

Spring 2024-
2025
ECON 205

Problem Set 7 Answers

- 1) A large lot of manufactured items contains 10% with exactly one defect, 5% with more than one defect, and the remainder with no defects. Ten items are randomly selected from this lot for sale. If Y_1 denotes the number of items with one defect and Y_2 , the number with more than one defect, the repair costs are $Y_1 + 3Y_2$. Find the mean and variance of the repair costs.

SOLUTION

Using formulas for means, variances, and covariances for the multinomial:

$$E(Y_1) = 10(.1) = 1$$

$$V(Y_1) = 10(.1)(.9) = .9$$

$$E(Y_2) = 10(.05) = .5$$

$$V(Y_2) = 10(.05)(.95) = .475$$

$$\text{Cov}(Y_1, Y_2) = -10(.1)(.05) = -.05$$

So,

$$E(Y_1 + 3Y_2) = 1 + 3(.5) = 2.5$$

$$V(Y_1 + 3Y_2) = .9 + 9(.475) + 6(-.05) = 4.875.$$

2)

A learning experiment requires a rat to run a maze (a network of pathways) until it locates one of three possible exits. Exit 1 presents a reward of food, but exits 2 and 3 do not. (If the rat eventually selects exit 1 almost every time, learning may have taken place.) Let Y_i denote the number of times exit i is chosen in successive runnings. For the following, assume that the rat chooses an exit at random on each run.

- a Find the probability that $n = 6$ runs result in $Y_1 = 3$, $Y_2 = 1$, and $Y_3 = 2$.
- b For general n , find $E(Y_1)$ and $V(Y_1)$.
- c Find $\text{Cov}(Y_2, Y_3)$ for general n .
- d To check for the rat's preference between exits 2 and 3, we may look at $Y_2 - Y_3$. Find $E(Y_2 - Y_3)$ and $V(Y_2 - Y_3)$ for general n .

SOLUTION

a. Using the multinomial distribution with $p_1 = p_2 = p_3 = 1/3$,

$$P(Y_1 = 3, Y_2 = 1, Y_3 = 2) = \frac{6!}{3!1!2!} \left(\frac{1}{3}\right)^6 = .0823.$$

b. $E(Y_1) = n/3$, $V(Y_1) = n(1/3)(2/3) = 2n/9$.

c. $\text{Cov}(Y_2, Y_3) = -n(1/3)(1/3) = -n/9$.

d. $E(Y_2 - Y_3) = n/3 - n/3 = 0$, $V(Y_2 - Y_3) = V(Y_2) + V(Y_3) - 2\text{Cov}(Y_2, Y_3) = 2n/3$.

3)

Let the joint density of two random variables x_1 and x_2 be given by

$$f(x_1, x_2) = \begin{cases} \frac{3}{8}x_1, & 0 \leq x_2 \leq x_1 \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

Find the covariance between x_1 and x_2 .

SOLUTION

First, we compute the expected value of X_1X_2 as follows.

$$\begin{aligned} E(X_1X_2) &= \int_0^2 \int_{x_2}^2 \frac{3}{8}x_1^2x_2 dx_1 dx_2 \\ E(X_1X_2) &= \int_0^2 \left(\frac{3}{24}x_1^3x_2 \Big|_{x_1=x_2}^{x_1=2} \right) dx_2 \\ E(X_1X_2) &= \int_0^2 \left(\frac{24}{24}x_2 - \frac{3}{24}x_2^4 \right) dx_2 = \int_0^2 \left(x_2 - \frac{1}{8}x_2^4 \right) dx_2 \\ E(X_1X_2) &= \left(\frac{x_2^2}{2} - \frac{1}{40}x_2^5 \right) \Big|_{x_2=0}^{x_2=2} = \frac{4}{2} - \frac{32}{40} = \frac{6}{5} \end{aligned}$$

Then, we compute the expected value of X_1 as follows,

$$\begin{aligned}
 E(X_1) &= \int_0^2 \int_0^{x_1} \frac{3}{8} x_1^2 dx_2 dx_1 \\
 E(X_1) &= \int_0^2 \left(\frac{3}{8} x_1^2 x_2 \Big|_0^{x_1} \right) dx_1 = \int_0^2 \left(\frac{3}{8} x_1^3 \right) dx_1 \\
 E(X_1) &= \frac{3}{32} x_1^4 \Big|_0^2 = \frac{3}{2}
 \end{aligned}$$

Similarly, we compute the expected value of X_2 as follows,

$$\begin{aligned}
 E(X_2) &= \int_0^2 \int_{x_2}^2 \frac{3}{8} x_1 x_2 dx_1 dx_2 \\
 E(X_2) &= \int_0^2 \left(\frac{3}{16} x_1^2 x_2 \Big|_{x_2}^2 \right) dx_2 = \int_0^2 \left(\frac{3}{4} x_2 - \frac{3}{16} x_2^3 \right) dx_2 \\
 E(X_2) &= \left(\frac{3}{8} x_2^2 - \frac{3}{64} x_2^4 \right) \Big|_0^2 = \frac{3}{4}
 \end{aligned}$$

The covariance is then given by

$$Cov(X_1, X_2) = E(X_1 X_2) - E(X_1)E(X_2)$$

$$Cov(X_1, X_2) = \frac{6}{5} - \left(\frac{3}{2} \right) \left(\frac{3}{4} \right) = \frac{3}{40}$$

4)

Let X and Y have a joint density function

$$f(x, y) = \begin{cases} 2, & 0 < x < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

- i) Find $f_X(x)$ and $f_Y(y)$.
- ii) Find $f_{Y|X}(Y|X = x)$ and $f_{X|Y}(X|Y = y)$.
- iii) Find $P(Y > 1/2|X = 1/2)$.
- iv) Find $P(X > 1/3|Y = 2/3)$.

SOLUTION

- i) Find $f_X(x)$ and $f_Y(y)$.

From the definition of marginal pdf,

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) \cdot dy = \int_x^1 2 \, dy = 2 - 2x \text{ for } 0 < x < 1$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y) \cdot dx = \int_0^y 2 \, dx = 2y \text{ for } 0 < y < 1$$

ii) Find $f_{Y|X}(Y|X = x)$ and $f_{X|Y}(X|Y = y)$.

$$\text{For } 0 < x < 1, f_{Y|X}(Y|X = x) = \frac{f(x,y)}{f_X(x)} = \begin{cases} \frac{1}{1-x}, & x < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\text{For } 0 < y < 1, f_{X|Y}(X|Y = y) = \frac{f(x,y)}{f_Y(y)} = \begin{cases} \frac{1}{y}, & 0 < x < y \\ 0, & \text{otherwise} \end{cases}$$

iii) Find $P(Y > 1/2|X = 1/2)$.

$$P(Y > 1/2|X = 1/2) = \int_{1/2}^1 \frac{1}{1 - \frac{1}{2}} dy = \frac{1}{1 - \frac{1}{2}} \cdot y \Big|_{y=1/2}^{y=1} = \frac{1}{2 \left(1 - \frac{1}{2}\right)} = 1$$

iv) Find $P(X > 1/3|Y = 2/3)$.

$$P(X > 1/3|Y = 2/3) = \int_{1/3}^{2/3} \frac{1}{\frac{2}{3}} dx = \frac{3x}{2} \Big|_{x=1/3}^{x=2/3} = \frac{3}{2 * 3} = \frac{1}{2}$$

5) The bivariate density function is defined as;

$$f(x, y) = \begin{cases} c(x^2 + xy) & \text{if } 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

- a) Find c and $P(X > Y)$.
- b) Find marginal densities of X and Y.
- c) Are X and Y independent?

SOLUTION

a. We find c from the condition.

$$\int_0^1 \int_0^1 f(x, y) dy dx = \int_0^1 \left(\int_0^1 f(x, y) dy \right) dx = 1.$$

We first evaluate the inner integral: if x is fixed then

$$\int_0^1 (x^2 + xy) dy = x^2 + x \int_0^1 y dy = x^2 + \frac{1}{2}x.$$

Now we evaluate the outer integral,

$$\int_0^1 \left(x^2 + \frac{1}{2}x \right) dx = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}.$$

Therefore, $c = 12/7$

The needed probability is

$$\Pr(X > Y) = \int_0^1 \int_0^x f(x, y) dy dx = \frac{12}{7} \int_0^1 \int_0^x (x^2 + xy) dy dx = \frac{12}{7} \int_0^1 \left(\int_0^x (x^2 + xy) dy \right) dx.$$

Again, calculate first the inner integral (x fixed)

$$\int_0^x (x^2 + xy)dy = x^3 + x \int_0^x ydy = x^3 + \frac{1}{2}x^3 = \frac{3}{2}x^3.$$

$$\Pr(X > Y) = \frac{12}{7} \times \frac{3}{2} \int_0^1 x^3 dx = \frac{12}{7} \times \frac{3}{2} \times \frac{1}{4} = \frac{9}{14}.$$

b. Find marginal densities of X and Y.

Solution: By definition (x fixed and y integrates out)

$$\begin{aligned} f_X(x) &= \int_0^1 f(x, y)dy = \frac{12}{7} \int_0^1 (x^2 + xy)dy \\ &= \frac{12}{7}x \left(x + \frac{1}{2} \right) \end{aligned}$$

for $0 \leq x \leq 1$.

Analogously (y fixed and x integrates out),

$$\begin{aligned} f_Y(y) &= \int_0^1 f(x, y)dx = \frac{12}{7} \int_0^1 (x^2 + xy)dx = \frac{12}{7} \int_0^1 x^2 dx + \frac{12}{7}y \int_0^1 x dx \\ &= \frac{12}{7} \times \frac{1}{3} + \frac{12}{7}y \times \frac{1}{2} = \frac{2}{7}(2 + 3y) \end{aligned}$$

for $0 \leq y \leq 1$.

c. Are X and Y independent?

Solution: X and Y are independent iff

$$f(x, y) = f_X(x)f_Y(y).$$

$$f_X(x) = \frac{12}{7}x \left(x + \frac{1}{2} \right),$$

for $0 \leq x \leq 1$.

$$f_Y(y) = \frac{2}{7}(2 + 3y),$$

for $0 \leq y \leq 1$.

$$f(x, y) = \frac{12}{7}x(x + y).$$

for $0 \leq x \leq 1$ and $0 \leq y \leq 1$. Then

$$f(x, y) \neq f_X(x) \times f_Y(y).$$

6) A valid joint probability density function is given by

$$f(y_1, y_2) = \begin{cases} 4y_1y_2, & 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

- a the marginal density functions for Y_1 and Y_2 .
- b $P(Y_1 \leq 1/2 | Y_2 \geq 3/4)$.
- c the conditional density function of Y_1 given $Y_2 = y_2$.
- d the conditional density function of Y_2 given $Y_1 = y_1$.
- e $P(Y_1 \leq 3/4 | Y_2 = 1/2)$.

f) Are Y_1 and Y_2 are independent?

SOLUTION

$$\text{a. } f_1(y_1) = \int_0^1 4y_1y_2 dy_2 = 2y_1, 0 \leq y_1 \leq 1;$$

$$f_2(y_2) = \int_0^1 4y_1y_2 dy_1 = 2y_2, 0 \leq y_2 \leq 1$$

$$\text{b) } P\left(Y_1 \leq \frac{1}{2} \mid Y_2 \geq \frac{3}{4}\right) = \frac{\int_0^{1/2} \int_{3/4}^1 4y_1y_2 dy_2 dy_1}{\int_{3/4}^1 2y_2 dy_2} = \frac{7}{64} * \frac{16}{7} = 1/4$$

c. $f(y_1 | y_2) = f_1(y_1) = 2y_1, 0 \leq y_1 \leq 1$

$f(y_1|y_2)$ is defined for $0 < y_2 \leq 1$

d. $f(y_2 | y_1) = f_2(y_2) = 2y_2, 0 \leq y_2 \leq 1$

$f(y_2|y_1)$ is defined for $0 < y_1 \leq 1$

e. $P(Y_1 \leq 3/4 | Y_2 = 1/2) = P(Y_1 \leq 3/4)$

$$= \int_0^{3/4} 2y_1 dy_1 = 9/16$$

f) $f_1(y_1) = 2y_1$ for $0 \leq y_1 \leq 1$ and $f_2(y_2) = 2y_2$ for $0 \leq y_2 \leq 1$. Then for $0 \leq y_1 \leq 1$ and $0 \leq y_2 \leq 1$, $f_1(y_1) * f_2(y_2) = 4y_1y_2 = f(y_1, y_2)$. Thus, Y_1 and Y_2 are independent.

7) For the given joint density function

$$f(y_1, y_2) = \begin{cases} 6(1 - y_2), & 0 \leq y_1 \leq y_2 \leq 1, \\ 0, & \text{elsewhere} \end{cases}$$

- a) Are Y_1 and Y_2 are independent?
- b) Find $E(Y_1)$ and $V(Y_1)$.
- c) Find $\text{Cov}(Y_1, Y_2)$.

SOLUTION

a)

The ranges of y_1 and y_2 depend on each other so Y_1 and Y_2 cannot be independent.

b)

$$f_1(y_1) = \int_{y_1}^1 6(1 - y_2) dy_2 = 3(1 - y_1)^2, 0 \leq y_1 \leq 1;$$

$$E(Y_1) = \int_0^1 3y_1(1-y_1)^2 dy_1 = 1/4.$$

$$E(Y_1^2) = \int_0^1 3y_1^2(1-y_1)^2 dy_1 = 1/10, \quad V(Y_1) = 1/10 - (1/4)^2 = 3/80$$

c)

$$f_2(y_2) = \int_0^{y_2} 6(1-y_2) dy_1 = 6y_2(1-y_2), \quad 0 \leq y_2 \leq 1.$$

$$E(Y_2) = \int_0^1 6y_2^2(1-y_2) dy_2 = 1/2.$$

$$E(Y_1 Y_2) = \int_0^1 \int_0^{y_2} 6y_1 y_2 (1-y_2) dy_1 dy_2 = 3/20.$$

$\text{Cov}(Y_1, Y_2) = E(Y_1 Y_2) - E(Y_1)E(Y_2) = 3/20 - 1/2 \cdot 1/4 = 1/40$. As $\text{Cov}(Y_1, Y_2)$ is different from 0, Y_1 and Y_2 cannot be independent.

8) For the following joint density function

$$f(y_1, y_2) = \begin{cases} e^{-y_1}, & 0 \leq y_2 \leq y_1 < \infty \\ 0, & \text{elsewhere} \end{cases}$$

- a) $P(Y_1 < 2, Y_2 > 1)$.
- b) $P(Y_1 \geq 2Y_2)$.
- c) $P(Y_1 - Y_2 \geq 1)$.
- d) Find the marginal density functions for Y_1 and Y_2 .
- e) What is the conditional density function of Y_1 given that $Y_2=y_2$? Be sure to specify the values of y_2 for which this conditional density is defined.
- f) What is the conditional density function of Y_2 given that $Y_1=y_1$? Be sure to specify the values of y_1 for which this conditional density is defined.

SOLUTION

$$\text{a. } P(Y_1 < 2, Y_2 > 1) = \int_1^2 \int_1^{y_1} e^{-y_1} dy_2 dy_1 = \int_1^2 \int_{y_2}^2 e^{-y_1} dy_1 dy_2 = e^{-1} - 2e^{-2}.$$

$$\text{b. } P(Y_1 \geq 2Y_2) = \int_0^\infty \int_{2y_2}^\infty e^{-y_1} dy_1 dy_2 = 1/2.$$

$$\text{c. } P(Y_1 - Y_2 \geq 1) = P(Y_1 \geq Y_2 + 1) = \int_0^\infty \int_{y_2+1}^\infty e^{-y_1} dy_1 dy_2 = e^{-1}.$$

d)

$$f_1(y_1) = \int_0^{y_1} e^{-y_1} dy_2 = y_1 e^{-y_1}, \quad y_1 \geq 0. \quad f_2(y_2) = \int_{y_2}^\infty e^{-y_1} dy_1 = e^{-y_2}, \quad y_2 \geq 0.$$

e) $f(y_1 | y_2)$ is defined for $0 \leq y_2$:

$$f(y_1 | y_2) = e^{-(y_1 - y_2)}, \quad y_1 \geq y_2.$$

f) $f(y_2 | y_1)$ is defined for $0 < y_1$:

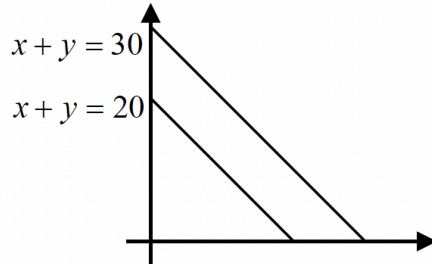
$$f(y_2 | y_1) = 1/y_1, \quad 0 \leq y_2 \leq y_1.$$

- 9) A health-food store stocks two different brands of a type of grain. Let X = the amount (lb) of brand A on hand and Y = the amount of brand B on hand. Suppose the joint pdf of X and Y is

$$f(x, y) = kxy \quad x \geq 0, y \geq 0, 20 \leq x + y \leq 30$$

- Draw the region of positive density and determine the value of k .
- Are X and Y independent? Answer by first deriving the marginal pdf of each variable.
- Compute $P(X + Y \leq 25)$.
- What is the expected total amount of this grain on hand?
- Compute $\text{Cov}(X, Y)$ and $\text{Corr}(X, Y)$.
- What is the variance of the total amount of grain on hand?

SOLUTION



$$\text{a.} \quad 1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = \int_0^{20} \int_{20-x}^{30-x} kxy dy dx + \int_{20}^{30} \int_0^{30-x} kxy dy dx = \frac{81,250}{3} \cdot k \Rightarrow k = \frac{3}{81,250}.$$

$$\text{b.} \quad f_X(x) = \begin{cases} \int_{20-x}^{30-x} kxy dy = k(250x - 10x^2) & 0 \leq x \leq 20 \\ \int_0^{30-x} kxy dy = k(450x - 30x^2 + \frac{1}{2}x^3) & 20 \leq x \leq 30 \end{cases}$$

and, by symmetry, $f_Y(y)$ is obtained by substituting y for x in $f_X(x)$. Since $f_X(25) > 0$ and $f_Y(25) > 0$ but $f(25, 25) = 0$, $f_X(x) \cdot f_Y(y) \neq f(x, y)$ for all x, y and so X and Y are not independent.

$$\text{c.} \quad P(X + Y \leq 25) = \int_0^{20} \int_{20-x}^{25-x} kxy dy dx + \int_{20}^{25} \int_0^{25-x} kxy dy dx = \frac{3}{81,250} \cdot \frac{230,625}{24} = .355$$

d. $E(X + Y) = E(X) + E(Y) = 2E(X) = 2 \int_0^{20} x \cdot k(250x - 10x^2) dx$

$$+ 2 \int_{20}^{30} x \cdot k(450x - 30x^2 + \frac{1}{2}x^3) dx = 2k(351,666.67) = 25.969 \text{ lb.}$$

e. $E(XY) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy \cdot f(x, y) dx dy = \int_0^{20} \int_{20-x}^{30-x} kx^2 y^2 dy dx$

$$+ \int_{20}^{30} \int_0^{30-x} kx^2 y^2 dy dx = \frac{k}{3} \cdot \frac{33,250,000}{3} = 136.4103, \text{ so}$$

$\text{Cov}(X, Y) = 136.4103 - (12.9845)^2 = -32.19$. Also, $E(X^2) = E(Y^2) = 204.6154$, so

$$\sigma_x^2 = \sigma_y^2 = 204.6154 - (12.9845)^2 = 36.0182 \text{ and } \rho = \frac{-32.19}{36.0182} = -.894.$$

f. $V(X + Y) = V(X) + V(Y) + 2\text{Cov}(X, Y) = 7.66$.