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Key

NAME:
SECTION:

Middle East Technical University
Department of Economics
ECON 205
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01.12.2003

MIDTERM EXAM
(Time Allowed: 120 minutes, each question accounts 20 pts.)

1. Answer the following questions:

a) (5 pts.) The cumulative distribution function for a random variable X is defined as

$$F(x)=\begin{cases}0 & \text{for } x<0\\ 4x^3-3x^4 & \text{for } 0\le x\le 1\\ 1 & \text{for } x>1\end{cases}$$

Find $P(\frac{1}{4}\le X\le\frac{3}{4})$ $= F(0.75)-F(0.25)$

$= (4(0.75)^3-3(0.75)^4)-(4(0.25)^3-3(0.25)^4)$

$= (0.75)^3(4-3(0.75))-(0.25)^3(4-3(0.25))$

$= (0.75)^3(4-2.25)-(0.25)^3(4-0.75)$

$= (0.75)^3(1.75)-(0.25)^3(3.25)=0.6875$

b) (5 pts.) Suppose that X and Y have the following joint p.d.f.

$$f(x,y)=\begin{cases}c & 0<x<1, 0<y<1\\ 0 & \text{elsewhere}\end{cases}$$

Find c and calculate $P(0\le X\le\frac{1}{2}, 0\le Y\le\frac{1}{4})$.

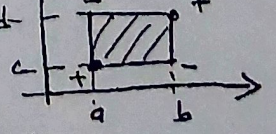
$1=\int_0^1\int_0^1 c\,dx\,dy \Rightarrow c=1/\int_0^1\int_0^1 dx\,dy = 1$ .. i.e. $\boxed{c=1}$

$P(0\le X\le 1/2,\ 0\le Y\le 1/4)=\int_0^{0.5}\int_0^{0.25} dx\,dy = 0.5(0.25)=\underline{0.125}$ (1/8)

c) (5 pts.) Show that $P(a<X\le b, c<Y\le d)=F(b,d)-F(a,d)-F(b,c)+F(a,c)$.

$P(a<X\le b, Y\le d)=F(b,d)-F(a,d)$

$P(a<X\le b, Y\le c)=F(b,c)-F(a,c)$



$P(a<X\le b, c<Y\le d)=P(a<X\le b, Y\le d)-P(a<X\le b, Y\le c)$

$=(F(b,d)-F(a,d))-(F(b,c)-F(a,c))$

$=F(b,d)+F(a,c)-F(a,d)-F(b,c)$

d) (5 pts.) Suppose that the random variables X, Y, and Z have the following multivariate p.d.f.

$$f(x,y,z)=\begin{cases}(x+y)e^{-z} & 0<x<1, 0<y<1, z>0\\ 0 & \text{elsewhere}\end{cases}$$

Find $f(x,y)$, $f(y,z)$, and $f(z)$.

$$f(x,y)=\int_0^\infty f(x,y,z)\,dz=\int_0^\infty (x+y)e^{-z}\,dz = x+y, \quad f(x,y)=\begin{cases}x+y, & 0<x<1,\ 0<y<1\\ 0, & \text{elsewhere}\end{cases}$$

$$f(y,z)=\int_0^1 (x+y)e^{-z}\,dx = e^{-z}\int_0^1 (x+y)\,dx = \left(\tfrac{1}{2}+y\right)e^{-z}$$

$$f(y,z)=\begin{cases}(y+1/2)e^{-z}, & 0<y<1,\ z>0\\ 0, & \text{elsewhere.}\end{cases}$$

$$f_z(z)=\int_0^1 f(y,z)\,dy=\int_0^1 e^{-z}(y+1/2)\,dy = e^{-z}\int_0^1 (y+1/2)\,dy$$

$$= e^{-z}\left(\tfrac{y^2}{2}+\tfrac{1}{2}y\right)\Big|_0^1 = e^{-z}, \quad f_z(z)=\begin{cases}e^{-z}, & z>0\\ 0, & \text{elsewhere.}\end{cases}$$

2. Let X and Y be the random variables with joint probability density function,

$$f(x,y)=\begin{cases}c\,e^{-2x}, & 0<y<x<\infty\\ 0, & \text{elsewhere}\end{cases}$$

a) (5 pts.) Find the constant c.

$$1=\int_{x=0}^{\infty}\int_{y=0}^{x} f(x,y)\,dy\,dx = c\int_{x=0}^{\infty}\int_{y=0}^{x} e^{-2x}\,dy\,dx$$

$$= c\int_0^\infty e^{-2x}\left(\int_0^x dy\right)dx = c\int_0^\infty x e^{-2x}\,dx$$

$$= c\left[-\tfrac{1}{2}xe^{-2x}-\tfrac{1}{4}e^{-2x}\right]_{x=0}^{\infty} = \frac{c}{4} \Rightarrow \boxed{c=4}$$

b) (15 pts.) Calculate the probability that the random variable X is greater than Y. That is, calculate $P(X>Y)$.

First marginals:

$$f_x(x)=\int_{y=0}^{x} f(x,y)\,dy = \int_0^x 4e^{-2x}\,dy = 4xe^{-2x}$$

$$f_x(x)=\begin{cases}4xe^{-2x}, & x>0\\ 0, & \text{elsewhere.}\end{cases}$$

$$f_y(y)=\int_{x=y}^{\infty} 4e^{-2x}\,dx = 2e^{-2y}$$

$$\rightarrow f_y(y)=\begin{cases}2e^{-2y}, & y>0\\ 0, & \text{elsewhere.}\end{cases}$$

Moreover,

$$P(X>y) = \int_y^\infty f_X(x)\,dx = \int_y^\infty 4x e^{-2x}\,dx$$

$$= 2y e^{-2y} + e^{-2y}$$

Therefore,

$$P(X>Y) = \int_{y=0}^\infty P(X>Y \mid Y=y) f_Y(y)\,dy$$

$$= \int_0^\infty P(X>y) f_Y(y)\,dy$$

$$= \int_0^\infty (2y e^{-2y} + e^{-2y})\, 2e^{-2y}\,dy$$

$$= 4\int_0^\infty y e^{-4y}\,dy + 2\int_0^\infty e^{-4y}\,dy$$

$$= 4\left(\frac{1}{16}\right) + 2\cdot\frac{1}{4}$$

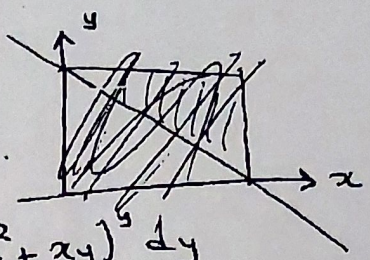
$$= \frac{4}{16} + \frac{2}{4}$$

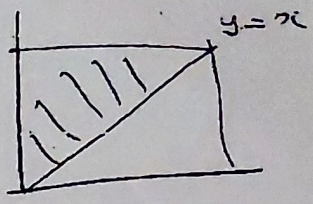
$$= \frac{1}{4} + \frac{1}{2}$$

$$= \frac{3}{4}$$

3. Let X and Y be two random variables with the following joint p.d.f.

$$f(x,y)=\begin{cases} c\,(x+y) & , \quad 0<x<y<1 \\ 0 & , \quad \textit{elsewhere} \end{cases}$$





a) **(5 pts.)** Find c.

$$1 = c\int_{y=0}^{1}\int_{x=0}^{y}(x+y)\,dx\,dy = c\int_0^1\left(\frac{x^2}{2}+xy\right)_0^y dy$$

$$= c\int_0^1\left(\frac{y^2}{2}+y^2\right)dy = c\int_0^1 \frac{3y^2}{2}\,dy = c\,\frac{3}{2}\,\frac{y^3}{3}\Big|_0^1 = \frac{c}{2}$$

$$\Rightarrow \boxed{c=2}$$

b) **(15 pts.)** What is the correlation between X and Y?

$$f(x,y)=\begin{cases} 2(x+y) & , \quad 0<x<y<1 \\ 0 & , \quad \text{else.} \end{cases}$$

$$f_X(x) = \int_{y=x}^{1} 2(x+y)\,dy = 2\left(xy+\frac{y^2}{2}\right)_{y=x}^{1} = 2xy+y^2\Big|_x^1 = 2x(1-x)+1-x^2$$

$$= 2x(1-x)+1-x^2 = 2x-2x^2+1-x^2 = 1+2x-3x^2$$

$$f_X(x)=\begin{cases} 1+2x-3x^2 & , \quad 0<x<1 \\ 0 & , \quad \text{else.} \end{cases}$$

$$f_Y(y) = \int_{x=0}^{y} 2(x+y)\,dx = x^2+2xy\Big|_0^y = y^2+2y^2 = 3y^2$$

$$f_Y(y)=\begin{cases} 3y^2 & , \quad 0<y<1 \\ 0 & , \quad \text{else.} \end{cases}$$

Now moments

$$E(Y)=\int_0^1 3y^3\,dy = \frac{3}{4}, \qquad E(Y^2)=\int_0^1 3y^4\,dy = \frac{3}{5}$$

$$\Rightarrow \operatorname{Var}(Y) = E(Y^2)-(E(Y))^2 = \frac{3}{5}-\frac{9}{16} = \frac{48-45}{80}$$

$$(16)\quad (5)$$

$$= \frac{3}{80}$$

$$E(X) = \int_0^1 x f_X(x)\,dx$$

$$= \int_0^1 x(1+2x-3x^2)\,dx = \int_0^1 (x+2x^2-3x^3)\,dx$$

$$= \frac{x^2}{2} + \frac{2}{3}x^3 - \frac{3}{4}x^4 \Big|_0^1 = \underset{(6)}{\frac{1}{2}} + \underset{(4)}{\frac{2}{3}} - \underset{(3)}{\frac{3}{4}} = \frac{6+8-9}{12} = \frac{5}{12}$$

$$E(X^2) = \int_0^1 x^2(1+2x-3x^2)\,dx = \int_0^1 (x^2+2x^3-3x^4)\,dx$$

$$= \frac{x^3}{3} + \frac{2}{4}x^4 - \frac{3}{5}x^5 \Big|_0^1 = \underset{(20)}{\frac{1}{3}} + \underset{(15)}{\frac{2}{4}} - \underset{(12)}{\frac{3}{5}} = \frac{20+30-36}{60}$$

$$= \frac{14}{60} = \frac{7}{30}$$

$$Var(X) = E(X^2) - (E(X))^2 = \frac{7}{30} - \frac{25}{144} = \frac{516}{8640}$$

$$E(XY) = 2\int_{y=0}^{1} \int_{x=0}^{y} xy(x+y)\,dx\,dy$$

$$= 2\int_0^1 \int_0^y (x^2y + xy^2)\,dx\,dy$$

$$= 2\int_0^1 \left[y\frac{x^3}{3} + x^2\frac{y^2}{2} \right]_{x=0}^{y} dy = 2\int_0^1 \left(\frac{y^4}{3} + \frac{y^4}{2}\right) dy$$

$$= 2\int_0^1 \frac{5y^4}{6}\,dy = \frac{10}{6}\int_0^1 y^4\,dy$$

$$= \frac{10}{6}\cdot\frac{y^5}{5}\Big|_0^1 = \frac{\cancel{10}^{2}}{6}\cdot\frac{1}{\cancel{5}_{1}} = \frac{2}{6} = \frac{1}{3}$$

$$Cov(X,Y) = E(XY) - \cancel{E(XY)} = E(X)\cdot E(Y) = \frac{1}{3} - \frac{5}{12}\cdot\frac{3}{4} = \underset{(16)}{\frac{1}{3}} - \frac{15}{48} = \frac{1}{48}$$

Therefore,

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}} = \frac{1/48}{\sqrt{\frac{3}{80}\cdot\frac{516}{8640}}} = \frac{0.0208}{\sqrt{(0.0375)(0.0597)}}$$

$$\simeq \frac{0.0208}{0.0472} \simeq 0.44$$

4. Let X be a random variable with probability mass function, for $\lambda > 0$

$$P(X = x) = e^{-\lambda} \lambda^x / x! \,, x = 0,1,2,3....$$

a) (10 pts.) Let μ_k denote the k^{th} moment of X. That is,

$$\mu_k = E(X^k), \; k = 1,2,3,....$$

then ~~show that~~

$$\mu_{k+1} = \lambda\left(\mu_k + \frac{d\mu_k}{d\lambda}\right), k = 0,1,2,3,4,...$$

use mathematical induction. First, note that $\mu_k = E(X^k) = \frac{d^k}{dt^k} M_X(t)\Big|_{t=}$

let $\mu_k(t) = \frac{d^k}{dt^k} M_X(t)$.

(*) show that the claim is true for $k=1$.

LHS $= \mu_2 = E(X^2) = \lambda + \lambda^2$

RHS $= \lambda\left(\mu_1 + \frac{d\mu_1}{d\lambda}\right) = \lambda\left(\lambda + \frac{d\lambda}{d\lambda}\right) = \lambda(\lambda+1) = \lambda + \lambda^2$

i.e. the claim is true for $k=1$.

(*) Now assume that the claim is true for k.

i.e.

$$\mu_{k+1} = \lambda\left(\mu_k + \frac{d\mu_k}{d\lambda}\right)$$

or $$\mu_{k+1}(t) = \lambda\left(\mu_k(t) + \frac{d\mu_k(t)}{d\lambda}\right).$$

(*) We need to show that, the claim is true for $k+1$

back to page.

b) (5 pts.) Show also that, for any function g,

$$E(Xg(X-1)) = E(\lambda g(X))$$

$$E(Xg(X-1)) = \sum_{x=0}^{\infty} x g(x-1) \frac{e^{-\lambda}\lambda^x}{x!} = \sum_{x=1}^{\infty} x g(x-1) \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=1}^{\infty} \frac{g(x-1)\, e^{-\lambda}\lambda^x}{(x-1)!}, \quad y = x-1$$

$$= \sum_{y=0}^{\infty} g(y) \frac{e^{-\lambda}\lambda^{y+1}}{y!} = \sum_{y=0}^{\infty} \lambda g(y) \frac{e^{-\lambda}\lambda^y}{y!} = \sum_{y=0}^{\infty} \lambda g(y) P(X=y)$$

$$= E(\lambda g(X))$$

i.e.

We need to show

$$\mu_{k+2}^{(t)} = \lambda\left(\mu_{k+1}^{(t)} + \frac{d\mu_{k+1}^{(t)}}{d\lambda}\right)$$

$$\mu_{k+2}^{(t)} = \frac{d^{k+2}}{dt^{k+2}} M_X(t) = \frac{d}{dt}\left(\frac{d^{k+1}}{dt^{k+1}} M_X(t)\right)$$

$$= \frac{d}{dt}\left[\lambda\left(\mu_k^{(t)} + \frac{d\mu_k^{(t)}}{d\lambda}\right)\right]$$

$$= \lambda \frac{d}{dt}\left(\mu_k^{(t)} + \frac{d\mu_k^{(t)}}{d\lambda}\right)$$

$$= \lambda\left[\frac{d}{dt}\mu_k^{(t)} + \frac{d}{dt}\frac{d\mu_k^{(t)}}{d\lambda}\right]$$

$$= \lambda\left[\mu_{k+1}^{(t)} + \frac{d}{d\lambda}\frac{d}{dt}\mu_k^{(t)}\right]$$

$$= \lambda\left(\mu_{k+1}^{(t)} + \frac{d}{d\lambda}\mu_{k+1}^{(t)}\right)$$

$$= \lambda\left[\mu_{k+1}^{(t)} + \frac{d\mu_{k+1}^{(t)}}{d\lambda}\right]$$

Thus, put $t=0$ to get the required identity. i.e.

$$\mu_{k+2} = \lambda\left(\mu_{k+1} + \frac{d\mu_{k+1}}{d\lambda}\right)$$

Therefore, the claim is true for any k.

i.e.

$$\mu_{k+1} = \lambda\left[\mu_k + \frac{d\mu_k}{d\lambda}\right]$$

c) (5 pts.) Use the result in (b) to calculate $E(X^4)$.

Remember that $E(X) = \lambda$, $E(X^2) = \lambda + \lambda^2$ and $E(X^3) = \lambda^3 + 3\lambda^2 + \lambda$ (from class notes)

Now, $E(X^4) = E(X X^3) = E(X(X-1+1)^3)$

$= E(X[(X-1)^3 + 3(X-1)^2 + 3(X-1) + 1])$

$= E(X(X-1)^3) + 3E(X(X-1)^2) + 3E(X(X-1)) + E(X)$

$= E(\lambda X^3) + 3E(\lambda X^2) + 3E(\lambda X) + E(X)$

$= \lambda[\lambda^3 + 3\lambda^2 + \lambda] + 3\lambda[\lambda + \lambda^2] + 3\lambda^2 + \lambda$

$= \lambda^4 + 3\lambda^3 + \lambda^2 + 3\lambda^2 + 3\lambda^3 + 3\lambda^2 + \lambda$

$= \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda$

5. **(20 pts.)** Akdi&Erdil Inc. is planning to market a new product and they are trying to decide how many to manufacture. They estimate that each item sold will return a profit of m dollars; each one not sold will result in an n dollars loss. Furthermore, they suspect the demand for the product, V, will have the following distribution

$$f(v) = \left(\frac{1}{\lambda}\right) e^{-v/\lambda}, \quad v > 0$$

How many items should the company produce if they want to maximize their expected profit assuming that n, m, and λ are known?

Note that, if a total x items are produced, the firm's profit is given by

$$Q(v) = \begin{cases} mv - n(x-v) & \text{if } v < x \\ mx & \text{if } v \geq x \end{cases}$$

→ Q5

Then the expected profit is;

$$E(Q(V)) = \int_0^{\infty} Q(v) f(v)\, dv$$

$$= \int_0^{x} [(n+m)v - nx] \frac{1}{\lambda} e^{-v/\lambda} dv + \int_x^{\infty} mx \frac{1}{\lambda} e^{-v/\lambda} dv$$

$$= \lambda(n+m) - \lambda(n+m) e^{-x/\lambda} - nx.$$

Now, the maximization (first order condition)

$$\frac{d\, E(Q(V))}{dx} = (n+m) e^{-x/\lambda} - n = 0$$

$$\Rightarrow n = (n+m) e^{-x/\lambda}$$

$$\Rightarrow \ln(n) = \ln(n+m) - \frac{x}{\lambda}$$

$$\Rightarrow \frac{x}{\lambda} = \ln(n+m) - \ln(n)$$

$$\Rightarrow \frac{x}{\lambda} = \ln\left(\frac{n+m}{n}\right)$$

$$\Rightarrow \boxed{x = \lambda \ln\left(\frac{n+m}{n}\right)}$$

OR $x = -\lambda \ln\left(\frac{n}{n+m}\right)$ $\left(\ln \frac{1}{x} = -\ln x\right)$

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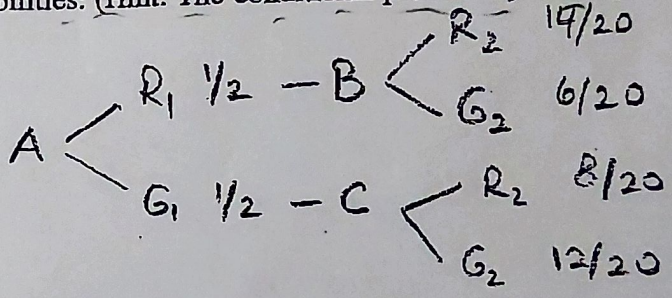
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MIDTERM EXAM
(Time Allowed: 120 minutes)
ATTEMPT FOUR FROM FIVE QUESTIONS

1. Consider a random experiment involving three boxes, each containing a mixture of red (R), and gren (G) balls with the following quantities:

Color	Box A	Box B	Box C
Red	5	14	8
Green	5	6	12

The first ball will be selected at random from box A. If that ball is red, the second ball will be drawn from box B, otherwise, the second ball will be drawn from box C. Let R_1 and G_1 represent the color of the first ball, R_2 and G_2 the color of the second ball. Determine the following probabilities: (Hint: The conditional probability identity will not work.)



a) $P(R_1)$

$$P(R_1) = \frac{1}{2}$$

b) $P(G_1)$

$$P(G_1) = \frac{1}{2}$$

c) $P(R_2 | R_1)$

$$P(R_2|R_1) = \frac{14}{20}$$

d) $P(R_2 | G_1)$

$$P(R_2|G_1) = \frac{8}{20}$$

e) $P(G_2 | G_1)$

$$P(G_2|G_1) = \frac{12}{20}$$

f) $P(G_2 | R_1)$

$$P(G_2|R_1) = \frac{6}{20}$$

2. a) Show that for any three events A, B and C with P(C)>0,

$$P(A \cup B \mid C) = P(A \mid C) + P(B \mid C) - P(A \cap B \mid C)$$

$$P(A \cup B \mid C) = \frac{P((A \cup B) \cap C)}{P(C)}$$

$$= \frac{P((A \cap C) \cup (B \cap C))}{P(C)}$$

$$= \frac{P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)}{P(C)}$$

$$= \frac{P(A \cap C)}{P(C)} + \frac{P(B \cap C)}{P(C)} - \frac{P(A \cap B \cap C)}{P(C)}$$

$$= P(A \mid C) + P(B \mid C) - P(A \cap B \mid C)$$

b) Show that ~~Cov(aX, bY)~~ = ab Cov(X,Y).

$$Cov(aX, bY) = E\{[aX - E(aX)][bY - E(bY)]\}$$

$$= E\{[aX - aE(X)][bY - bE(Y)]\}$$

$$= E\{a[X - E(X)]b[Y - E(Y)]\}$$

$$= ab\, E\{[X - E(X)][Y - E(Y)]\}$$

$$= ab\, Cov(X,Y)$$

$E(axby) - E(ax)E(by)$

$ab(E(xy) - a(E(x)b(E(y))$

$ab(E(xy) - E(x)E(y))$

$Cov(x,y)$

3. The total number of hours, in units of 100 hours, that a family runs a vacuum cleaner over a period of one yaer is a random variable X having the density function

$$f(x)=\begin{cases} c\frac{x}{2}, & 0<x<1 \\ c\frac{2-x}{3}, & 1\le x<2 \\ 0, & \text{elsewhere} \end{cases} \Rightarrow g(x)=\begin{cases} \frac{6}{5}x, & 0<x<1 \\ \frac{4}{5}(2-x), & 1\le x<2 \\ 0, & \text{elsewhere} \end{cases}$$

a) Determine the value c so that $f(x)$ is a probability density function of random variable X.

$$\int_{-\infty}^{\infty} g(x)\,dx = 1 \Rightarrow \int_0^1 \frac{cx}{2}\,dx + \int_1^2 \frac{c(2-x)}{3}\,dx$$

$$= \frac{c}{2}\left[\frac{x^2}{2}\Big|_0^1\right] + \frac{c}{3}\left[2x-\frac{x^2}{2}\Big|_1^2\right]$$

$$= \frac{c}{4} + \frac{c}{6} = \frac{5c}{12} \Rightarrow \frac{5c}{12} = 1 \Rightarrow c = \frac{12}{5}$$

b) Find $P(\frac{1}{2}<X<\frac{3}{2})$.

$$P\left(\tfrac{1}{2}<X<\tfrac{3}{2}\right) = \int_{1/2}^{3/2} g(x)\,dx = \int_{1/2}^{1} \frac{6}{5}x\,dx + \int_1^{3/2} \frac{4}{5}(2-x)\,dx$$

$$= \frac{3}{5}x^2\Big|_{1/2}^{1} + \frac{4}{5}\left(2x-\frac{x^2}{2}\right)\Big|_1^{3/2}$$

$$= \frac{3}{5}\left(1-\frac{1}{4}\right) + \frac{4}{5}\left(3-\frac{9}{8}-2+\frac{1}{2}\right) = \frac{9}{20}+\frac{3}{10} = \frac{9}{10}$$

c) Find the variance and Standard deviation of X.

$$E(X) = \int_0^1 x\,\frac{6}{5}x\,dx + \int_1^2 x\,\frac{4}{5}(2-x)\,dx$$

$$= \frac{6}{5}\cdot\frac{x^3}{3}\Big|_0^1 + \frac{4}{5}\left(x^2-\frac{x^3}{3}\right)\Big|_1^2 = \frac{2}{5}+\frac{8}{45} = \frac{26}{45}$$

$$E(X^2) = \int_0^1 x^2\,\frac{6}{5}x\,dx + \int_1^2 x^2\,\frac{4}{5}(2-x)\,dx$$

$$= \frac{6}{5}\cdot\frac{x^4}{4}\Big|_0^1 + \frac{4}{5}\left(\frac{2}{3}x^3-\frac{x^4}{4}\right)\Big|_1^2 = \frac{3}{10}+\frac{11}{15} = \frac{31}{30}$$

$$\sigma^2 = Var(X) = E(X^2) - [E(X)]^2$$

$$= \frac{31}{30} - \left(\frac{26}{45}\right)^2 = \frac{31}{30} - \frac{676}{2025} = \frac{42495}{60750} = 0.69$$

$$\sigma = \sqrt{0.6995} = 0.8364$$

4. A restaurant prepares a tossed salad containing on the average 5 vegetables. Find the probability that the salad contains more than 5 vegetables

a) on a given day;

X = # of vegetables

$X \sim \text{Poisson}(\lambda = 5)$

$P(X > 5) = 1 - P(X \leq 5)$

$= 1 - \sum_{x=0}^{5} \frac{e^{-5} 5^x}{5!}$ from Poisson table

$= 1 - 0.6160$

$= 0.3840$

$\frac{e^{-\lambda} \lambda^x}{x!}$

b) on 3 of the next 4 days;

Using the Binomial distribution with $p = 0.38$

$b(3; 4; 0.3840) = \binom{4}{3} (0.3840)^3 (0.6160)^1 = 0.1395$

c) for the first time in May on May 5.

Using the Geometric distribution with $p = 0.3840$

$g(5; 0.3840) = (0.3840)(0.6160)^4 = 0.0553$

$(1-p)^{5-1} \cdot p \qquad (1-p)^4 \cdot p$

5. If X is a random variable distributed as $N(\mu,\sigma^2)$, find the value of c, in terms of μ and σ, for which

$$P(X<c)=2-9P(X>c).$$

$$P(X<c) = 2 - 9P(X>c)$$
$$= 2 - 9[1 - P(X<c)]$$
$$= 2 - 9 + 9P(X<c)$$

$$8P(X<c) = 7$$
$$P(X<c) = \frac{7}{8} = 0.875$$

$$X \sim N(\mu, \sigma^2)$$
$$P(X<c) = P\left(z < \frac{c-\mu}{\sigma}\right) = 0.875$$

From standard normal table,

$$\frac{c-\mu}{\sigma} = 1.15 \qquad 2 < \frac{c-\mu}{\sigma}$$

$$c = \mu + 1.15\sigma$$

$$P(Y<c) = 2 - 9P(X>c)$$
$$2 - 9P(1 - P(Y<c))$$
$$P(Y<c) = 2 - 9 + 9P(X<c)$$
$$7 = 8P(Y<c)$$
$$P(Y<c) = 7/8$$

GOOD LUCK☺

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Sınav Gözcüleri
21.11.05 6 Sayfa

Middle East Technical University
Department of Economics
ECON 205
Akdi, Erdil

11.11.2005

FIRST MIDTERM EXAMINATION
(Time Allowed: 150 minutes)

Part A: Answer ALL of the following questions:

1. (20 pts.) Let A and B be two events with $P(A)=0.5$, $P(A\mid B)=0.4$ and $P(B)=0.6$. Define the following random variables:

$$X:\Omega\to R \qquad\qquad Y:\Omega\to R$$

$$w\to X(w)=\begin{cases}1, & w\in A\\ 0, & w\notin A\end{cases} \qquad w\to Y(w)=\begin{cases}1, & w\in B\\ 0, & w\notin B\end{cases}$$

a) Find the joint distribution of X and Y.

b) Find the correlation between X and Y.

Solution:

(a) First of all, let us calculate $P(A\cap B)$, $P(A^c\cap B)$, $P(A\cap B^c)$ and $P(A^c\cap B^c)$

$P(A\cap B)=P(B)P(A\mid B)=0.6(0.4)=0.24$

$P(A^c\cap B)=P(B)-P(A\cap B)=0.6-0.24=0.36$

$P(A\cap B^c)=P(A)-P(A\cap B)=0.5-0.24=0.26$

$P(A^c\cap B^c)=1-P(A\cup B)=1-[P(A)+P(B)-P(A\cap B)]=1-[0.5+0.6-0.24]=0.14$.

Note also that, $D_{X,Y}=\{(0,0),(0,1),(1,0),(1,1)\}$. Now calculate the probabilities one by one.

$P(X=0,Y=0)=P(A^c\cap B^c)=0.14$, $P(X=1,Y=0)=P(A\cap B^c)=0.26$

$P(X=0,Y=1)=P(A^c\cap B)=0.36$, $P(X=1,Y=1)=P(A\cap B)=0.24$.

So, the joint probability function will be

X/Y	Y=0	Y=1
X=0	0.14	0.36
X=1	0.26	0.24

(b) The marginals are:

$P(X=0)=0.5$, $P(X=1)=0.5$ and $P(Y=0)=0.4$, $P(Y=1)=0.6$ and their expected values and variances will be

$E(X)=0.5$. $Var(X)=0.25$ and $E(Y)=0.6$, $Var(Y)=0.24$

and the product moment

$E(XY)=\sum_{x=0}^{1}\sum_{y=0}^{1}xyP(X=x,Y=y)$ $=$ $P(X=1,Y=1)=P(A\cap B)=0.24$. Therefore the correlation is:

$$\rho_{X,Y}=\frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}=\frac{E(XY)-E(X)E(Y)}{\sqrt{Var(X)Var(Y)}}=\frac{0.24-(0.5)(0.6)}{\sqrt{(0.25)(0.24)}}=-\frac{0.06}{0.5\sqrt{4(0.06)}}=-\sqrt{0.06}$$

$=-0.245$

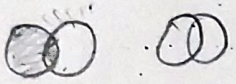
2. (20 pts.) Let A, B and C be events with $0 < P(C) < 1$.

If $P(A \mid C) \geq P(B \mid C)$ and $P(A \mid C^c) \geq P(B \mid C^c)$, then show that $P(A) \geq P(B)$.

Solution:

Note that

$$A = (A \cap C) \cup (A \cap C^c) \text{ and } B = (B \cap C) \cup (B \cap C^c)$$

and 

$$P(A) = P(A \cap C) + P(A \cap C^c) \text{ and } P(B) = P(B \cap C) + P(B \cap C^c).$$

$$P(A \mid C) = \frac{P(A \cap C)}{P(C)} \geq P(B \mid C) = \frac{P(B \cap C)}{P(C)} \Rightarrow P(A \cap C) \geq P(B \cap C)$$

and

$$P(A \mid C^c) = \frac{P(A \cap C^c)}{P(C^c)} \geq P(B \mid C^c) = \frac{P(B \cap C^c)}{P(C^c)} \Rightarrow P(A \cap C^c) \geq P(B \cap C^c)$$

therefore,

$$P(A) = P(A \cap C) + P(A \cap C^c) \geq P(B \cap C) + P(B \cap C^c) = P(B)$$

That is, $P(A) \geq P(B)$.

3. (20 pts.) if the joint probability density of X and Y is given by

$$f(x,y)=\begin{cases} c(2x+y), & 0<x<1, 0<y<2 \\ 0 & ,otherwise \end{cases}$$

Find

a) the constant c,

b) the marginal density of X,

c) the marginal density of Y,

d) the conditional density of Y given X =1/4,

e) the conditional density of X given Y =1.

Solution:

(a) Since. $\int_{x=0}^{1}\int_{y=0}^{2}(2x+y)dydx=\int_{x=0}^{1}\left[2xy+y^2/2\right]_{y=0}^{2}dx=\int_{x=0}^{1}(4x+2)dx=(2x^2+2x)_0^1=4$,

$c=1/4$.

(b) $f_X(x)=\int_{y=0}^{2}f(x,y)dy=\frac{1}{4}\int_{y=0}^{2}(2x+y)dy=\left[\frac{xy}{2}+\frac{y^2}{8}\right]_{y=0}^{2}=x+1/2$.

That is,

$$f_X(x)=\begin{cases} x-1/2 & , \quad 0<x<1 \\ 0 & , \quad o.w. \end{cases}$$

(c) $f_Y(y)=\int_{x=0}^{1}f(x,y)dx=\frac{1}{4}\int_{x=0}^{1}(2x+y)dy=\left[\frac{x^2}{4}+\frac{xy}{4}\right]_{x=0}^{1}=\frac{y+1}{4}$

$$f_Y(y)=\begin{cases} (y+1)/4 & , \quad 0<y<2 \\ 0 & , \quad o.w \end{cases}$$

(d) $f_{Y|X=1/4}(y\,|\,1/4)=\frac{f(1/4,y)}{f_X(1/4)}=\frac{(2(1/4)+y)/4}{(1/4)+(1/2)}=\frac{(y+1/2)/4}{3/4}=\frac{(y+1/2)}{3}=\frac{2y+1}{6}$

That is,

$$f_{Y|X=1/4}(y\,|\,x=1/4)=\begin{cases} (2y+1)/6 & , \quad 0<y<2 \\ 0 & , \quad o.w. \end{cases}$$

(e) $f_{X|Y=1}(x\,|\,y=1)=\frac{f(x,1)}{f_Y(1)}=\frac{(2x+1)/4}{(1+1)/4}=\frac{(2x+1)/4}{1/2}=\frac{(2x+1)}{2}$

$$f_{X|Y=1}(x\,|\,y=1)=\begin{cases} (2x+1)/2 & , \quad 0<x<1 \\ 0 & , \quad o.w. \end{cases}$$

4. (20 pts.) A building contains two elevators, one fast and one slow. The average waiting time for the slow elevator is 3 minutes and the average waiting time of the fast elevator is 1 minute. If a passenger chooses the fast elevator with probability 2/3 and the slow elevator with probability 1/3, what is the expected waiting time?

Solution:

$X = Elevator$, $Y =$ Waiting time

$$E(Y) = E(Y \mid Slow\ Elevator)P(Slow) + E(Y \mid Fast\ \ Elevator)P(Fast)$$

$$= 3\left(\frac{1}{3}\right) + 1\left(\frac{2}{3}\right) = 1 + \frac{2}{3} = \frac{5}{3}$$

Part B: Answer ONE of the following questions:

5. (20 pts.) The stock prices of two companies at the end of any given year are modeled with random variables X and Y that follow a distribution with joint density function

$$f(x,y)=\begin{cases}2x & , \quad 0<x<1,\ x<y<x+1 \\ 0 & , \quad \text{otherwise}\end{cases}$$

What is the conditional variance of Y given that $X = x$?

What is the significance of this variance for the person who plans to buy these stocks? Explain.

Solution:

First, we need to find the marginal pdf of X and then conditional pdf of Y given $X = x$.

$f_X(x)=\int_{y=x}^{x+1} 2x\,dy = 2x$, the marginal pdf of X is $f_X(x)=\begin{cases}2x & , \quad 0<x<1 \\ 0 & , \quad o.w.\end{cases}$

Now the conditional pdf of Y given $X = x$ is,

$f_{Y|X=x}(y\mid x)=\dfrac{f(x,y)}{f_X(x)}=\dfrac{2x}{2x}=1$ That is, the conditional probability density function is:

$$f_{Y|X=x}(x)=\begin{cases}1 & , \quad x<y<x+1 \\ 0 & , \quad o.w.\end{cases}$$

and the conditional variance can be calculated as follows:

$$E(Y\mid X=x)=\int_{y=x}^{x+1} y\, f_{Y|X=x}(y\mid x)\,dy=\int_{y=x}^{x+1} y\,dy=\left[\frac{y^2}{2}\right]_{y=x}^{x+1}=\frac{(x+1)^2-x^2}{2}=\frac{2x+1}{2}$$

$$E(Y^2\mid X=x)=\int_{y=x}^{x+1} y^2\, f_{Y|X=x}(y\mid x)\,dy=\int_{y=x}^{x+1} y^2\,dy=\left[\frac{y^3}{3}\right]_{y=x}^{x+1}=\frac{(x+1)^3-x^3}{3}=\frac{3x^2+3x+1}{3}$$

and the variance

$$Var(Y\mid X=x)=E(Y^2\mid X=x)-[E(Y\mid X=x)]^2=\frac{3x^2+3x+1}{3}-\left[\frac{2x+1}{2}\right]^2$$

$$=\frac{3x^2+3x+1}{3}-\frac{4x^2+4x+1}{4}=\frac{12x^2+12x+4-12x^2-12x-3}{12}=\frac{1}{12}$$

Note that the variance is constant whatever the value of X is taken.

6. (20 pts.) In actuarial calculations, one of the simpler probability function used for describing mortality in the general population is the quadratic polynomial:

$$f(y)=\begin{cases} 3\,10^{-6}\,y^2(100-y)^2 & , \quad 0\le y\le 100 \\ \\ 0 & , \quad \textit{elsewhere} \end{cases}$$

where y denotes a person's age at death (Y is a random variable which measures the person's age). Based on this model,

a) Is it more likely that a person will die between the ages 50 and 60?

b) Is it more likely that a person will live more than 70?

Solution:

(a)

$$P(50<Y<60)=\int_{50}^{60} f(y)dy=\int_{50}^{60}[3\,(10)^{-9}\,y^2(100-y)^2\,dy=\frac{1141}{6250}\cong 0.18$$

(b)

$$P(Y>70)=\int_{70}^{100} f(y)dy=\frac{4077}{25000}\cong 0.16$$

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Department of Economics
ECON 205
Akdi

30.04.2007

MIDTERM EXAMINATION
(Time Allowed: 100 minutes)

1. (25 pts.) Let X be a random variable with probability density function $f(x)$, where

$$f(x)=\begin{cases} c_1 x & , \quad 0<x<1 \\ c_2 & , \quad 1<x<2 \\ 0 & , \quad \textit{elsewhere} \end{cases}$$

a) Find the constants c_1 and c_2 when $E(X)=13/12$

$$1=\int_0^1 c_1 x\,dx+\int_1^2 c_2\,dx=\frac{c_1}{2}+c_2 \Rightarrow c_1=2(1-c_2)$$

$$\frac{13}{12}=E(X)=c_1\int_0^1 x^2\,dx+c_2\int_1^2 x\,dx=c_1\frac{x^3}{3}\Big|_0^1+c_2\frac{x^2}{2}\Big|_1^2=\frac{c_1}{3}+\frac{3}{2}c_2 \Rightarrow \frac{c_1}{3}+\frac{3}{2}c_2=\frac{13}{12}$$

then solve for c_1 and c_2.

$$\frac{c_1}{3}+\frac{3}{2}c_2=\frac{13}{12} \quad \text{and} \quad c_1=2(1-c_2)$$

$$\frac{2(1-c_2)}{3}+\frac{3}{2}c_2=\frac{13}{12} \Rightarrow \frac{2}{3}+\left(\frac{3}{2}-\frac{2}{3}\right)c_2=\frac{13}{12} \Rightarrow \frac{5}{6}c_2=\frac{13}{12}-\frac{2}{3} \Rightarrow \frac{5}{6}c_2=\frac{5}{12} \Rightarrow c_2=\frac{1}{2}=0.5$$

$$c_1=2(1-c_2) \Rightarrow c_1=2(1-0.5)=1$$

That is, $c_1=1$; $c_2=\frac{1}{2}$

b) Calculate $Var(X)$

$$E(X^2)=\int_0^1 x^3\,dx+\frac{1}{2}\int_1^2 x^2\,dx=\frac{x^4}{4}\Big|_0^1+\frac{x^3}{6}\Big|_1^2=\frac{1}{4}+\frac{7}{6}=\frac{3+14}{12}=\frac{17}{12}$$

Thus,

$$Var(X)=E(X^2)-(E(X))^2=\frac{17}{12}-\left(\frac{13}{12}\right)^2=\frac{35}{144}$$

$$Var(X)=\frac{35}{144}$$

2. (25 pts.) Let X and Y be two random variables with joint probability mass function

$$f(x,y)=\begin{cases} c(x^2+y) & , \quad x=1,2\,;y=1,2 \\ \\ 0 & , \quad \textit{elsewhere} \end{cases}$$

a) Find the constant c

$$1=\sum_{x=1}^{2}\sum_{y=1}^{2} f(x,y) = c\sum_{x=1}^{2}\sum_{y=1}^{2}(x^2+y) = c\sum_{x=1}^{2}\left[(x^2+1)+(x^2+2)\right]= c\sum_{x=1}^{2}(2x^2+3)$$

$$= c\big((2+3)+(8+3)\big)= 16\,c$$

$$\Rightarrow\ c=\frac{1}{16}$$

b) Find the correlation between X and Y

$$f_X(x)=P(X=x)=\frac{1}{16}\sum_{y=1}^{2}(x^2+y)=\frac{1}{16}\left[(x^2+1)+(x^2+2)\right]=\frac{2x^2+3}{16}\ ,\ x=1,\,2$$

$$E(X)=\frac{1}{16}\sum_{x=1}^{2}x\left[2x^2+3\right]=\frac{27}{16}\ ;\ E(X^2)=\frac{1}{16}\sum_{x=1}^{2}x^2\left[2x^2+3\right]=\frac{49}{16}$$

$$Var(X)=E(X^2)-(E(X))^2=\frac{49}{16}-\left(\frac{27}{16}\right)^2=\frac{55}{256}$$

$$f_Y(y)=P(Y=y)=\frac{1}{16}\sum_{x=1}^{2}(x^2+y)=\frac{1}{16}\left[(1+y)+(4+y)\right]=\frac{2y+5}{16}\ ,\ y=1,\,2$$

$$E(Y)=\frac{1}{16}\sum_{y=1}^{2}y(2y+5)=\frac{25}{16}\ ;\ E(Y^2)=\frac{1}{16}\sum_{y=1}^{2}y^2\left[2y+5\right]=\frac{43}{16}$$

$$Var(Y)=E(Y^2)-(E(Y))^2=\frac{43}{16}-\left(\frac{25}{16}\right)^2=\frac{63}{256}$$

$$E(XY)=\sum_{x=1}^{2}\sum_{y=1}^{2}xy\,P(X=x,Y=y)=\frac{1}{16}\sum_{x=1}^{2}\sum_{y=1}^{2}xy\,(x^2+y)=\frac{21}{8}$$

Thus,

$$\rho_{XY}=Corr\{X,Y\}=\frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}=\frac{E(XY)-E(X)E(Y)}{\sqrt{Var(X)Var(Y)}}=\frac{\frac{21}{8}-\left(\frac{27}{16}\right)\left(\frac{25}{16}\right)}{\sqrt{\frac{55}{256}\frac{63}{256}}}$$

$$=-\frac{3}{\sqrt{55(63)}}=-\frac{3}{\sqrt{3465}}=-\frac{1}{\sqrt{385}}\cong -0.05096\cong -0.051$$

c) Calculate $E(Y|X=x)$

First, let us find the conditional probability mass function of Y given $X=x$

$$P(Y=y|X=x)=\frac{P(X=x,Y=y)}{P(X=x)}=\frac{(x^2+y)/16}{(2x^2+3)/16}=\frac{(x^2+y)}{(2x^2+3)} \quad ; \; y=1,2$$

Thus,

$$E(Y|X=x)=\sum_{y=1}^{2} y\,P(Y=y|X=x)=\frac{1}{2x^2+3}\sum_{y=1}^{2} y(x^2+y)$$

$$=\frac{1}{2x^2+3}\left[(x^2+1)+2(x^2+2)\right] \;=\; \frac{3x^2+5}{2x^2+3}$$

d) Now assume that X and Y are any two random variable (either discrete or continuous) then show that $E(X)=E(E(X|Y))$

$$E(X)=\int_{x\in D_X} x\,f_X(x)\,dx=\int_{x\in D_X} x\left[\int_{y\in D_Y} f(x,y)\,dy\right]dx=\int_{x\in D_X}\int_{y\in D_y} x\,f(x,y)\,dy\,dx$$

$$=\int_{x\in D_X}\int_{y\in D_y} x\,f(x|y)\,f_Y(y)\,dy\,dx=\int_{y\in D_Y}\int_{x\in D_X} x\,f(x|y)\,f_Y(y)\,dy\,dx$$

$$=\int_{y\in D_Y}\left[\int_{x\in D_X} x\,f(x|y)\,dx\right]f_Y(y)\,dy=\int_{y\in D_Y}[E(X|Y=y)]f_Y(y)\,dy$$

$$=E(E(X|Y))$$

3. **(25 pts.)** Let X and Y be two independent random variables with the same probability density function $f(x)$ where

$$f(x)=\begin{cases} e^{-x} & , \quad x>0 \\ \\ 0 & , \quad elsewhere \end{cases}$$

Find the expected value of $U = X/(X+Y)$

Solution:

First, the joint pdf of X and Y is (because of the independence)

$$f(x,y)=f_X(x)\,f_Y(y)=\begin{cases} e^{-(x+y)} & , \quad x>0\,,\ y>0 \\ \\ 0 & , \quad elsewhere \end{cases}$$

$$U=\frac{X}{X+Y}\ ,\ V=X+Y \Rightarrow X=UV\ ,\ Y=V(1-U);\ 0<u<1,\ v>0$$

and the Jocobien matrix and its determinant is

$$J=\begin{bmatrix} \partial X/\partial U & \partial X/\partial V \\ \\ \partial Y/\partial U & \partial Y/\partial V \end{bmatrix}=\begin{bmatrix} V & U \\ \\ -V & 1-U \end{bmatrix}\Rightarrow \det(J)=V$$

Therefore the joint pdf of U and V is

$$f_{U,V}(u,v)=|J|\,f(x(u,v),y(u,v))=ve^{-v}\ ,\ 0<u<1\,,\ v>0$$

Thus,

$$f_U(u)=\int_{v=0}^{\infty} f_{U,V}(u,v)\,dv=\int_0^{\infty} ve^{-v}\,dv=1$$

That is,

$$f_U(u)=\begin{cases} 1 & , \quad 0<u<1 \\ \\ 0 & , \quad elsewhere \end{cases}$$

$$E(U)=\int_0^1 u\,du=\frac{u^2}{2}\Big|_{u=0}^{1}=\frac{1}{2}$$

4. **(25 pts.)** Let X and Y be two independent random variables with probability mass functions

$$f_X(x) = P(X = x) = e^{-\lambda}\lambda^x / x! \quad , \; x = 0,1,2,3,....$$

$$f_Y(y) = P(Y = y) = e^{-\theta}\theta^y / y! \quad , \; y = 0,1,2,3,....$$

a) Find the probability mass function of $U = X + Y$

$M_X(t) = e^{\lambda(e^t-1)}$, $M_Y(t) = e^{\theta(e^t-1)}$ and

$$M_{X+Y}(t) = M_X(t)M_Y(t) = e^{\lambda(e^t-1)}e^{\theta(e^t-1)} = M_{X+Y}(t) = e^{(\lambda+\theta)(e^t-1)}$$

That is, the moment generating function of $X+Y$ has the same form. Therefore the probability mass function will have the same form.

$$f_U(u) = P(U = u) = \frac{e^{-(\lambda+\theta)}(\lambda+\theta)^u}{u!} \; , \; u = 0,1,2,3,...$$

b) Calculate $P(X = x|U = u)$

$$P(X = x|U = u) = \frac{P(X = x, U = u)}{P(U = u)} = \frac{P(X = x, x+Y = u)}{P(U = u)} = \frac{P(X = x, Y = u-x)}{P(U = u)}$$

$$= \frac{P(X = x)PY = u-x)}{P(U = u)} = \frac{\left(e^{-\lambda}\lambda^x / x!\right)\left(e^{-\theta}\theta^{u-x}/(u-x)!\right)}{e^{-(\lambda+\theta)}(\lambda+\theta)^u / u!} = \frac{u!}{x!(u-x)!}\frac{\lambda^x\theta^{u-x}}{(\lambda+\theta)^u}$$

$$= \binom{u}{x}\frac{\lambda^x\theta^{u-x}}{(\lambda+\theta)^u} = \binom{u}{x}\frac{\lambda^x\theta^{u-x}}{(\lambda+\theta)^{u-x+x}} = \binom{u}{x}\left(\frac{\lambda}{\lambda+\theta}\right)^x\left(\frac{\theta}{\lambda+\theta}\right)^{u-x} = \binom{u}{x}\left(\frac{\lambda}{\lambda+\theta}\right)^x\left(1-\frac{\lambda}{\lambda+\theta}\right)^{u-x}$$

$$\boxed{P(X = x|U = u) = \binom{u}{x}\left(\frac{\lambda}{\lambda+\theta}\right)^x\left(1-\frac{\lambda}{\lambda+\theta}\right)^{u-x} \; , \; x = 0,1,2,..,u}$$

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Ne yasadıysam boşuna

ECON 205 Md (7)

FALL 2011

ERDIL, AKDİ

KEY

03.11.2011

21 NOV 2011

(TIME ALLOWED 100 MIN)
NO PARTIAL CREDIT

PART A: ANSWER ALL OF THE FOLLOWING FIVE QUESTIONS (Total 90 pts.)

1. **(20 pts.)** Let A and B be any two independent events such that $P(A \cap B)=6/25$ and $P(A|B) + P(B|A) = 1$. Assume that $P(A) < P(B)$, then find $P(A)$.

Note that

$$1 = P(A|B) + P(B|A) = \frac{P(A \cap B)}{P(B)} + \frac{P(A \cap B)}{P(A)} = \frac{P(A)P(B)}{P(B)} + \frac{P(A)P(B)}{P(B)}$$

$$= P(A) + P(B) \quad /10$$

i.e. $P(A) + P(B) = 1$ and $P(A \cap B) = 6/25$

$$(P(A) - P(B))^2 = (P(A) + P(B))^2 - 4P(A)P(B)$$

$$= 1 - 4P(A \cap B)$$

$$= 1 - 4 \cdot \frac{6}{25} = 1 - \frac{24}{25} = \frac{1}{25}$$

i.e. $(P(A) - P(B))^2 = \frac{1}{25}$

$$\Rightarrow P(A) - P(B) = \pm \frac{1}{5}$$

since $P(A) < P(B)$ $\quad P(A) - P(B) = -\frac{1}{5}$

since $P(A) + P(B) = 1$

we have $\boxed{P(A) = 2/5}$ /10

$$\begin{cases} P(A) - P(B) = -\frac{1}{5} \\ P(A) + P(B) = 1 \end{cases}$$

$$2P(A) = \frac{4}{5}$$

2. a) (10 pts.) Let A and B be any two Independent events with $P(A) = 0.6$ and $P(B) = 0.3$.

Calculate $P(A \cup B)$, $P(A \cup B^c)$ and $P(A^c \cup B^c)$.

b) (10 pts.) Let A, B and C be independent events such that $P(A) = P(B) = P(C) = 0.25$.

Calculate $P[(A^c \cap B^c) \cup C]$.

a) $P(A) = 0.6$, $P(B) = 0.3$, A and B indep.

i) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= P(A) + P(B) - P(A)P(B) = 0.6 + 0.3 - 0.18$

$= 0.72$

ii) $P(A \cup B^c) = P(A) + P(B^c) - P(A \cap B^c)$

$= P(A) + P(B^c) - P(A) \cdot P(B^c)$

$= 0.6 + 0.7 - (0.6)(0.7) = 0.88$

iii) $P(A^c \cup B^c) = P(A \cap B)^c = 1 - P(A \cap B)$

$= 1 - P(A)P(B)$

$= 1 - (0.6)(0.3) = 0.82$

b) $P((A^c \cap B^c) \cup C) = P(A^c \cap B^c) + P(C) - P(A^c \cap B^c \cap C)$

$= P(A^c)P(B^c) + P(C) - P(A^c)P(B^c)P(C)$

$= \left(\frac{3}{4} \cdot \frac{3}{4}\right) + \frac{1}{4} - \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{1}{4}$

$= \frac{9}{16} + \frac{9}{64} + \frac{1}{4} = \frac{36 + 16 - 9}{64} = \frac{43}{64}$

(4) (1) (16)

3. **(20 pts.)** Erdil&Akdi Ltd. is trying to establish name recognition in a community consisting of 60,000 households. They estimate that 5,000 of those households would do business with the firm if they were contacted and informed about the services of the company. With that in mind, the company has hired a staff of telemarketers to place 1,000 calls. Write a formula for the probability that at least 100 new customers will be identified.

Let Y = # of new customers identified.

$N = 60,000$, $r = 5,000$ and $n = 1,000$

$$p(y) = \frac{\binom{5,000}{y}\binom{55,000}{1000-y}}{\binom{60,000}{1,000}} \qquad 10$$

$$P(Y \geq 100) = 1 - P(Y \leq 99)$$

$$= 1 - \sum_{y=0}^{99} \frac{\binom{5000}{y}\binom{55000}{1000-y}}{\binom{60,000}{1,000}} \quad 10$$

4. **(20 pts.)** Two distinct integers are chosen at random from the first five positive integers. Find the expected value of the absolute value of the difference of the two numbers.

Sample	Absolute value of Difference
1, 2	1
1, 3	2
1, 4	3
1, 5	4
2, 3	1
2, 4	2
2, 5	3
3, 4	1
3, 5	2
4, 5	1

If Y is absolute value of difference, then

$P(1) = 4/10$, $P(2) = 3/10$, $P(3) = 2/10$, $P(4) = 1/10$

$E(Y) = 1.(4/10) + 2(3/10) + 3(2/10) + 4(1/10) = \boxed{2}$

5. **(10 pts.)** Let A, B and C be events with $0<P(A)<1$. If $P(B|A)\geq P(C|A)$ and $P(B|A^c)\geq P(C|A^c)$, then show that $P(B)\geq P(C)$.

Remark: $B=(B\cap A)\cup(B\cap A^c)$ and $C=(C\cap A)\cup(C\cap A^c)$

$P(B)=P[(B\cap A)\cup(B\cap A^c)]=P(B\cap A)+P(B\cap A^c)-P(B\cap A\cap A^c)$

$\rightarrow P(B)=P(B\cap A)+P(B\cap A^c)$ and $P(C)=P(C\cap A)+P(C\cap A^c)$

$P(B|A)=P(B\cap A)/P(A)\geq P(C|A)=P(C\cap A)/P(A)$ $\rightarrow P(B\cap A)\geq P(C\cap A)$

$P(B|A^c)=P(B\cap A^c)/P(A^c)\geq P(C|A^c)=P(C\cap A^c)/P(A^c)$ $\rightarrow P(B\cap A^c)\geq P(C\cap A^c)$

$+$

Summing by these two expressions $P(B\cap A)+P(B\cap A^c)\geq P(C\cap A)+P(C\cap A^c)$

$P(B)\geq P(C)$

PART B: ANSWER ONLY ONE OF THE FOLLOWING TWO QUESTIONS (Total 20 pts.)

6. **(20 pts.)** An urn contains four chips numbered 1 through 4. Two are drawn without replacement. Let the random variable *X* denote the larger of the two. Find *E(X)*.

Sample	Larger of two
1, 2	2
1, 3	3
1, 4	4
2, 3	3
2, 4	4
3, 4	4

$p(2) = 1/6$, $p(3) = 2/6$, $p(4) = 3/6$

$E(X) = 2(1/6) + 3(2/6) + 4(3/6)$

$= 20/6 = \boxed{10/3}$

7. **(20 pts.)** Let X be a random variable with a probability function

$$P(X = x) = p\,q^{x-1},\ x = 1, 2, 3, 4, \ldots.$$

where $P(X = 1) = p$ and $q = 1 - p$. For any positive integers s and t such that $t<s$, verify the following equality.

$$P(X>s \mid X>t) = P(X > s-t)$$

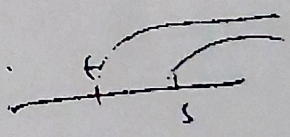
Note that

$$P(X>n) = 1 - P(X \le n)$$

$$= 1 - \sum_{i=1}^{n} P(X=i) = 1 - \sum_{x=1}^{n} p\,q^{x-1}$$

$$= 1 - p \sum_{x=0}^{n-1} q^{x} = 1 - p\,\frac{1-q^n}{(1-q)}$$

$$= 1 - \frac{p(1-q^n)}{p} = q^n \quad 5$$



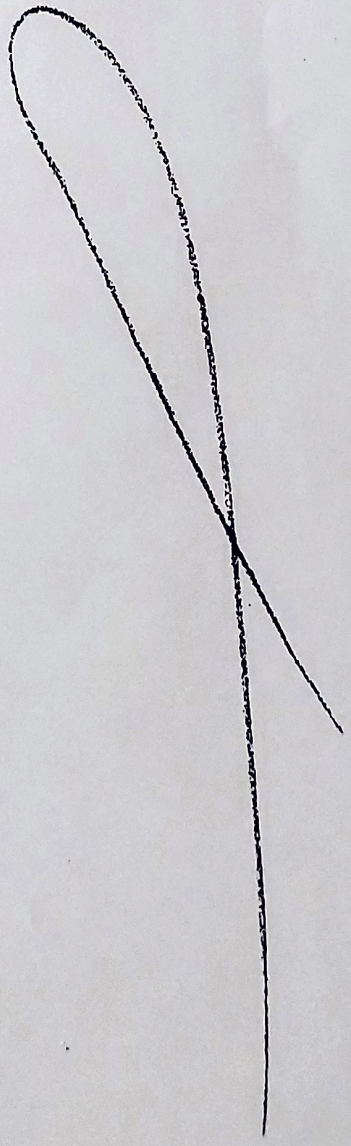
Thus,

$$P(X>s \mid X>t) = \frac{P(X>s,\ X>t)}{P(X>t)}$$

$$= \frac{P(X>s)}{P(X>t)} = \frac{q^s}{q^t} = q^{s-t}$$

$$= P(X>s-t) \quad 15$$

as required



ECON 205 Md II

KEY

(6)

NAME:.....
FALL 2011
ECON 205
ERDİL, AKDİ

20.12.2011

MIDTERM EXAM II
(TIME ALLOWED 120 MIN)
NO PARTIAL CREDIT

1. (15 pts.) The p^{th} percentile of a continuous probability density is the value y_p for which $P(y \le y_p) = p/100$. For the following pdf

$$f(y) = \begin{cases} ky^3 & , \quad 1 \le y \le 2 \\ 0 & , \quad elsewhere \end{cases}$$

a) (5 pts.) Find k.

$$\int_1^2 k y^3 \, dy = 1 \Rightarrow k \frac{y^4}{4} \Big|_1^2 = 1$$

$$\left(\frac{16}{4} - \frac{1}{4}\right) k = 1 \qquad \frac{15}{4} k = 1 \Rightarrow \boxed{k = \frac{4}{15}}$$

b) (10 pts.) Find the 30^{th} percentile.

$$\int_1^{y_{30}} \frac{4}{15} y^3 \, dy = 0.30$$

$$\frac{4y^4}{60} \Big|_1^{y_p} = \frac{y_p^4}{15} - \frac{1}{15} = 0.30$$

$$y_p^4 = 5.5 \Rightarrow \boxed{y_p = 1.53}$$

2. **(20 pts.)** The length of a time, Y, that a customer stands in line in front of PTT window for paying natural gas bill before being served is described by the following pdf

$$f(y)=\begin{cases}0.2e^{-0.2y}, & y\geq 0\\ 0, & elsewhere\end{cases}$$

where Y is measured in minutes.

However, the customer will leave if the wait would be longer than 10 minutes. Suppose the customer goes to PTT twice in the next month. Let the random variable X denote the number of times the customer leaves because the line is too long. Calculate the probability that $X=1$.

Prob. that a customer leaves on any visit is;

$$\int_{10}^{\infty} 0.2e^{-0.2y}\,dy = e^{-0.2y}\Big|_{10}^{\infty} = 1-e^{-2} = \boxed{0.865}$$

If A = prob. customer leaves on first visit
B = prob. customer leaves on second visit.

Then, $P(A)=P(B)=0.865$.

$$P(X=1)=P(A)\cdot P(B^c)+P(A^c)\cdot P(B)$$
$$=2(0.865)(0.135)=\boxed{0.23355}$$

$$\int_{10}^{\infty} 0.2e^{-0.2y}$$

3. **(25 pts.)** Let X and Y be two random variables with the following joint pdf

$$f(x,y)=\begin{cases} x+y\ , & 0<x<1,\ 0<y<1 \\ 0\ , & \textit{elsewhere} \end{cases}$$

Calculate the correlation coefficient between X and Y.

$f_X(x) = x + \frac{1}{2} \Longrightarrow E(X) = \int_0^1 x\left(x+\frac{1}{2}\right)dx.$

$\boxed{E(X) = 7/12}$ 2

$E(X^2) = \int_0^1 x^2\left(x+\frac{1}{2}\right)dx = \boxed{5/12}$ 3

$Var(X) = 5/12 - (7/12)^2 = \boxed{11/144}$ 3

$f_Y(y) = y + 1/2 \Longrightarrow E(Y) = \int_0^1 y\left(y+\frac{1}{2}\right)dy$

$\boxed{E(Y) = 7/12}$ 2

$E(Y^2) = \int_0^1 y\left(y+\frac{1}{2}\right)dy = \boxed{5/12}$ 3

$Var(Y) = 5/12 - (7/12)^2 = \boxed{11/144}$ 3

$E(XY) = \int_0^1\int_0^1 xy(x+y)\,dy\,dx$ 3

$= \int_0^1 \left(\frac{x^2}{2}+\frac{x}{3}\right)dx = \frac{x^3}{6}+\frac{x^2}{6}\Big|_0^1 = \boxed{1/3}$

$Cov(X,Y) = 1/3 - (7/12)(7/12) = \boxed{-1/144}$ 3

$\rho = \frac{-1/144}{\quad} = \boxed{-1/11}$ 3

4. (20 pts.) For the random variables X and Y,

a) Show that $Cov(X+Y, X-Y) = Var(X) - Var(Y)$.

$$Cov(X+Y, X-Y) = E[(X+Y)(X-Y)] - E(X+Y)E(X-Y)$$

$$= E[X^2 - Y^2] - (\mu_X + \mu_Y)(\mu_X - \mu_Y)$$

$$= E(X^2) - (\mu_X)^2 - E(Y^2) + (\mu_Y)^2$$

$$= Var(X) - Var(Y). \quad \therefore$$

b) Suppose $Cov(X, Y) = 0$, show that

$$\rho(X+Y, X-Y) = \frac{Var(X) - Var(Y)}{Var(X) + Var(Y)}$$

$$\rho(X+Y, X-Y) = \frac{Cov(X+Y, X-Y)}{\sqrt{Var(X+Y)\,Var(X-Y)}}$$

$Cov(X+Y, X-Y) = Var(X) - Var(Y)$ from part (a)

$$Var(X+Y) = Var(X) + Var(Y) + 2\cancel{Cov(X,Y)}.$$
$$= Var(X) + Var(Y).$$

$$Var(X-Y) = Var(X) + Var(Y) - 2\cancel{Cov(X,Y)}$$
$$= Var(X) + Var(Y).$$

$$\rho(X+Y, X-Y) = \frac{Var(X) - Var(Y)}{\sqrt{[Var(X)+Var(Y)][Var(X)+Var(Y)]}}$$

$$\rho(X+Y, X-Y) = \frac{Var(X) - Var(Y)}{Var(X) + Var(Y)} \quad \therefore$$

5. (15 pts.) Show that the moment-generating function of the random variable Y having pdf

$$f(y) = \begin{cases} 1/3 , & -1 < y < 2 \\ 0 , & \text{elsewhere} \end{cases}$$

is

$$m(t) = \begin{cases} \frac{e^{2t} - e^{-t}}{3t} , & t \neq 0 \\ 1 , & t = 0 \end{cases}$$

$$M_y(t) = \int_{-1}^{2} e^{ty} \frac{1}{3} dy = \frac{1}{3t} e^{ty} \Big|_{-1}^{2}$$

$$= \frac{1}{3t} (e^{2t} - e^{-t}) \quad \text{for } t \neq 0.$$

$$M_y(0) = \lim_{t \to 0} \frac{1}{3t} (e^{2t} - e^{-t}) = \lim_{t \to 0} \frac{\frac{d}{dt}(e^{2t} - e^{-t})}{\frac{d}{dt} 3t}$$

by L'Hospital's rule. 10

Thus,

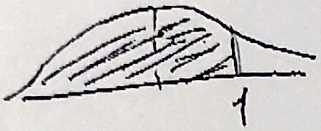
$$M_y(0) = \lim_{t \to 0} \frac{2e^{2t} + e^{-t}}{3} = \frac{1}{2}$$

6. (15 pts.) Let $X \sim N(50, 100)$

a) (6 pts.) Find the following probabilities.

i. $P(X \leq 60)$

$$P(X < 60) = P\left(\frac{X-50}{10} < \frac{60-50}{10}\right) = P(Z < 1)$$



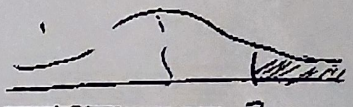
$$= 0.5 + P(0 < Z < 1) = 0.5 + 0.3413$$

$$= \boxed{0.8413}$$

ii. $P\left(\frac{X-30}{5} > 8\right)$

$$P\left(\frac{X-30}{5} > 8\right) = P(X - 30 > 40) = P(X > 70)$$

$$= P\left(\frac{X-50}{10} > \frac{70-50}{10}\right) = P(Z > 2)$$



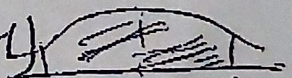
$$= 0.5 - P(0 < Z < 2) = 0.5 - 0.4772 = \boxed{0.0228}$$

iii. $P(|X - 40| \leq 20)$

$$P(|X-40| \leq 20) = P(-20 \leq X - 40 \leq 20)$$

$$= P(-20 - 10 \leq X - 50 \leq 20 - 10) = P(-30 < X - 50 \leq 10)$$

$$= P(-3 \leq Z \leq 1) = P(0 < Z < 3) + P(0 \leq Z \leq 1)$$

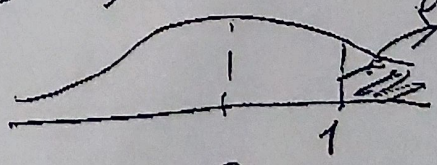


$$= 0.4987 + 0.3413 = \boxed{0.84}$$

b) (9 pts.) For each case find x_0.

i. $P(X > x_0) = 0.1585$

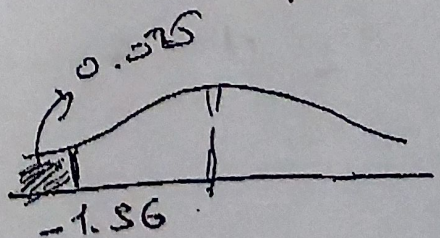
$$P(X > x_0) = P\left(\frac{X-50}{10} > \frac{x_0-50}{10}\right) = 0.1585$$



$$\frac{x_0-50}{10} = 1 \Rightarrow \boxed{x_0 = 60}$$

ii. $P(X < x_0) = 0.025$

$$P(X < x_0) = 0.025$$



$$P\left(\frac{X-50}{10} < \frac{x_0-50}{10}\right) = 0.025$$

$$\frac{x_0-50}{10} = -1.96 \Rightarrow x_0 - 50 = -19.6$$

$$\boxed{x_0 = 30.4}$$

NAME:.....

KEY

FALL 2014
ECON 205
ERDIL, AKDI

18.12.2014

MIDTERM EXAM II
(TIME ALLOWED 120 MIN)
NO PARTIAL CREDIT

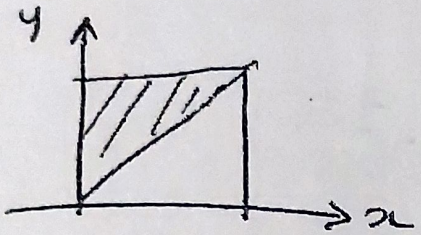
1. (25 pts.) Let X and Y be two random variables with joint probability density function

$$f(x,y)=\begin{cases} cx & , \quad 0<x<y<1 \\ 0 & , \quad elsewhere \end{cases}$$

a) (4 pts.) Find the constant c

b) (7 pts.) Find the marginal probability density functions of X and Y. Are these two random variables Independent?

$$f(x,y) = \begin{cases} cx & , 0 < x < y < 1 \\ 0 & , \text{else.} \end{cases}$$



$$1 = \int_{y=0}^{1} \int_{x=0}^{y} cx \, dx \, dy$$

$$= c \int_{y=0}^{1} \frac{x^2}{2}\Big|_0^y \, dy$$

$$= c \int_{y=0}^{1} \frac{y^2}{2} \, dy = c \frac{y^3}{6}\Big|_0^1$$

$$= \frac{c}{6}$$

$$\frac{c}{6} = 1 \Rightarrow \boxed{c = 6}$$

b) find the marginals.

$$f(x,y) = \begin{cases} 6x & , 0 < x < y < 1 \\ 0 & , \text{else.} \end{cases}$$

$$f_y(y) = \int_{x=0}^{y} 6x \, dx = 3x^2\Big|_0^y = 3y^2$$ (5)

$$f_y(y) = \begin{cases} 3y^2 & , 0 < y < 1 \\ 0 & , \text{else.} \end{cases}$$

$$f_x(x) = \int_{y=x}^{1} 6x \, dy = 6xy\Big|_{y=x}^{1} = 6x(1-x)$$

$$f_x(x) = \begin{cases} 6x(1-x) & , 0 < x < 1 \\ 0 & , \text{else.} \end{cases}$$

They are not indep. (2)

c) (7 pts.) Calculate $E(Y \mid X = x)$

d) (7 pts.) Calculate the correlation between X and Y

c) Find $E(Y|X=x)$

$$f_{Y|X=x}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{6x}{6x(1-x)} = \frac{1}{1-x}$$ (2)

$$f_{Y|X=x}(y|x) = \begin{cases} \frac{1}{1-x} & , \ x<y<1 \\ 0 & , \ \text{else}. \end{cases}$$

$$E(Y|X=x) = \int_{y=x}^{1} y \frac{1}{1-x}\, dy = \frac{1}{1-x} \left.\frac{y^2}{2}\right|_x^1$$

$$= \frac{1-x^2}{2(1-x)} = \frac{(1-x)(1+x)}{2(1-x)} = \frac{x+1}{2}$$

$$\boxed{E(Y|X=x) = \frac{x+1}{2}}$$ (6)

$\frac{1}{1}$

d) Calculate the correlation between X & Y.

$$E(Y) = \int_0^1 y f_Y(y)\, dy = \int_0^1 y\, 3y^2\, dy = 3\int_0^1 y^3\, dy = \frac{3}{4} y^4 \Big|_0^1$$

$$= \frac{3}{4}$$

$$E(Y^2) = 3\int_0^1 y^4\, dy = \frac{3}{5}$$

$$Var(Y) = E(Y^2) - E(Y)^2 = \frac{3}{5} - \frac{9}{16} = \frac{48-45}{80}$$

$$= \frac{3}{80}$$

$$E(X) = \int_0^1 x f_X(x)\, dx = \int_0^1 x\, 6x(1-x)\, dx$$

$$= 6\int_0^1 x^2(1-x)\, dx = \frac{1}{2}$$

$$6 \times \text{Beta}(3,2) = \frac{\Gamma(3)\Gamma(2)}{\Gamma(5)} = \frac{6 \cdot 2 \cdot 1}{4!} = \frac{6/\cdot 2}{1 \cdot 2 \cdot 3 \cdot 4} = \frac{2}{4} = \frac{1}{2}$$

$$E(X^2) = \int_0^1 x^2 f_X(x)\,dx = \int_0^1 6x^3(1-x)\,dx$$

$$= 6\,\frac{\Gamma(4)\Gamma(2)}{\Gamma(6)} = 6\,\frac{3!\cdot 1!}{5!} = \frac{1\cdot 2\cdot 3}{1\cdot 2\cdot 3\cdot 4\cdot 5} = \frac{3}{10}$$

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = \underset{(2)}{\frac{3}{10}} - \underset{(5)}{\frac{1}{4}} = \frac{6-5}{20} = \frac{1}{20}$$

$$E(XY) = \int_{y=0}^{1}\int_{x=0}^{y} xy\, f(x,y)\,dx\,dy$$

$$= \int_{y=0}^{1}\int_{x=0}^{y} xy\cdot 6x\,dx\,dy$$

$$= \int_{y=0}^{1} 6y\,\frac{x^3}{3}\Big|_{x=0}^{y} dy = 2\int_{y=0}^{1} y^4\,dy = \frac{2}{5}$$

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y)$$

$$= \frac{2}{5} - \frac{1}{2}\cdot\frac{3}{4} = \underset{(8)}{\frac{2}{5}} - \underset{(5)}{\frac{3}{8}} = \frac{16-15}{40} \quad (5)$$

$$= \frac{1}{40}$$

$$\rho_{X,Y} = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}} = \frac{1/40}{\sqrt{\frac{3}{80}\cdot\frac{1}{20}}} = \frac{1/40}{\sqrt{\frac{3}{1600}}} \quad (2)$$

$$= \frac{1}{\sqrt{3}} \approx 0.5773$$

2. (10 pts.) Let $X \sim N(\mu = 50,\ \sigma^2 = 100)$

a) **(5 pts.)** Calculate the following probabilities

i) $P(X > 45)$

ii) $P(40 < X < 65)$

iii) $P(|X - 50| > 15)$

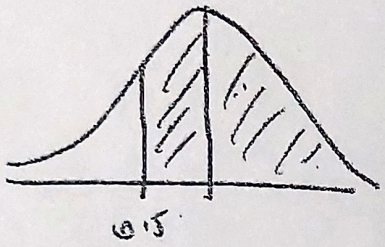
(2) $X \sim N(50, 100)$

i) $P(X > 45) = P\left(Z > \frac{45-50}{10}\right)$

$= P(Z > -0.5) = 0.5 + P(0 < Z < 0.5)$

$= 0.5 + 0.1915$

$= 0.6915$

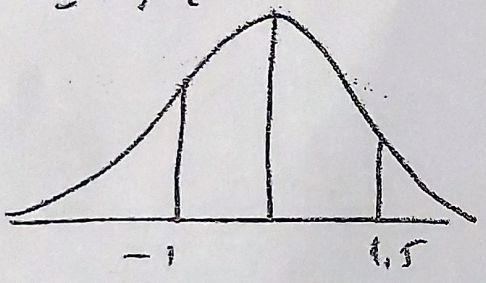


ii) $P(40 < X < 65) = P\left(\frac{40-50}{10} < Z < \frac{65-50}{10}\right)$

$= P(-1 < Z < 1.5) = P(0 < Z < 1) + P(0 < Z < 1.5)$

$= 0.3413 + 0.4332$

$= 0.7745$



iii) $P(|X-50| > 15) = 1 - P(|X-50| \leq 15)$

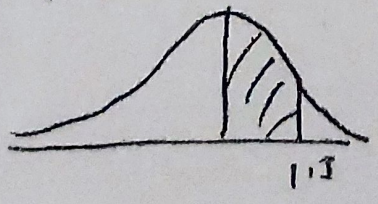
$= 1 - P(-15 < X-50 < 15)$

$= 1 - P\left(-\frac{15}{10} < Z < \frac{15}{50}\right)$

$= 1 - P(-1.5 < Z < 1.5) = 1 - 2P(0 < Z < 1.5)$

$= 1 - 2(0.4332)$

$= 0.1336$



b) **(5 pts.)** Find the value of x_0 for each case

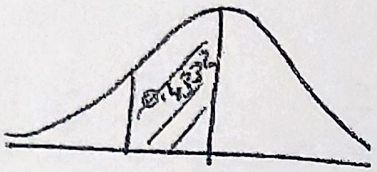
i) $P(X > x_0) = 0.9332$

ii) $P(35 < X < x_0) = 0.9104$

b)

i) $P(X > x_0) = 0.9332$

$$P(X > x_0) = P\left(Z > \frac{x_0 - 50}{10}\right) = 0.9332$$



$$\frac{x_0 - 50}{10} = -1.5$$

$$x_0 - 50 = -15$$

$$\boxed{x_0 = 35}$$

ii) $P(35 < X < x_0) = 0.9104$

$$P(35 < X < x_0) = P\left(\frac{35-50}{10} < Z < \frac{x_0-50}{10}\right) = 0.9104$$

$$\Rightarrow P\left(-1.5 < Z < \frac{x_0-50}{10}\right) = 0.9104$$

$$\Rightarrow P(-1.5 < Z < 0) + P\left(0 < Z < \frac{x_0-50}{10}\right) = 0.9104$$

$$\Rightarrow 0.4332 + P\left(0 < Z < \frac{x_0-50}{10}\right) = 0.9104$$

$$\Rightarrow P\left(0 < Z < \frac{x_0-50}{10}\right) = 0.4772$$

$$\Rightarrow \frac{x_0-50}{10} = 2 \Rightarrow \boxed{x_0 = 70}$$

3. (20 pts.) Let X be a continuous random variable with the cummulative distribution function

$$F(x)=\begin{cases}1-e^{x}-xe^{-x} & , \quad x>0 \\ 0 & , \quad \textit{elsewhere.}\end{cases}$$

Find the mode of the random variable X.

(3) Let X be a continuous random variable with the cummulative distribution function,

$$F(x) = \begin{cases} 1 - e^{-x} - xe^{-x} & , x > 0 \\ 0 & , \text{elx.} \end{cases}$$

Find the mode of X.

Solution:

pdf of X $= 0 - [(1)e^{-x} + (x+1)(-e^{-x})]$
$= -e^{-x} + (x+1)e^{-x} = xe^{-x}$.

$$f(x) = \frac{d}{dx}F(x) = \begin{cases} xe^{-x} & , x > 0 \\ 0 & , \text{elx.} \end{cases}$$ 10

Mode of X is the one that

$$\frac{d}{dx}f(x) = 0$$

$$\Rightarrow \frac{d}{dx}f(x) = \frac{d}{dx}(xe^{-x}) = e^{-x} - xe^{-x} = 0$$

$$\Rightarrow e^{-x}(1-x) = 0$$

$$\Rightarrow \boxed{x = 1}$$ 10

4. (25 pts.) Let

$$f(x_1,x_2)=\begin{cases}21x_1^2x_2^3 & , \quad 0<x_1<x_2<1\\ 0 & , \quad \textit{otherwise}\end{cases}$$

a) (7 pts.)Find the conditional mean of X_1, given $X_2 = x_2$, $0<x_2<1$

$$f(x_2)=\int_0^{x_2} 21x_1^2x_2^3\,dx_1 = 7x_2^6, \quad 0<x_2<1$$

$$f(x_1|X_2=x_2)=\frac{21x_1^2x_2^3}{7x_2^6}=\frac{3x_1^2}{x_2^3}, \quad 0<x_1<x_2<1$$

$$E(X_1|X_2)=\int_0^{x_2} x_1\left(\frac{3x_1^2}{x_2^3}\right)dx_1=\boxed{\frac{3}{4}x_2}$$

b) (7 pts.)Find the conditional variance of X_1, given $X_2 = x_2$, $0<x_2<1$

$$E(X_1^2|X_2)=\int_0^{x_2} x_1^2\left(\frac{3x_1^2}{x_2^3}\right)dx_1=\int_0^{x_2}\frac{3x_1^4}{x_2^3}dx_1$$

$$=\frac{3x_1^5}{5x_2^3}\Big|_0^{x_2}=\boxed{\frac{3}{5}x_2^2}$$

$$Var(X_1|X_2)=\frac{3}{5}x_2^2-\left(\frac{3}{4}x_2\right)^2=\frac{3}{5}x_2^2-\frac{9}{16}x_2^2$$

$$=\frac{48x_2^2-45x_2^2}{80}=\boxed{\frac{3x_2^2}{80}}$$

c) (11 pts.)Let $Y = E(X_1 | X_2)$, find the E(Y) and Var(Y) compare these to $E(X_1)$ and $Var(X_1)$ respectively.

$$E(Y) = E(X_1) \Longrightarrow E(E(X_1|X_2)) = E(X_1)$$

$$f(x_1) = \int_{x_1}^{1} 21 x_1^2 x_2^3 \, dx_2 = \frac{21 x_1^2 x_2^4}{4}\Big|_{x_1}^{1} = \frac{21}{4}\left[\frac{x_1^2}{4} - \frac{x_1^6}{4}\right]$$

$$E(X_1) = \frac{21}{4}\int_0^1 (x_1^2 - x_1^6) x_1 \, dx_1 = \frac{21}{4}\left[\frac{x_1^4}{4} - \frac{x_1^8}{4}\right]_0^1, \quad 0 < x_1 < 1.$$

$$= \frac{21}{4}\left[\frac{1}{4} - \frac{1}{8}\right] = \boxed{21/32} = E(Y) = \boxed{0.656} \quad (3)$$

$$E(X_1^2) = \frac{21}{4}\int_0^1 (x_1^2 - x_1^6) x_1^2 \, dx_1 = \frac{21}{4}\left[\frac{x_1^5}{5} - \frac{x_1^9}{9}\right]_0^1$$

$$= \frac{21}{4}\left[\frac{1}{5} - \frac{1}{9}\right] = \frac{21}{45}$$

$$Var(X_1) = \frac{21}{45} - \left(\frac{21}{32}\right)^2 = \frac{553}{15360} = \boxed{0.036} \quad (3)$$

$$G(y) = P\left(\frac{3}{4} x_2 \leq y\right) = \int_0^{4y/3} 7 x_2^6 \, dx_2 = \left(\frac{4y}{3}\right)^7, \quad 0 < y < 3/4$$

$$g(y) = G'(y) = 7 \cdot \left(\frac{4}{3}\right)^7 y^6, \quad 0 < y < 3/4$$

$$E(Y^2) = 7 \cdot \left(\frac{16384}{2187}\right) \int_0^{3/4} y^6 y^2 \, dy = \frac{114688}{2187} \cdot \frac{y^3}{3}\Big|_0^{3/4}$$

$$= \frac{114688}{2187} \cdot \frac{(3/4)^3}{3} \simeq \boxed{0.4375} \quad (3)$$

$$Var(Y) = E(Y^2) - [E(Y)]^2$$

$$= 0.4375 - \left(\frac{21}{32}\right)^2 = 0.4375 - 0.4306$$

$$\boxed{= 0.0068} \quad (3)$$

So, $E(Y) = E(X_1)$ but $Var(Y) < Var(X_1)$.

5. (20 pts.) A man and a woman decide to meet at a certain location. If each person independently arrives at a uniformly distributed between noon 12 noon and 1 p.m. Find the probability that the first to arrive has to wait longer than 10 minutes.

Let X & Y denote respectively the time past 12 that the man and woman arrive then X & Y are indep. r.vs. each of which uniformly distributed over (0, 60).

Desired probability is $P(X+10<Y) + P(Y+10<X)$ which by symmetry equals $2P(X+10<Y)$, then

$$2P(X+10<Y) = 2\iint_{x+10<y} f(x,y)\,dx\,dy.$$

$$= 2\iint_{x+10<y} f(x)\cdot f(y)\,dx\,dy.$$

$$= 2\int_{10}^{60}\int_{0}^{y-10} \left(\frac{1}{60}\right)^2 dx\,dy.$$

$$= \frac{2}{60^2}\int_{10}^{60} (y-10)\,dy.$$

$$= 25/36 = 0.6944$$

(1)

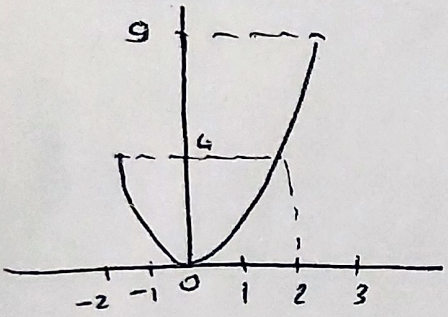
4. (25 pts.) Let Ω be the sample space such that $\Omega=[-2,3]$ and for any A, $A\subset\Omega$ define a probability measure as $P(A)=$"$length\ of\ A$"$/5$. The random variable X is defined by

$$X:\Omega\to\mathbb{R}$$

$$w\to X(w)=w^2.$$

Since the range of X, $D_X=[0,9]$ is an uncountable subset of Ω, X is a continuous random variable.

Calculate $E(X)$.



First we need the cdf since $D_x=[0,9]$.

$F(x)=0$ for $x<0$

and $F(x)=1$ for $x>9$

$0<x<4$ $\quad F(x)=P(\{w:\ w^2\le x\})=P(\{w:\ -\sqrt{x}\le w\le\sqrt{x}\})=\frac{2\sqrt{x}}{5}$

$4<x<9$ $\quad F(x)=P(\{w:\ w^2\le x\})=P(\{w:\ -2\le w\le\sqrt{x}\})=\frac{\sqrt{x}+2}{5}$

Since $f(x)=\frac{d}{dx}F(x)$, the cdf and pdf are

$$F_X(x)=\begin{cases}0, & x\le 0\\ \frac{2\sqrt{x}}{5}, & 0<x\le 4\\ \frac{\sqrt{x}+2}{5}, & 4<x\le 9\\ 1, & x>9\end{cases}$$

$$f_X(x)=\frac{d}{dx}F(x)=\begin{cases}\frac{1}{5\sqrt{x}}, & 0<x<4\\ \frac{1}{10\sqrt{x}}, & 4<x<9\\ 0, & e/x\end{cases}$$

$$E(X)=\int_{x\in D_x} x f(x)\,dx=\int_0^4 x\frac{1}{5\sqrt{x}}\,dx+\int_4^9 x\frac{1}{10\sqrt{x}}\,dx$$

$$=\frac{1}{5}\int_0^4\sqrt{x}\,dx+\frac{1}{10}\int_4^9\sqrt{x}\,dx=\frac{1}{5}\cdot\frac{2}{3}x^{3/2}\Big|_0^4+\frac{1}{10}\cdot\frac{2}{3}x^{3/2}\Big|_4^9$$

$$=\frac{2}{15}x^{3/2}\Big|_0^4+\frac{1}{15}x^{3/2}\Big|_4^9=\frac{16}{15}+\frac{19}{15}=\frac{35}{15}=\frac{7}{3}$$

$$\boxed{E(X)=\frac{7}{3}}$$

5. (25 pts.) Let X and Y be two random variables with joint probability density function

$$f(x,y)=\begin{cases} cx & , \quad 0<x<1, x<y<x+1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

a. (5 pts.) Find the constants c

$$1=\int_{x=0}^{1}\int_{y=x}^{x+1} cx\,dy\,dx = c\int_{x=0}^{1} x\,y\Big|_{x}^{x+1}dx = c\int_0^1 x\,dx = c\,\frac{x^2}{2}\Big|_0^1$$

$$= \frac{c}{2} \Rightarrow \boxed{c=2}$$

b. (5 pts.) Are the random variables X and Y independent?

Marginals

$$f_X(x)=\int_{y=x}^{x+1} 2x\,dy = 2x\,y\Big|_x^{x+1} = 2x$$

$$f_X(x)=\begin{cases} 2x & , 0<x<1 \\ 0 & , \text{e/w.} \end{cases}$$

For the marginal pdf of Y see back page.

c. (5 pts.) Calculate $E(Y \mid X=x)$

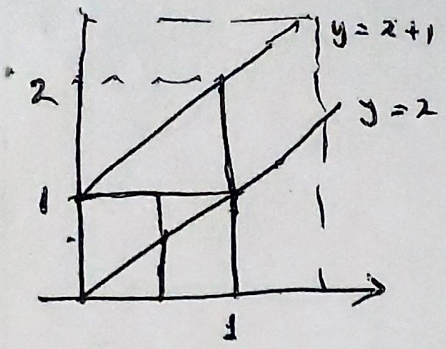
$$f_{Y/X=x}(y/x) = \frac{f(x,y)}{f_X(x)} = \frac{2x}{2x} = 1, \quad x<y<x+1$$

$$E(Y/X=x) = \int_{y=x}^{x+1} y\, f_{Y/X=x}(y/x)\,dy = \int_x^{x+1} y\,dy = \frac{y^2}{2}\Big|_x^{x+1}$$

$$= \frac{(x+1)^2-x^2}{2} = \frac{2x+1}{2}$$

So

$$\boxed{E(Y/X=x) = \frac{2x+1}{2}}$$



$0 < y < 1 \Rightarrow f_y(y) = \int_0^y 2x\,dx$

$= x^2\Big|_0^y = y^2$

$1 < y < 2 \Rightarrow f_y(y) = \int_{x=y-1}^{1} 2x\,dx$

$= x^2\Big|_{y-1}^{1} = 1 - (y-1)^2$

$$f_y(y) = \begin{cases} y^2 & , 0<y<1 \\ 1-(y-1)^2 & , 1<y<2 \\ 0 & , \text{else.} \end{cases}$$

Note that

$$\int_0^1 y^2\,dy + \int_1^2 1-(y-1)^2\,dy = \frac{1}{3} + \frac{2}{3} = 1$$

Since

$$f(x,y) \neq f_x(x) \cdot f_y(y)$$

the r.v's X and Y are NOT independent.

d. (10 pts.) Calculate the correlation between X and Y

We need the following moments

$E(X)$, $E(Y)$, $E(XY)$ $\text{Var}(X)$ $\text{Var}(Y)$

From the marginals:

$$E(X) = \int_0^1 x f(x)\,dx = \int_0^1 x\, 2x\,dx = \frac{2}{3}$$

$$E(X^2) = \int_0^1 x^2 \cdot 2x\,dx = 2\int_0^1 x^3\,dx = \frac{1}{2}$$

$$\text{Var}(X) = E(X^2) - (E(X))^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$$

$$\boxed{E(X) = \frac{2}{3}, \quad \text{Var}(X) = \frac{1}{18}}$$

$$E(Y) = \int_0^1 y\, y^2\,dy + \int_1^2 y\,(1-(y-1)^2)\,dy$$

$$= \frac{1}{4} + \frac{11}{12} = \frac{3+11}{12} = \frac{14}{12} = \frac{7}{6}$$

$$E(Y^2) = \int_0^1 y^4\,dy + \int_1^2 y^2\,(1-(y-1)^2)\,dy$$

$$= \frac{1}{5} + \frac{13}{10} = \frac{2+13}{10} = \frac{15}{10} = \frac{3}{2}$$

$$\text{Var}(Y) = \frac{3}{2} - \frac{49}{36} = \frac{54-49}{36} = \frac{5}{36}$$

$$\boxed{E(X) = \frac{7}{6}, \quad \text{Var}(X) = \frac{5}{36}}$$

(5)

$$E(XY) = \int_{x=0}^{1} \int_{y=x}^{x+1} xy\, f(x,y)\, dy\, dx$$

$$= \int_{x=0}^{1} \int_{y=x}^{x+1} 2x^2 y\, dy\, dx = \frac{5}{6}$$

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

$$= \frac{5}{6} - \frac{2}{3} \cdot \frac{7}{6} = \frac{15-14}{18} = \frac{1}{18}$$

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}} = \frac{1/18}{\sqrt{\frac{1}{18} \cdot \frac{5}{36}}}$$

$$= \frac{\sqrt{18(36)}}{\sqrt{5}(18)} = \frac{6}{\sqrt{5}} \cdot \frac{\sqrt{18}}{18} = \frac{2}{\sqrt{10}}$$

NAME:.....
FALL 2015-2016
ECON 205
ERDIL, AKDI

18.01.2016

FINAL EXAM
(TIME ALLOWED 120 MIN)
NO PARTIAL CREDIT

1. (20 pts.) Let X and Y be two random variables with joint probability function

$$f(x,y)=\begin{cases} c(x+y) & , \quad x=0,1\ ,\ y=0,1 \\ 0 & , \quad elsewhere \end{cases}$$

a) Find the constant c

b) Calculate $E(Y \mid X=x)$

c) Calculate the correlation between X and Y

1. Let X and Y be two random variables with joint probability function

$$f(x,y)=\begin{cases} c(x+y) & , \ x=0,1 \ , \ y=0,1 \\ 0 & , \quad \text{elsewhere} \end{cases}$$

a) Find the constant c

b) Calculate $E(Y|X=x)$

c) Calculate the correlation between X and Y

a) $1=\sum_{x=0}^{1}\sum_{y=0}^{1} c(x+y) = c\sum_{x=0}^{1}(x+x+1) = c\sum_{x=0}^{1}(2x+1)$

$= c[1+3] = 4c,$

$4c=1 \Rightarrow \boxed{c=\frac{1}{4}}$

$$f(x,y)=\begin{cases} \frac{x+y}{4} & , x,y=0,1 \\ 0 & , \text{ elx.} \end{cases}$$

b) $f_X(x) = \sum_{y=0}^{1} \frac{x+y}{4} = \frac{1}{4}[x+x+1] = \frac{2x+1}{4}, \ x=0,1$

$$f_X(x)=\begin{cases} \frac{2x+1}{4} & , \ x=0,1 \\ 0 & , \ \text{elx.} \end{cases}$$

the conditional dist^n of Y given $X=x$ is

$$f_{Y|X=x}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{\frac{x+y}{4}}{\frac{2x+1}{4}} = \frac{x+y}{2x+1}, \ y=0,1.$$

so, $E(Y|X=x) = \sum_{y=0}^{1} y\, f_{Y|X=x}(y|x) = \sum_{y=0}^{1} y\,\frac{x+y}{2x+1}$

$$= 0 + \frac{x+1}{2x+1} = \frac{x+1}{2x+1}$$

i.e.

$$\boxed{E(Y|X=x) = \frac{x+1}{2x+1}}$$

c) Note that the marginal prob. functions are the same

i.e. $f_X(x) = \frac{2x+1}{4}, \ x=0,1, \quad f_Y(y) = \frac{2y+1}{4}, \ y=0,1$

or we can write these prob. functions

$X=x$	0	1
$P(X=x)$	$\frac{1}{4}$	$\frac{3}{4}$

$Y=y$	0	1
$P(Y=y)$	$\frac{1}{4}$	$\frac{3}{4}$

→

The joint prob. function

$$f(x,y) = \frac{x+y}{4}, \quad x = 0,1, \quad y = 0,1$$

$y \backslash x$	0	1	$P(Y=y)$
0	0	$\frac{1}{4}$	$\frac{1}{4}$
1	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$
$P(X=x)$	$\frac{1}{4}$	$\frac{3}{4}$	1

From the marginals (Bern($\frac{3}{4}$))

$$E(X) = E(Y) = \frac{3}{4}$$

$$Var(X) = Var(Y) = \frac{3}{16}$$

$$E(XY) = \sum_{x=0}^{1} \sum_{y=0}^{1} xy\, P(X=x, Y=y) = 0+0+0+1 \cdot 1 \cdot P(X=1,Y=1)$$

$$= P(X=1, Y=1) = \frac{1}{2}$$

Therefore,

$$Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{2} - \frac{3}{4}\cdot\frac{3}{4} = \frac{1}{2} - \frac{9}{16}$$

$$(8) \quad (1)$$

$$= -\frac{1}{16}$$

and

$$\rho_{X,Y} = Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}} = \frac{-1/16}{\sqrt{\frac{3}{16}\cdot\frac{3}{16}}}$$

$$= \frac{-1/16}{3/16} = -\frac{1}{3}$$

$$\boxed{\rho_{X,Y} = -\frac{1}{3}}$$

2. (20 pts.) Let X and Y be two independent random variables such that $X \sim Gamma(\alpha,\theta)$ and $Y \sim Gamma(\beta,\theta)$.

Note: If $X \sim Gamma(\alpha,\theta)$, then the probability density function of X is given by

$$f(x)=\frac{1}{\Gamma(\alpha)\theta^{\alpha}}x^{\alpha-1}e^{-x/\theta}, \quad x>0$$

a) (7 pts.) Find the distribution of $U=X+Y$

b) (7 pts.) Find the distribution of $V=X/(X+Y)$

c) (6 pts.) Show that the random variables U and V are independet.

FINAL EXAM QUESTIONS

2. Let X and Y be two independent random variables such that $X \sim Gamma(\alpha,\theta)$ and $X \sim Gamma(\alpha,\theta)$.

a) Find the distribution of $U = X+Y$

b) Find the distribution of $V = X/(X+Y)$

c) Show that the random variables U and V are independet.

Note: If $X \sim Gamma(\alpha,\theta)$, then the probability density function of X is given by

$$f(x) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}} x^{\alpha-1} e^{-x/\theta}, \quad x>0$$

a) if $X \sim$ Gamma (α,θ), then the mgf. of X is given by

$M_X(t) = \left(\frac{1}{1-\theta t}\right)^{\alpha}$, $\theta < \frac{1}{t}$

if X & Y are indep. r.v. then $M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$

So $M_{X+Y}(t) = M_X(t) \cdot M_Y(t) = \left(\frac{1}{1-\theta t}\right)^{\alpha+\beta}$

$\therefore U = X+Y \sim$ Gamma $(\alpha+\beta, \theta)$

i.e. the pdf of U is

$$f_U(u) = \frac{1}{\Gamma(\alpha+\beta)\,\theta^{\alpha+\beta}} u^{\alpha+\beta-1} e^{-u/\theta}, \quad u>0$$

b) $\left.\begin{aligned} U &= X+Y \\ V &= \frac{X}{X+Y} \end{aligned}\right\} \Rightarrow \begin{aligned} X &= UV \\ Y &= U(1-V) \end{aligned}$; $D_U = \mathbb{R}^+$ and $D_V = (0,1)$

$$J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} v & u \\ 1-v & -u \end{bmatrix}, \quad \det(J) = -uv - u(1-v) = -uv - u + uv = -u$$

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)\,\theta^{\alpha+\beta}} x^{\alpha-1} y^{\beta-1} e^{-(x+y)/\theta}, \quad x,y>0$$

$$f_{U,V}(u,v) = |J|\, f_{X,Y}(x(u,v), y(u,v))$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)\,\theta^{\alpha+\beta}} |-u| (uv)^{\alpha-1} (u(1-v))^{\beta-1} e^{-u/\theta}, \quad \begin{matrix} u>0 \\ 0<v<1 \end{matrix}$$

$\rightarrow$

$$f_{U,V}(u,v) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)\theta^{\alpha+\beta}} u\, u^{\alpha-1} u^{\beta-1} v^{\alpha-1}(1-v)^{\beta-1} e^{-u/\theta}$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)\theta^{\alpha+\beta}} u^{\alpha+\beta-1} v^{\alpha-1}(1-v)^{\beta-1} e^{-u/\theta}, \quad \begin{matrix} u>0 \\ 0<v<1 \end{matrix}$$

But we need $f_V(v)$

$$f_V(v) = \int_{u\in D_u} f_{U,V}(u,v)\,du = \frac{1}{\Gamma(\alpha)\Gamma(\beta)\theta^{\alpha+\beta}} v^{\alpha-1}(1-v)^{\beta-1} \int_0^\infty u^{\alpha+\beta-1} e^{-u/\theta}\,du$$

$$= \frac{v^{\alpha-1}(1-v)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)\theta^{\alpha+\beta}} \underbrace{\int_0^\infty u^{\alpha+\beta-1} e^{-u/\theta}\,du}_{\Gamma(\alpha+\beta)\,\theta^{\alpha+\beta}}$$

$$= \frac{v^{\alpha-1}(1-v)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)\theta^{\alpha+\beta}} \cdot \Gamma(\alpha+\beta)\,\theta^{\alpha+\beta}$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} v^{\alpha-1}(1-v)^{\beta-1}, \quad 0<v<1$$

i.e $V \sim \text{Beta}(\alpha,\beta)$

c) Note that $U \sim \text{Gamma}(\alpha+\beta,\theta)$ and $V \sim \text{Beta}(\alpha,\beta)$

from part (b) the joint pdf of U and V is found that

$$f_{U,V}(u,v) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)\theta^{\alpha+\beta}} u^{\alpha+\beta-1} v^{\alpha-1}(1-v)^{\beta-1} e^{-u/\theta}, \quad \begin{matrix} u>0 \\ 0<v<1 \end{matrix}$$

multiply and divide by $\Gamma(\alpha+\beta)$, then,

$$f_{U,V}(u,v) \underbrace{\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} v^{\alpha-1}(1-v)^{\beta-1}}_{f_V(v)} \underbrace{\frac{1}{\Gamma(\alpha+\beta)\theta^{\alpha+\beta}} u^{\alpha+\beta-1} e^{-u/\theta}}_{f_U(u)}$$

so, $f_{U,V}(u,v) = f_U(u)\cdot f_V(v)$

and therefore the r.v's U and V are indep.

3. Let X be a discrete random variable with probability function

$$P(X=x)=pq^{x-1},\ x=1,2,3,\ldots..\ 0<p<1 \text{ and } q=1-p$$

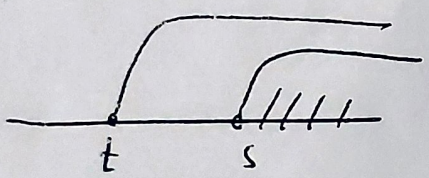
show that $P(X>s|X>t)=P(X>s-t)$, $s>t>0$

Note that $X \sim$ Geometric (p).

$$P(X>n) = 1 - P(X \le n) = 1 - \sum_{x=1}^{n} p(X=x) = 1 - \sum_{x=1}^{n} p q^{x-1}$$

$$= 1 - \sum_{y=0}^{n-1} p q^{y} = 1 - p\,\frac{1-q^n}{1-q} = 1 - 1 + q^n = q^n$$

i.e. $P(X>n) = q^n$



$$P(X>s, X>t) = P(X>s)$$

$$P(X>s|X>t) = \frac{P(X>s, X>t)}{P(X>t)} = \frac{P(X>s)}{P(X>t)} = \frac{q^s}{q^t}$$

$$= q^{s-t} = P(X>s-t)$$

i.e.

$$\boxed{P(X>s|X>t) = P(X>s-t)}$$

4. (20 pts.) Let X and Y be two independent random variables with the following probability functions

$$f(x) = P(X = x) = e^{-\lambda}\lambda^x / x!, \quad x = 0,1,2,3,.....$$
$$f(y) = P(Y = y) = e^{-\theta}\theta^y / y!, \quad y = 0,1,2,3,.....$$

Hint: Mgfs of the above probability functions are as follows:

$$M_X(t) = e^{\lambda(e^t - 1)}$$
$$M_Y(t) = e^{\theta(e^t - 1)}$$

a) (10 pts.) Find the probability function of $U = X + Y$.

b) (10 pts.) Calculate $P(X = x \mid U = u)$.

4 a)

$$M_X(t) = e^{\lambda(e^t-1)}, \quad M_Y(t) = e^{\theta(e^t-1)}$$

If X and Y are indep. then

$$M_{X+Y}(t) = M_X(t) \cdot M_Y(t).$$

a) $M_U(t) = M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$

$$= e^{\lambda(e^t-1)} \; e^{\theta(e^t-1)} = e^{(\theta+\lambda)(e^t-1)}$$

which is the moment generating function of Poisson $(\lambda+\theta)$.

Therefore

$$P(U=u) = e^{-(\lambda+\theta)} (\lambda+\theta)^u / u! \quad , u = 0,1,2\ldots$$

b) $P(X=x \mid U=u) = \dfrac{P(X=x, U=u)}{P(U=u)}$

$$= \frac{P(X=x, X+Y=u)}{P(U=u)} = \frac{P(X=x, Y=u-x)}{P(U=u)}$$

$$= \frac{P(X=x)\,P(Y=u-x)}{P(U=u)}$$

$$= \frac{\dfrac{e^{-\lambda}\lambda^x}{x!} \; \dfrac{e^{-\theta}\theta^{u-x}}{(u-x)!}}{\dfrac{e^{-(\theta+\lambda)}(\theta+\lambda)^u}{u!}} = \frac{u!}{x!(u-x)!} \frac{\lambda^x \theta^{u-x}}{(\lambda+\theta)^u}$$

→

$$P(X=x \mid U=u) = \frac{u!}{x!\,(u-x)!} \cdot \frac{\lambda^x}{(\theta+\lambda)^u}\,\theta^{u-x}$$

$$= \binom{u}{x}\left(\frac{\lambda}{\lambda+\theta}\right)^x \left(\frac{\theta}{\lambda+\theta}\right)^{u-x}$$

$$= \binom{u}{x}\left(\frac{\lambda}{\lambda+\theta}\right)^x \left(1-\frac{\lambda}{\lambda+\theta}\right)^{u-x}$$

which is the prob. function of

Binom (u, p)

where $p = \dfrac{\lambda}{\lambda+\theta}$

i.e.

$X \mid U=u \sim \text{Binom}\left(u, \dfrac{\lambda}{\lambda+\theta}\right)$

5. (20 pts.) Erkan is working as a quality inspector in a cookie shop. According to latest measurements 2 of 20 boxes has a defect. Erkan chooses 3 boxes from a lot of 20 boxes. Find the probability that

a) **(6 pts.)** at least one box has a defect.

b) **(6 pts.)** the second selected box has a defect.

c) **(8 pts.)** Let's change the plot of the question in such a way. The probability of being defective for each box is 0.10. Erkan chooses boxes from lot of 20 boxes. If the first defective box is the one chosen at the third draw then the lot of 20 boxes is not loaded to truck for delivery. Calculate the probability that the lot of 20 boxes is loaded for delivery.

5) $P(\text{box has a defect}) = \frac{2}{20} = 0.1 = p$.

X, # of boxes has a defect (out of 3)
3 boxes have been chosen.

a) $X \sim \text{Binom}(3, 1/10)$

$$P(X \geq 1) = 1 - P(X < 1) = 1 - P(X=0)$$
$$= 1 - \binom{3}{0}(0.1)^0(0.9)^{3-0}$$
$$= 1 - (0.9)^3$$

b) X : # of trials to choose a defected box

$X \sim \text{Geo}(1/10)$

$$P(X=2) = (0.1)(0.9)^{2-1} = (0.1)(0.9)$$
$$= \underline{\underline{0.09}}$$

c) $Y_i = \begin{cases} 1, & X=3 \\ 0, & \text{o.w.} \end{cases}$

$$P(X=3) = (0.1)(0.9)^{3-1} = p \text{ (say)}$$

$Y_i \sim \text{Bernoulli}(p)$

$Y = \sum Y_i \sim \text{Binom}(20, p)$

$$P(Y=20) = \binom{20}{20} p^{20} (q)^0 = p^{20}$$

6. (15 pts.) Let X and Y be two random variables with the following probability function

Y\X	-4	-2	2	4
-3	5/32	3/32	1/32	1/32
-1	4/32	2/32	0	1/32
1	0	1/32	5/32	4/32
3	0	0	2/32	3/32

Let's define a new random variable $Z = Maximum\{XY,5\}$.

a) (10 pts.) Find the probability distribution of Z, $P(Z = z)$.

b) (5 pts.) Calculate mean of Z.

(6) $D_X = \{-4, -2, 2, 4\}$

$D_Y = \{-3, -1, 1, 3\}$

XY :

12	4	−4	−12
* 6	2	−2	−6
−6	−2	2	6 *
−12	−4	4	12

$Z = \max\{XY, 5\}$

$D_Z = \{5, 6, 12\}$

$P(Z=12) = P(X=-4, Y=-3) + P(X=4, Y=3)$

$= \frac{5}{32} + \frac{3}{32} = \frac{8}{32}$

$P(Z=6) = P(X=-2, Y=-3) + P(X=2, Y=3)$

$= \frac{3}{32} + \frac{2}{32} = \frac{5}{32}$

$P(Z=5) = 1 - [P(Z=12) + P(Z=6)]$

$= 1 - \left[\frac{8}{32} + \frac{5}{32}\right] = 1 - \frac{15}{32}$

$= \frac{19}{32}$

→

So, the prob. function of Z can be written as

$z=z$	5	6	12
$P(Z=z)$	$\frac{19}{32}$	$\frac{5}{32}$	$\frac{8}{32}$

b) $E(Z) = \sum_{z \in D_Z} z\, P(Z=z)$

$= 5 \cdot P(Z=5) + 6\, P(Z=6) + 12 P(Z=12)$

$= 5 \cdot \frac{19}{32} + 6 \cdot \frac{5}{32} + 12 \frac{8}{32}$

$= \frac{95}{32} + \frac{30}{32} + \frac{96}{32}$

$= \frac{221}{32}$