \quad \text{define } w_t = \frac{x_t}{\sum x_t^2}$$

$$\hat{\beta}_1 = \sum w_t y_t = \sum w_t (y_t - \bar{y}) = \sum w_t y_t - \bar{y} \sum w_t$$

$$\text{where } \sum w_t = \frac{\sum x_t}{\sum x_t^2} = \frac{\sum (x_t - \bar{x})}{\sum x_t^2} = \frac{T\bar{x} - T\bar{x}}{\sum x_t^2}$$

$$\hat{\beta}_1 = \sum w_t y_t = \sum w_t (\beta_0 + \beta_1 x_t + u_t) = \sum w_t = 0$$

$$\hat{\beta}_1 = \beta_0 \sum w_t + \beta_1 \sum w_t x_t + \sum w_t u_t$$

$$\text{where } \sum w_t x_t = \frac{\sum x_t x_t}{\sum x_t^2} = \frac{\sum (x_t - \bar{x}) x_t}{\sum x_t^2}$$

$$= \frac{\sum x_t^2 - \bar{X} \sum x_t}{\sum x_t^2} = \frac{\sum x_t^2 - T \bar{X}^2}{\sum x_t^2} = \frac{\sum (x_t - \bar{X})^2}{\sum x_t^2}$$

$$= \frac{\sum x_t^2}{\sum x_t^2} = 1$$

$$\hat{\beta}_1 = \beta_1 + \sum w_t u_t$$

$$E(\hat{\beta}_1) = \beta_1 + E(\sum w_t u_t)$$

$$E(\hat{\beta}_1) = \beta_1 + \sum w_t \underbrace{E(u_t)}_0 \quad E(\hat{\beta}_1) = \beta_1$$

then,

$$\text{var}(\hat{\beta}_1) = E[\hat{\beta}_1 - \underbrace{E(\hat{\beta}_1)}_{\beta_1}]^2 = E(\hat{\beta}_1 - \beta_1)^2 = E(\sum w_t u_t)^2$$

$$\text{var}(\hat{\beta}_1) = E(w_1 u_1 + w_2 u_2 + \dots + w_T u_T)^2$$

$$\text{var}(\hat{\beta}_1) = \underbrace{w_1^2 E(u_1^2)}_{\sigma^2} + \underbrace{w_2^2 E(u_2^2)}_{\sigma^2} + \dots + \underbrace{w_T^2 E(u_T^2)}_{\sigma^2} + \underbrace{w_1 w_2 E(u_1 u_2)}_{=0} + \underbrace{w_1 w_{T-1} E(u_T u_{T-1})}_{=0}$$

$$\text{var}(\hat{\beta}_1) = \sigma^2 \sum w_t^2 = \sigma^2 \sum \left(\frac{x_t}{\sum x_t^2} \right)^2 = \sigma^2 \sum \frac{x_t^2}{(\sum x_t^2)^2}$$

$$\text{var}(\hat{\beta}_1) = \sigma^2 \frac{\sum x_t^2}{(\sum x_t^2)^2} = \frac{\sigma^2}{\sum x_t^2}$$